Issue 20 | November 2024
Triannual | Bangalore
An Azim Premji University publication

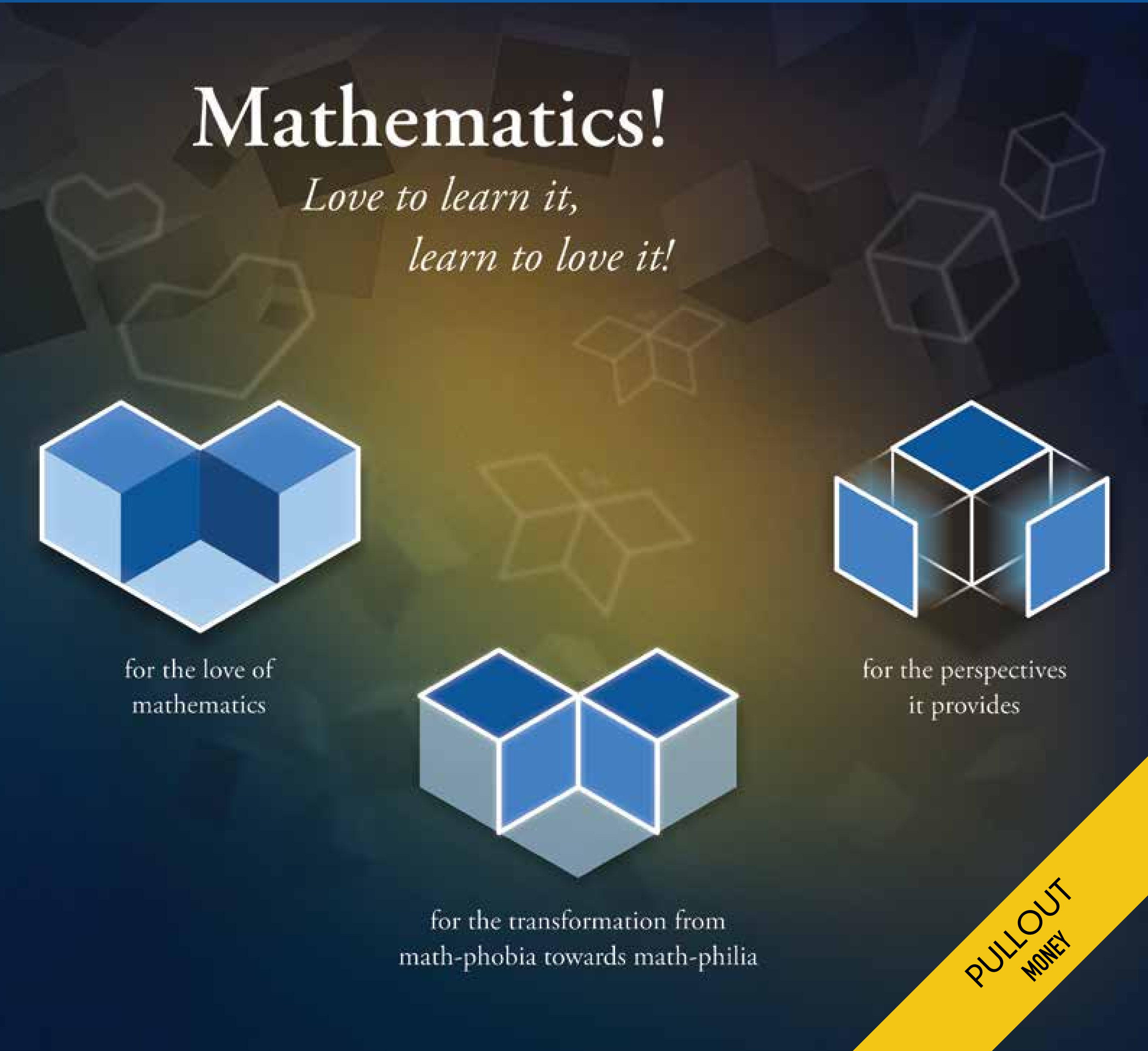
Azim Premji University

RNI No.: KARENG/2018/76619

Azim Premji University At Right Angles

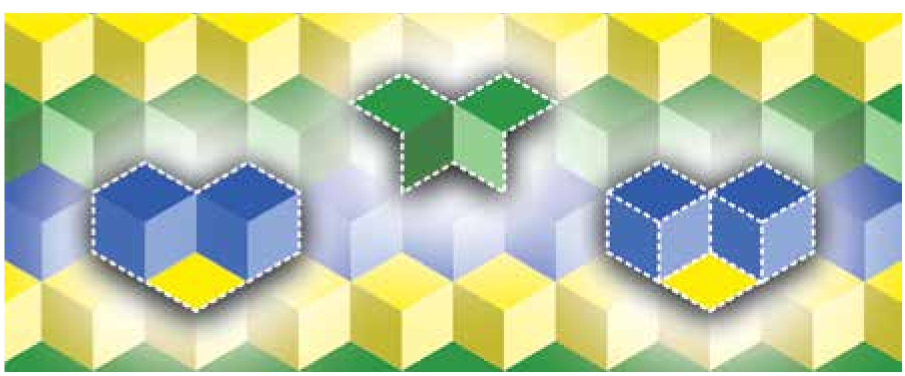
A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873



For the love of mathematics

Mathematics is usually associated with the brain- and no wonder that prowess in mathematics is a matter of pride, of ranking, of discarding those who don't excel at it. But what about a heart for mathematics? A gentler, less threatening way of seeing it? Of transformations in the classroom because of changed perspectives? How can we create inclusive, non-threatening spaces that draw students in, to the love of mathematics?



From the Editors Desk . . .

Celebrating a Decade of Mathematical Discovery

Dear Readers,

As we turn the pages of this issue, we celebrate not just the rich experiences of teaching and learning mathematics but also a remarkable milestone—the 10th anniversary of Math Space at Azim Premji University. Over the last decade, this space has grown into a vibrant hub for exploration, learning, and inspiration in mathematics education.

A Decade of Mathematical Inspiration

In our *Features* section, we mark this milestone through two engaging pieces. Nandita brings us a thought-provoking interview that reflects on the journey of Math Space and its impact over the years. Complementing this is a photo feature showcasing the unique *mat(h)erials* housed at Math Space, offering a visual journey through the resources that have inspired countless learners. Together, these articles capture the spirit of curiosity and deep understanding that defines Math Space.

From Classroom to Playground: Mathematics in Action

Our *Classroom* section offers a diverse range of articles, each addressing important aspects of elementary mathematics education. Narayana takes on the challenge of teaching word problems involving addition and subtraction, providing practical strategies for educators. Asma and Jivesh share their classroom experience of introducing patterns to young learners. Kshama highlights Montessori mathematics tools and offers guidance on how they can be adapted to regular classrooms. Anushka brings fresh insights into introducing the concept of algorithms to young children, making these complex ideas accessible at an early age.

Joy of Mathematics: Playful Explorations

Learning continues beyond the classroom in our *Joy of Mathematics* section, which showcases how mathematics can be both fun and intellectually stimulating. The section begins with a twist—an 'online' conversation among teachers debating how best to introduce fractions to students. Ajaykumar explores line symmetry through hands-on paper-cutting activities, making this abstract concept engaging and tactile. Tejas shares a personal story of encountering and solving a fascinating maths problem, concluding with a few problems for our readers to puzzle over.

Tools for Teaching, Materials for Understanding

In the *Review* section, we continue our focus on teaching tools with two insightful contributions. Mokhtar draws from his classroom experience to explore the use of *arrow cards* as a teaching tool, shedding light on their effectiveness in helping students grasp mathematical concepts. Math Space offers a comparative analysis of *Dienes blocks* and *static beads*, providing educators with practical guidance on how these tools can be used to make abstract ideas more concrete for learners.

Mathematics Beyond the Textbook

In this issue's *Pullout*, Padmapriya continues her series on the concept of money, offering activities that bridge classroom learning with real-life applications. The activities are designed to help children relate mathematical concepts to everyday financial experiences, making learning more meaningful and practical.

Fresh Perspectives from Our Online Section

Finally, our *Online* section features two contributions that highlight the creativity and innovation of younger voices in the mathematics community. Diksha, a student, presents an intriguing geometric construction that challenges readers to n-sect a square in new and unexpected ways. Gauri, a student-teacher, shares her experience of posing a problem about square roots that engaged her students and sparked lively discussions.

As we celebrate a decade of meaningful work at Math Space and look ahead to the future, we are reminded of the endless possibilities that teaching and learning mathematics offer. Let us continue to explore, question, and engage with mathematics, creating spaces where curiosity thrives and learning deepens.

Happy reading, and may your mathematical journey be as exciting as ever!

Warm regards,

Mohan R
Associate Editor

At Right Angles is a publication of Azim Premji University. It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an ‘academic’ and ‘practitioner’ oriented magazine.

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Nifty, Thrifty, and Crafty: Math Space Turns 10

Nandita Jayaraj

Over the decade of its existence, Math Space at Azim Premji University has been slowly, but surely, integrating itself into the fabric of the vibrant university ecosystem.

Walking into Azim Premji University's Math Space for the first time, I immediately felt like a kid in a toy store. Before me were shelves, tables and drawers overflowing with seemingly handmade contraptions; colourful stationery was scattered around virtually every available surface; discarded tea boxes and laptop packaging were stacked up across the floor space. The room was a great example of organised chaos, I thought, the kind that gets your fingers fidgety, eager to settle into a spot and create something. Needless to say, I was only too happy to allow Swati Sircar, the custodian of this space, to finish up a meeting while I browsed.

A mathematician by qualification, Swati worked as a teacher for a few years before she joined the university's School of Continuing Education and University Resource Centre in 2013. Passionate about maths education, Swati and her colleague and "partner-in-crime" Sneha Titus (now the Chief Editor of At Right Angles) envisioned and brought to life Math Space.

Confined at first to a tiny room in the PES University campus (where Azim Premji University was functioning before the permanent campus was completed), Math Space first shifted to a larger shared room in the same campus, and then to the sunny room in the Sarjapura campus where I now stood. By the end of next year, Math Space is due to move to its final home.

The journey of Math Space, Swati told me, began with a small cupboard in a room that was designated as a science laboratory for the upcoming undergraduate programmes. "The cupboard was full of some material from Jodogyan, a maths resource group based in Delhi. The key to the room used to be with the security staff, so every time we wanted to access the material, we spent 20 minutes retrieving the key and another 20 minutes to hand it over after we were done. This was not going to work out."

At this point, Swati and team recognised that these resources had a lot of potential in mathematics education. But before this potential could be realised, they would need to effectively document how teachers could use them and disseminate this through workshops and publications. Where would they do all this?

"At the time, the campus had a pottery studio. So I thought why can't we have a similar space for maths?" Swati said. The duo began to explore possibilities by which they

Keywords: Teaching Learning Materials, mathematics, laboratory, pedagogy, free play, cost-effectiveness, recycle, re-use.

could share the room with the science team. This was not approved, however, they were assigned a small room from which they could start a maths lab. "It was one of the tiniest rooms, smaller than even that pottery studio! And that was our very, very humble beginning," she reminisced.

It didn't take long for Math Space to get its first taste of success. Back in 2014, the university had only two programmes - MA Education and MA Development. As part of their curriculum, the Education students had weekly engagements at nearby government and low fee private schools. Apart from this, the foundation also used to run a couple of centres for children of migrant labourers. "We had these schools with whom we have an existing relationship. So we thought: why not do some workshops with them," said Swati. And so the duo began lugging their stash of resources into a larger room, where they conducted day-long workshops for teachers of these schools. These took place on a monthly basis and while they tried repeatedly to cap participants at 30, it was common for the workshop strength to cross 40. Other subjects began their own similar workshops the following year, and these took place across primary and middle school levels. Kannada versions of the workshops were also organised for the benefit of municipality schools. "The workshops were a big hit. It was much, much beyond our imagination," said Swati.

In a couple of years, it was universally agreed upon that they needed more space. As it turned out, they were re-assigned to a larger room, the same one they had initially been denied. The room began to function as a combined maths and science laboratory, with teachers participating in workshops and students performing science experiments. "Everyone was happy about this as the space was finally being fully utilised," said Swati.

In addition to managing Math Space, Sneha and Swati were also teaching an elective called Curricular Material Development in Mathematics to MA Education students. During the course, they would encourage their students to create games, worksheets and other material. "We reminded them that this would not only get them grades and experience, but whatever they produce - their babies - would become part of the lab," she recalled. This motivation worked. Students created various maths teaching resources such as a game to learn fractions, a model to visualise trigonometric functions, etc. "Sometimes one batch would make something, and the next batch would get inspired to build on the same thing," she said, while proudly showing me a shelf full of resources that had been built by her former students.

Along the way Math Space also established a valuable partnership with the organisation Mantra4Change, which ended up hiring numerous alumni from the university. Alumni, now employed by various organisations, would return to Math Space to brainstorm new ideas with Swati. People from various NGOs working in education, too, would visit them, using their space to tinker and create.

Math Space further evolved with the initiation of the Student Assistantship programme which permitted students to have part-time jobs at the university. "We took advantage of this, and started employing a few students during semester breaks," said Swati. The employed students, representing various disciplines, busied themselves with tasks related to the production of math teaching resources. Depending on their skills, some would draw, some would cut, others would manage, and so on. This streamlining allowed Math Space to bulk produce this *mat(h)erial* that would eventually go to schools across the country.

Job openings at Math Space became so much in-demand among students that last summer, 16 of them signed up! It was gratifying for Swati to see students so engaged in the pursuit of accessible and high quality maths education, and even more so to see them come up with newer ideas and improvements for existing kits. "People were wondering what on earth was going on here," Swati laughed. After a round of auditing, the authorities were reassured that all was in order.

"Now we have very strict limits. We can employ only up to 10 students, and we have a huge waiting list."

Meanwhile, another exciting partnership was developing, with the university's Infrastructure Management Function (IMF) team. The IMF is responsible for a range of activities fundamental to the functioning of the university, including land acquisition, construction, furnishing of offices, administration and facility management and purchases. "We have a good tie up with the IMF, through which we were able to source waste materials such as tea bag boxes, discarded posters, etc." said Swati. After a point, she observed with amusement that Math Space was turning into an unofficial archive of sorts. "Nothing is thrown away here. So sometimes IMF people come in here looking for old posters they need for documentation. It's really funny!" More recently, when the university finished its landmark 40-storey student residence building, the IMF enlisted Math Space to create 40 key boxes for them to keep the keys organised in a floor-wise fashion. "With the IMF as one of our clients, so to speak, sourcing material has become much easier. They are a huge partner in all this."

The impact of Math Space is getting increasingly evident. The maths resources they produced were taken to various schools in the northeast by university educators working in the area. Some of them even found their way to state board textbooks.

In this age of 3D printing and mass manufacturing, I asked Swati if she has been tempted by the allure of shifting to toolkits made of plastic and other synthetic materials. Wouldn't these be quicker to produce, last longer and relieve her of the trouble of having to source waste and recycle? "Our resources last long enough," she affirmed. Besides, she pointed out, the whole point was empowering teachers and students to know they can recreate these materials. "You break it, you fix it. It's as simple as that. It's not like if they break it, they have to stand in the corner as punishment. That's why we make it with low-cost or no-cost materials available around us."

The matter of cost is particularly relevant here. "For our material to be effective, a class of 30 needs to have at least 6 sets. Which government school has the budget to buy six sets of every material?" she asked. Swati illustrated this with the case of one of her former BSc-BEd maths students, who shared with her the scenario at the "maths lab" in his school. "I call it a 'maths temple'! Because the students were taken there in a line, not allowed to touch anything, they just do one round of *darshan* and then go back to class." Such stories are a thorn in the side of educators like Swati. "What is the point of such a maths lab? Students should be playing with the material, moving things around, putting them together, and saying 'Oh! This happened!' or 'See what I made!'."

Ten years since it came into being, Math Space has become an integral part of the fabric of the vibrant university ecosystem. And there is more to come. "We are working on explanations, explorations, posters, videos, worksheets, and animated presentations for introducing various concepts and constructions. We also hope to include games in the future," Swati elaborated. There is also a website now, where many of these resources are being documented https://sites.google.com/apu.edu.in/mathspace/home. Clearly, this is only the beginning. "Oh and this is not going to be our last location," Swati reminded me. "The upcoming Campus Commons will be our permanent home."

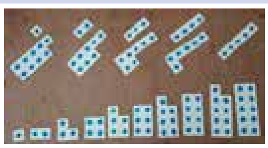
NANDITA JAYARAJ is a freelance science writer, author and co-founder of Lab Hopping Science Media Forum and TheLifeofScience.com. She is currently also a science communication consultant at Azim Premji University. You can reach her at nandita.jayaraj@apu.edu.in

A Walk Through Math Space

Ready 1… 2… 3…

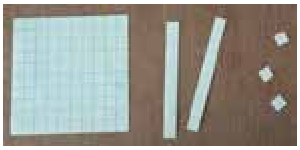
Numeracy: Counting to measuring

Counters



Tens frames
(handmade with carefully chosen dimensions)

Big FLU for Foundational stage



Small FLU for Preparatory stage

Decimal FLU

Arrow cards

Decimal Arrow Cards

Manipulatives for fractions...

Fraction-strips

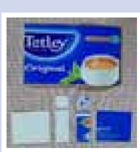
Converting laptop boxes to key organizers – we recently took this order for 40 boxes, explored possibilities, finalized design after one round of feedback and delivered all boxes within 3 days!

We are very proud that the core idea of cutting the laptop box in order to preserve its locking feature came from a BSc-BEd math student, who had just joined Math Space team!

Shapes and Space

...and more

Fraction sectors



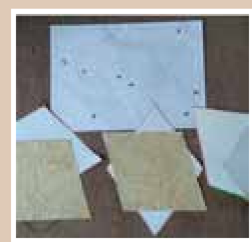
Algebra tiles

Triangles kit

Playing with
Polyominoes and Tangrams

Boxes and packets
made to keep the
mat(h)erials sorted

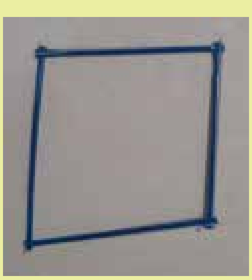
Geometry and Mensuration kit



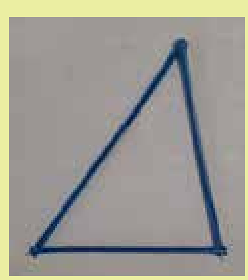
Cutouts

Straw models

Nets

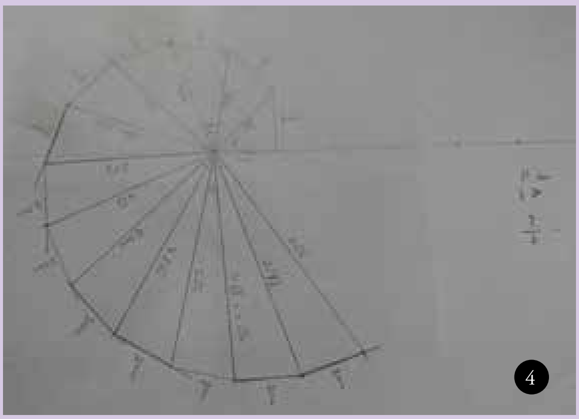


One model, all
possible quadrilaterals



Same for triangles –
came few years later!



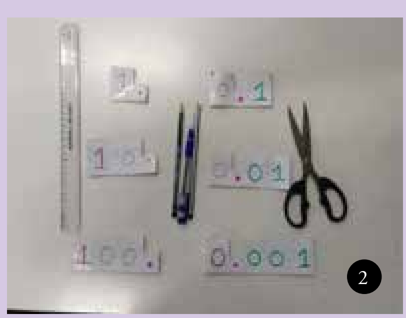


Workshop Series

1. Primary math workshop
2. Tray including stationery and mat(h)erials
3. 3-day Investigation workshop after the primary series. Note the upcycling of the A4 paper box cover to a tray
4. Square root spiral made by a participant during the upper primary series; this spiral led to one article by a participant, which further generated a Low Floor High Ceiling (LFHC) article after the 3-day Investigation workshop following this series [See reference for links to both articles]
5. Model for algebraic identities (quadratic): each participant made these post lunch, the session reserved for mat(h)erial making

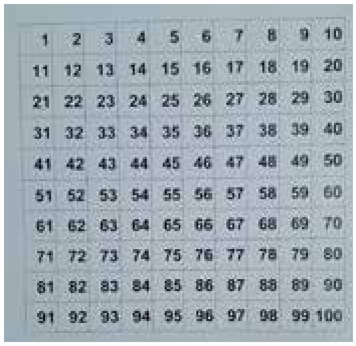
Math Space ran three series of monthly one-day workshops, around 7-9 sessions each – for Primary (Class 1-5), Upper Primary (Class 6-8) and Secondary (9-10) from 2015-2018. Post lunch sessions were usually reserved for mat(h)erial making, one set per school. The Primary and Upper Primary series were followed by 3-day Investigations workshops during summer break.

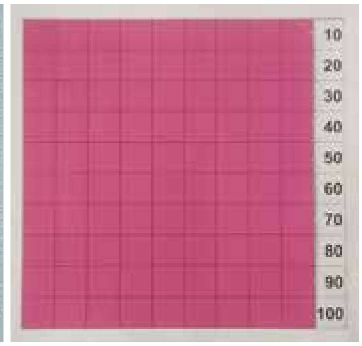
Math Space was invited to conduct several workshops with government school teachers from several states with a strong emphasis on mat(h)erial making.

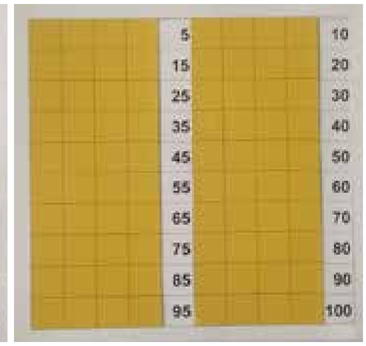



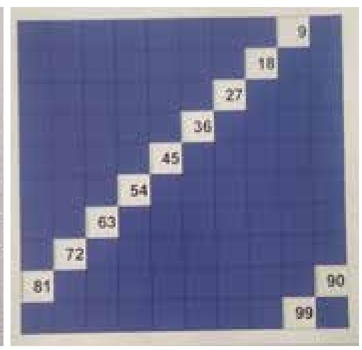
1. Fractions sectors made by government school teachers in Uttarkashi
2. Decimal arrow cards made by a similar set of teachers in Udham Singh Nagar

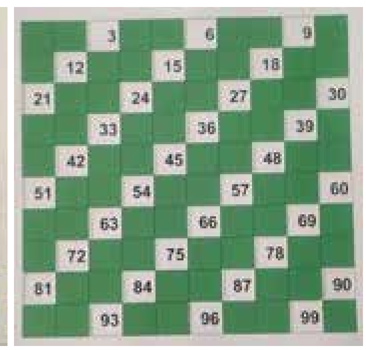
CMD-Mat(h)erials

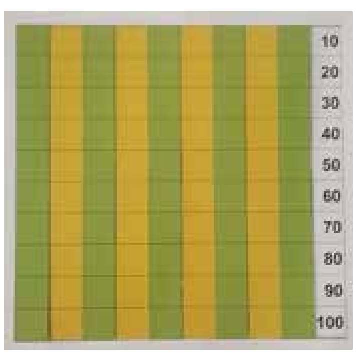




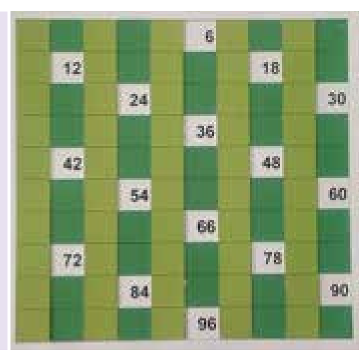








Exploring multiples and common multiples with frames – leads to divisibility rules



Intersecting circles for angles

Spinners of various shapes and distributions

1. Fraction sectors made from ceiling fan boxes and packed in the plastic cover for the blades – each long packet is sealed at the short end, long end cut open and stapled to create pockets for different sectors - Pokhrama, Bihar
2. Teachers engaging with polyominoes after attending an online Math Space workshop
3. Straw branches showing power of 2, from northeast India
4. Ganitmalas made by students at a government school – initiated by Mantra4 Change (M4C) after they attended primary math workshops by Math Space

M4C carried the spirit of Math Space to many states all over the country… ☺

Montessori Mat(h)erials:
To know more about these, read the accompanying article in the same issue

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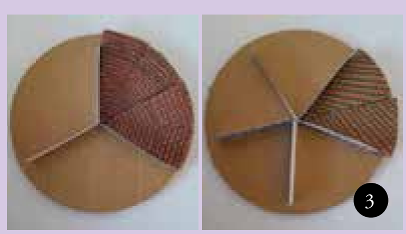
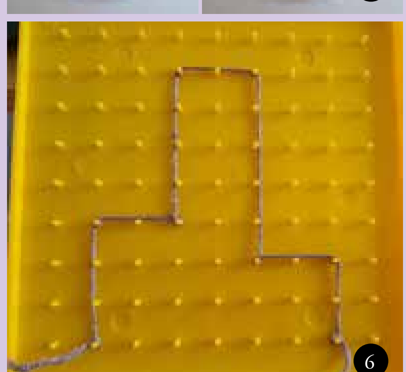
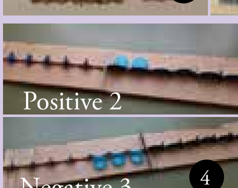
First set of pink tower, brown stairs, long rods, number rods and cylinders with the makers







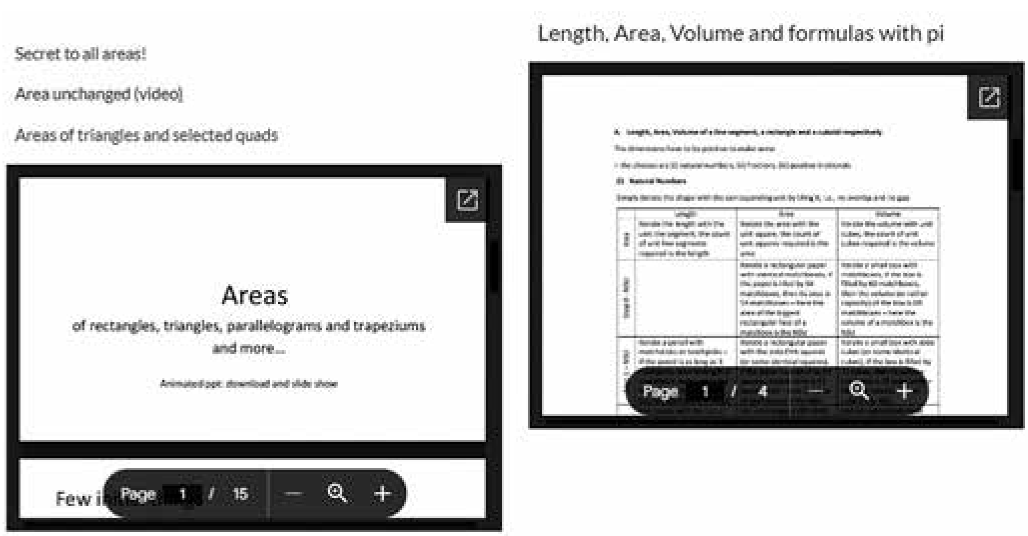
Tactile Mat(h)erials for Visually Impaired Learners

1. Tactile algebra tiles
2. Tactile fraction wall
3. Tactile fraction circle and sectors
4. Tactile number line with bottle caps
5. Tactile protractor, every 15°
6. Histogram on geoboard

Website: https://sites.google.com/apu.edu.in/mathspace/home

Math Space website has the following sections:

- **Mat(h)erials –** textbooks, websites, visuals, low-cost Montessori ones, various kits and more includes details of how to make (with layouts) and how to use
- **Numbers and Operations –** whole numbers, fractions, decimals, integers, rational numbers, exponent and roots – the thrust is meaning making and a step-by-step build up from first principles
- **Geometry and Spatial Understanding –** 2D and 3D shapes, symmetry, mapping 3D to 2D includes a range of resources (animated ppts, worksheets and more) especially for middle stage
- **Measurement –** length, weight, capacity etc. **– and Mensuration –** 2D: perimeter, area etc. and 3D: surface area, volume etc. including exploration of formulas involving pi (π)
- **Patterns and Algebra –** patterns, expressions, equations, identities as well as ratio-proportion and percentages
- **Data Handling, Statistics –** an in depth look at mean, median, mode **– and Probability –** spinners, Bayes' theorem, independence



The work is far from complete. Many resources, especially (i) exploring fractions through various models, (ii) using the notion of algebra tiles to solve equations and (iii) several visuals, are in the pipeline.

There are explanations, explorations, posters, videos, worksheets, and many animated ppts for demonstrations as well as introducing various concepts and constructions. There are a few Proofs Without Words (PWW) – more will be included. We also hope to include a **game** section in future.

References:

1. Khushboo Awasthi (2017, March). Drawing A Spiral of Square Roots, At Right Angles 6(1). 39-45. https://bit.ly/3Ui2Br6
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Unveiling the Magic of Patterns

Asma Memon & Jivesh Panchbhai

In this article, the authors talk about their experience in planning and conducting sessions on patterns in a Grade 3 classroom. Two Teaching Learning Materials (TLMs) - Rangometry and Aakaar Parivar sets were used, and unexpected learnings were seen!

Mathematics is often described as the language of patterns. Observing and extending patterns is a valuable skill at any level of mathematics education. It is, therefore, not surprising that the National Council of Educational Research and Training (NCERT) emphasises the importance of observing and extending both geometrical and numerical patterns as a key learning outcome. (NCERT, 2017)

To meet these learning outcomes, the NCERT textbook for Grade 3 has a chapter "Play with Patterns". The chapter starts with repetitive patterns that are based on colour, a sequence, and rotation by a fixed angle. Then, it introduces some growing patterns. This is followed by introducing repetitive number patterns and growing number patterns which are based on adding small numbers or adding 10.

This article is based on teaching this chapter to thirty Grade 3 students in a school in a town in Uttarakhand, India. The children were largely from the middle-class or lower-income families of the town.

Session 1

The students were expected to draw patterns in their notebooks that would be similar to or inspired by the textbook patterns.

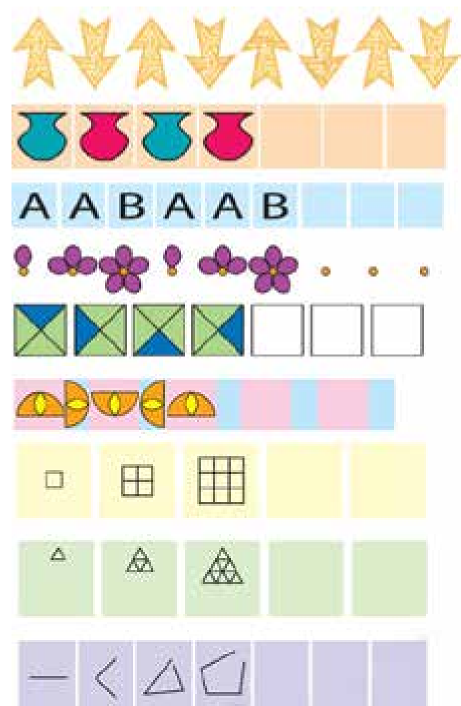

Figure 1: Patterns given in the NCERT textbook for Grade 3

Keywords: Patterns, Rangometry sets, Aakaar Parivar sets, growing patterns, increasing patterns, decreasing patterns.

To make the session interesting and hands-on, the teacher divided the students into six groups of 5 students each and distributed Rangometry sets to them. They were asked to make their own patterns with these sets.

Exploring patterns using Rangometry

Students created various patterns using the Rangometry shapes. Some focussed on the colours being in a pattern while some others focussed on the shapes. Watching them work together, sharing the sets and being completely engaged was a treat to the eyes!

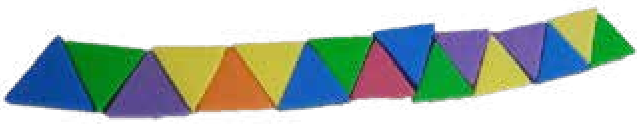

Figure 2: Alternate repeating 2D pattern made using inversion of triangles

***Note:** The student has decided to ignore the colour of the shapes to make this pattern.*

The textbook and the crayon drawn patterns in notebooks both focused on two-dimensional (2D) patterns. Thus, the teacher had expected similar 2D patterns from Rangometry as well. However, to the teacher's surprise, students started stacking the shapes to make three-dimensional (3D) patterns.

Figure 3: Alternate repeating 3D pattern made using colour and positioning of squares

Session 2

In this session, the teacher planned to introduce increasing and decreasing patterns to students. Due to a shortage of Rangometry sets, the teacher decided to mix Aakar Parivar sets with Rangometry. Unlike Rangometry, Aakar Parivar sets have shapes in different sizes, leading to students making increasing and decreasing patterns on their own.

Exploring patterns using Aakar Parivar

Here are some such patterns:

Figure 4: Decreasing pattern (starting from the bottom) made using squares in Aakar Parivar

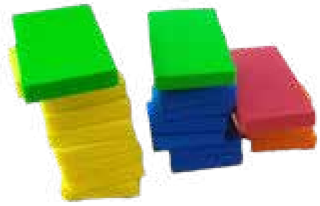

Figure 5: Growing pattern (from right to left) made by varying the number of rectangles in the stack.

***Note:** Here, students most likely didn't focus on the number of rectangles that were being added in each subsequent stack.*

Initially, the student increased the number of squares by 1, that is, 1, 2, 3, 4 but then there was no stack with 5 squares and thus, it seems like the focus was just on increasing the height and not on the numbers.

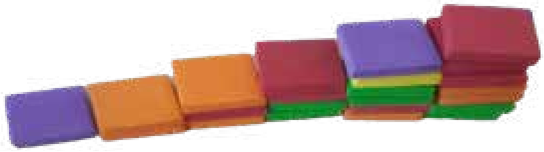

Figure 6: Growing pattern made by increasing the number of squares in each stack.

A concern and strategies to address it

Students quickly noticed that these TLMs can be used to make images such as faces, peacocks, etc. and many students started making them!

Figure 7: Face made by a student using Rangometry and Aakar Parivar sets

In such situations, it is not advisable to reprimand the student for not following instructions. Lecturing them on the meaning of patterns and the use of the materials may turn out to be futile. However, this can be seen as an opportunity to develop an intuitive understanding of what a pattern is. Creating the first few shapes of the pattern and asking the student to continue it may help. Picking a shape and asking why one can or cannot place it next in the sequence can help build curiosity. Giving examples of peer's work to develop an intuitive understanding can also be helpful, although a point to keep in mind for the teacher is to ensure that this does not lead to any comparison. Thus the stray instances of wrong usage of TLMs may be mined for learning opportunities.

To keep the students engaged throughout the session it is important that the teacher has rich conversations with the students, and have relevant follow-up questions. Conversations around the patterns each student made, how two patterns are similar or different, and questions such as: "Keeping the first three shapes the same can a different pattern be made?", "What exactly is increasing or decreasing here?", "What can you do to change this pattern into a growing pattern?", can engage students more with patterns.

The use of TLMs led to a lot of Aha! Moments during the sessions. The below-mentioned patterns provide evidence of learning resulting from the doing stage.

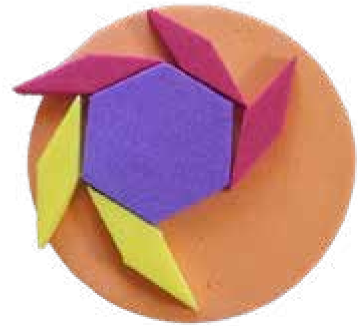

Figure 8: Pattern made by rotating quadrilaterals in an anticlockwise direction

Here, a student has made this pattern in which each quadrilateral is rotated anticlockwise to get the next quadrilateral.

Also, the student has used two criteria simultaneously, that is, colour and rotation. Similarly, in the below-mentioned pattern, multiple criteria have been used simultaneously.

Figure 9: Decreasing pattern made using rectangles where colour and positioning also have a pattern.

This is beyond what a teacher would have expected from a pattern made by a third-grade student. In fact, the teacher's educational judgement of letting students explore the material on their own led students to be creative and unveil the potential of the materials. This gave them the freedom to choose to explore their environment as Montessori's theory of learning suggests. (Faryadi, 2007)

Figure 10: A display of a variety of students' work

Conclusion

To conclude, this classroom sets a good example of how a suitable choice of activity plan and Teaching Learning Materials can create a learning environment where students learn by using their creativity and exploring their potential. At the same time, it's important to note here that there should be adequate material available for every student. For example, in this classroom, we tried to give at least one set of Rangometry and one set of Aakaar Parivar to each group of 5 students. This allowed them enough scope to play with colours, shapes, and sizes. However, in case of shortage, students helped each other by sharing the required colour and shape pieces. Thus, this activity helped to improve teamwork and classroom dynamics as well. These sessions highlighted children's sensitivity towards the factors of colour and shapes, and incorporating these in any TLM can enhance the learning experience.

Acknowledgement: The authors would like to thank Kshama Chakravarthy for her help in editing this article.

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Introduction to Algorithms

Anushka Tonapi

This article explores the concept of algorithms, their significance, and guides us through various everyday instances of algorithms in use, as well as mathematical examples highlighting their applications.

What is an Algorithm?

Have you ever wondered how your favourite video games work? These devices or apps work by understanding code in a special programming language that the computer understands. This code provides instructions for the computer to execute, allowing you to use the app or play the game. This set of instructions to perform a particular task is called an algorithm. However, to be called an algorithm, the set of instructions must have a clear starting and ending, else you will have an algorithm that goes on forever!

Algorithms aren't just for computers; we humans use them all the time in our daily lives! Let's see how with a simple example: Imagine you want to make a tasty snack – a roti with jaggery (gur) and ghee. You follow these steps to make it:

1. Get a roti.
2. Spread a spoonful of ghee on it.
3. Grate some jaggery.
4. Sprinkle grated jaggery over the roti.
5. Roll up the roti.

Now you have yourself a tasty snack. Enjoy!

Roti with ghee and jaggery

Why are algorithms important?

Algorithms are everywhere! They help computers, robots, and even people solve problems efficiently. An algorithm is a set of precise instructions that ensures that a task is performed consistently and accurately, every time it is followed. They are intended to be carried out even by a machine and hence should be unambiguous - they cannot have more than one meaning!

Algorithms are particularly useful for automating repetitive tasks, and ensuring consistent output. An algorithm is like a recipe: if the instructions are clear and precise, the dish should taste the same every

Keywords: Algorithm, sequence, flow, procedures, code

time, no matter who cooks it. Machines can do repetitive tasks when given a well-defined sequence of instructions – in that case, the output will be precisely the same in each iteration. Using an algorithm to automate tasks ensures that there is no human error due to boredom or fatigue as the machine executes the instructions exactly the same way each time.

This means the current algorithm for making roti with jaggery isn't really an algorithm because different people might interpret the instructions in various ways. For example, the first instruction, "get a roti," doesn't give enough detail. One possible way to improve this instruction is to specify the size, thickness, and ingredients of the roti. We can rewrite it as: "Get one cooked atta roti with a radius of about 5 cm and a thickness of 2 mm." The second instruction can also be clearer by stating: "Take a teaspoon of Amul ghee and spread it evenly on one side of the roti." We encourage readers to refine the remaining steps to make them clear and precise.

As you can see, without algorithms, our world would be much more chaotic and much less fun. By learning about algorithms, you can start to understand how to solve problems better, whether it is in math, science, or everyday life.

Getting started with algorithms

Question: Can there be multiple algorithms to solve one problem?

Yes, there can be multiple algorithms to solve a single problem! For example, suppose you are trying to get from your house to school. You could take the school bus, or you could ride your bicycle. Both methods will get you to school, but each one requires a different set of steps – or in other words, a different algorithm.

Question: Can we solve a problem without an algorithm?

Well, we might think we can solve problems without an algorithm, this is often because we are not consciously recording the steps we are taking. Without a clear algorithm, we might find ourselves going back and forth while solving a problem. Not just that, we may not identify or remember the most efficient way to solve the problem and we may have to repeat all our reasoning when we solve a similar problem. While solving a problem, we should use an organized sequence of steps. We read the problem, analyse the information, and make a plan to solve the problem. Then, we carry out this plan and find the answer. Let us dive into some fun activities to learn more about algorithms. We will explore through stories, games, and exercises that make learning about algorithms as exciting as playing your favourite game.

Activity 1: Story Time - "The Thirsty Crow"

Objective: Use stories to communicate how characters use algorithms to solve problems.

Story: *It was a hot summer afternoon and Kalia the crow was thirsty. He found a mud pot filled halfway through and perched on the edge of the pot. He put his long beak in and tried to sip the water, but it was useless, as the water level was too low for him to drink from!*

The thirsty crow.

Kalia looked around him, his beady eyes scanning the ground. He found a pile of pebbles nearby. He hopped off the mud pot's edge, picked a pebble up with his beak, and hopping onto the pot's edge again, dropped the pebble into the pot with a plop! He kept dropping pebbles into the pot until the water level reached high enough for him to drink from. Kalia the crow drank to his heart's content and flew away refreshed!

Let's arrange Kalia's solution to his problem into an algorithm:

Kalia's algorithm

1. Find a pebble.
2. Drop it into the pot.
3. Measure the water level by putting your beak into the pot.
4. If the water level is not high enough for your beak to reach, repeat steps 1-3.
5. If the water level is high enough for your beak to reach, drink enough water to satisfy your thirst!

Discussion: This list of steps is Kalia's algorithm to satisfy his thirst. By following each step precisely, Kalia can prevent getting dehydrated!

Questions

What do you think would happen if Kalia forgot to do one of the steps in his algorithm, such as not checking the water level after dropping a pebble? Why is it important to follow each step in the right order?

Can you think of a different algorithm Kalia could have used to solve the problem? What if there were no pebbles around? For example, could he have pecked the earthen pot and made a hole with his beak to get the water?

Activity 2: Dance algorithm

Objective: Learn about sequences and order of steps.

Game: We write down the steps to a simple dance and then follow them together. We can even follow the steps of popular nursery rhymes/songs. For example, for the song 'Looby Loo', we can put the lyrics into the following dance algorithm:

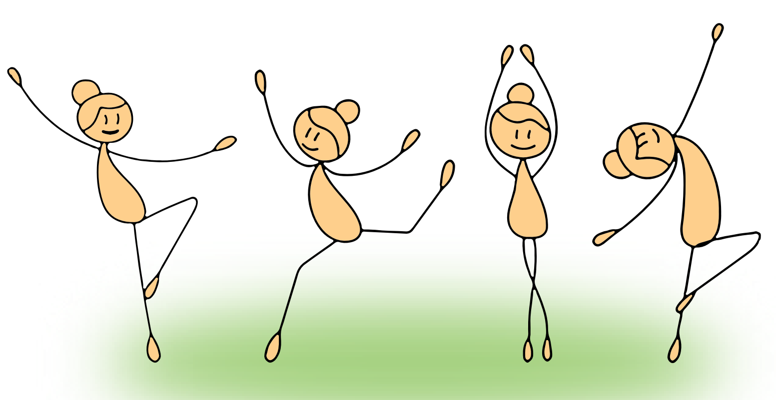
Dance Steps

1. Stand with both your hands horizontally outstretched along your sides.
2. Raise your right hand up.
3. Shake it up and down.
4. Spin around once.
5. Raise your left hand up.
6. Shake it up and down.
7. Spin around again.

Instructions

1. Write the steps on a board or paper.
2. Practice each step slowly according to a song or rhythm.
3. Put all the steps together and dance.

Discussion: By following the dance algorithm, you can remember the dance moves and perform them in the right order. If you mix up the steps, the dance will not look the same. This shows why the order of steps in an algorithm is important. Make sure to dance to the algorithm along with some music! You can use any music of your choice, but it would be helpful if it was a song with lyrics that included dance steps or body movements.



Dance steps

Activity 3: Planting seeds algorithm

Objective: Use an algorithm to plant a seed and observe the growth of a plant.

The "Planting Seed" Algorithm

1. Collect 15-20 seeds.

2. Take a pot of diameter 30 cm and height 20 cm and fill upto 70% of it with soil.

3. Add 20% of compost.

4. Make 5-6 small holes with your index finger.

5. Place 3-4 seeds in each hole.

6. Cover the holes with soil.

7. Water the seed with a watering can.

8. Place the pot where there is gentle sunlight and water regularly.

9. Observe the plant sprout and grow.

Discussion: Importance of the sequence and of each step.

- What happens if Step 3 is omitted?
- What happens if Step 8 is omitted?
- What happens if Step 6 is done before Step 5?

Activity 4: Everyday Algorithms

Objective: Identify algorithms in daily activities.

Exercise: Let's think about some daily tasks and break them down into algorithms.

Example 1: Brushing Your Teeth

1. Take your toothbrush.
2. Apply toothpaste to the brush.
3. Wet the toothbrush under the tap.
4. Brush your teeth for two minutes. (**Note:** You can make an algorithm for this step too!)
5. Rinse your mouth with water.
6. Clean your toothbrush and put it back in its place.

Example 2: Packing Your School Bag

1. Gather all your schoolbooks.
2. Check your timetable.
3. Put the right books in your bag.
4. Add your pencil case.
5. Check if your pencils are sharpened, and you have an eraser.
6. Check if you have your lunch box and water bottle.
7. Zip up your bag.

Discussion: By breaking down these tasks into simple steps, you can complete them more efficiently and remember everything you need to do.

Math games with Algorithms

Game 1: Sum of the first n natural numbers via pattern recognition

Objective: Finding the sum of the first 100 natural numbers

Algorithm

1. You write the problem on the board: 1 + 2 + 3 + ... + 100
2. Ask the students if they recognize any patterns in the sequence
3. Add a line below the problem: 100 + 99 + 98 + ….. + 1
4. Ask the students if they recognize any patterns in the sequence
5. Demonstrate the pairing of numbers on the board: (1 + 100), (2 + 99), (3 + 98), ..., (50 + 51)
6. Recognize the pattern that all these pairs sum to the same number 101
7. Discuss how many such pairs there are
8. Find the product of the number of pairs and 101
9. Identify how this product is related to the sum of the sequence.
10. Try and generalize the pattern for other numbers by applying the same pattern and coming up with a formula!

Note to the Teacher

Discuss how recognizing patterns helps to design the steps of an algorithm. When students recognize a pattern, they are essentially breaking down the problem into simpler and repeatable parts. For example, in the sequence (5, 10, 15, 20…), if the students notice that there is an increment of 5 with every term, this pattern can help them solve problems related to this sequence. Similarly, the purpose of an algorithm is to break down a complex problem into a series of instructions or steps that, when followed, help them arrive at an efficient solution to the problem. In the context of algorithms, pattern recognition is extremely important, as it helps students break down complex problems into simpler and manageable components. Sorting and recognizing patterns in data helps students understand the concept of categorization.

Conclusion: The Power of Algorithms

Algorithms might sound complex, but they are just a way of thinking about solving problems step by step. Whether you are making a snack, solving a problem, or playing a game, algorithms help you do things more efficiently and effectively.

By learning about algorithms, we are not just getting better at using computers or doing math—we are becoming better problem solvers. So, the next time you are faced with a tricky problem, remember to think like an algorithm: break it down into smaller steps, follow each step carefully, and you will find your solution!

Algorithms are like treasure maps leading you to the right answers. The more you practise, the better you will get at finding your way.

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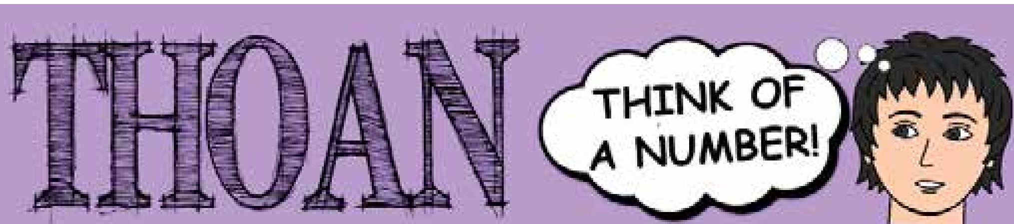

1. Choose any two digit number.
2. Reverse it.
3. Subtract it from the first number you had chosen.
4. What answer did you get?

What happens if the digits are the same? What happens if the digits are different? Can you explain the pattern you observe?

Look out for more THOAN activities from **Yathiraj Sharma** in upcoming issues of At Right Angles.

Montessori Approach: An Introduction to Selected Materials and How to re-Create Them (Part 1)

Kshama Chakravarthy

In this article, we examine various Montessori materials, discussing their introduction to children, the lessons they facilitate, the connections between mathematical concepts, supplementary activities that can accompany each material, and a budget-friendly alternative for creating these resources. This is a 2-part article, covering a total of six Montessori materials.

The National Education Policy 2020 (NEP 2020) makes clear recommendations for education at all levels, starting with education for children of age 3 to higher education. Schooling has been divided into four stages based on the styles of learning best suited for those age groups — Foundational Stage for ages 3-8, Preparatory Stage for ages 8-11, Middle Stage for ages 11-14, and Secondary Stage for ages 14-18 (1).

The National Curriculum Framework for School Education (NCFSE, 2023) mentions that *currently, a large proportion of students in the early grades are not achieving Foundational Literacy and Numeracy. Mathematics learning has traditionally been more 'robotic' and 'procedural' rather than creative and aesthetic*. Students also tend to develop a sense of fear around the subject of Mathematics. NCFSE 2023 suggests interactive teaching-learning methods involving play, exploration, discovery, discussion, games, and puzzles to help counter this fear. The content suggested to achieve the Learning Standards is predominantly concrete play materials, such as toys, puzzles, picture books, and manipulatives during the first three years. Textbooks/ playbooks/ workbooks are recommended only from Grade 1. *The pedagogy is largely play-based and emphasises nurturing, caring relationships between the teacher and the children. There should be a balance between self-paced individual learning and group activities.* (2)

Among the various methods of early education that are predominantly followed in India and across the world, Montessori education, started by Dr. Maria Montessori is a century-old educational philosophy and practice, based on hands-on learning, self-paced and self-directed activities and collaborative play. In Montessori classrooms,

Keywords: Patterns, Montessori, TLMs, low-cost materials, DIY

children can make choices in their learning activities, with the environment and the teacher who is trained specifically in the Montessori ways ensuring age-appropriate activities are offered to the child to choose from. Children work individually and also do group activities, discovering and exploring the world around them, gaining knowledge and developing to their maximum potential. These classrooms are well designed and thoroughly thought through to meet the specific needs of the child based on their age. Based on her experience of working with children directly, Dr. Maria Montessori discovered that experiential learning in such an environment led to a deeper understanding of mathematics, language, geography, science and a lot more. It helps develop and promote curiosity in a child and builds a strong foundation for lifelong learning. One can see that the overall vision for early education listed in NEP 2020 is achieved to a great extent in the Montessori method of education.

The Montessori Curriculum consists of five key areas of study: Practical Life, Sensorial, Mathematics, Language, and Culture. There is a set of Montessori materials for each area, focussing on a key knowledge or skill. In this article, we will look at some of the sensorial materials, specifically those that allow for visual discrimination (size, length, breadth, and width of objects). As you read through, you will realise that there's so much more to them- mathematics, language, focus and a lot more that is intertwined! It will hopefully give a glimpse of what can be done in the class and what to expect as an outcome from it.

Material 1: Pink Tower

Pink tower is one of the initial materials given to children in the Montessori environment. It consists of 10 pink (no prizes for guessing!) cubes, of volume varying from 1000 cubic centimetres to 1 cubic centimetre, from the bottom to the top. It is typically introduced to children in the age group of 2 ½ to 3 years, to help develop a sense of dimension, and familiarise them with terms like "big" and "small" (and later to bring in the comparative and superlative degrees of bigger, biggest and smaller, smallest). The child is asked to bring the cubes **one at a time** starting with the smallest cube (the top-most cube). Bringing one cube at a time allows them to, either consciously or subconsciously, realise that the cube size is increasing. Once all the cubes are placed on the mat, the child has to place the biggest cube in front of them. The teacher mixes up the rest of the cubes and then asks the child to place the next biggest cube. This goes on till they have placed all the cubes one above the other in decreasing order of size. As they do this, they are comparing as well as ordering similar objects by size.

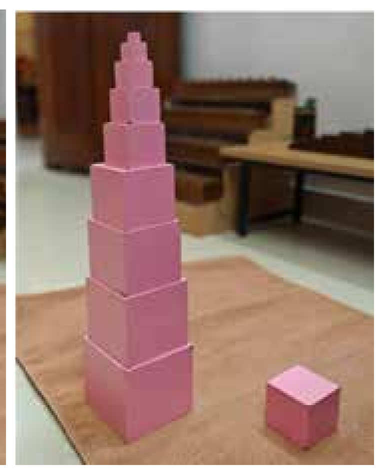

Figure 1: Pink Tower Figure 2: Where does this cube fit?

Other activities to try: Once they familiarise themselves with the material and are able to place the cubes correctly, there can be a few more activities that can be performed. One of them is for the teacher to pick out a cube at random without the knowledge of the child and ask the child to place it back on the tower correctly (Figure 2). This activity requires the child to notice where there is a sudden difference in the cube sizes within the tower and place the removed cube there, or compare the size of the removed cube with those on the tower and find its correct position, which is above the cube that is bigger than and below the cube that is smaller than this one.

Another activity is to mix up all the cubes in a tray and ask the child to bring one cube at a time, starting from the biggest to the smallest, to form the tower on the mat. Each time the child will have to look for the biggest cube among what is left on the tray. This may also be a good time for the teacher to introduce comparative and superlative degrees. Place two cubes on the mat and ask, "Which cube is bigger?", "Which one is smaller?" After repeating this a few times, place any three cubes on the mat and ask, "Which one is the biggest?", "Which cube is the smallest?" The teacher continues till the child is familiar with the idea and meaning of these words.

In another activity, the teacher makes 10 square cards, each card matching the face of each of the cubes and then asks the child to match the card to its respective cube.

Material 2: Brown stairs

Brown stairs is a material that is introduced around the same time as pink tower (between 2 ½ and 3 years of age). It consists of brown cuboids each of the same length (l) but varying in width (w) and height (h) (and so its thickness), thus providing the effect of stairs (Figure 3).

Figure 3: Brown Stairs

The same set of activities that are performed with pink tower can be done with brown stairs too. The vocabulary that children are introduced to are thick, thicker, thickest, thin, thinner, and thinnest. An interesting point to note is that the faces of the pink tower cubes match the faces of the brown stairs cuboids! As children continue to explore these two materials, this realisation dawns on them automatically! There are multiple arrangements of these two materials that children can make, and in the process learn so much about patterns, balance, and focus.

Figure 4: The cubes and the cuboids

Figure 5: The cubes and the cuboids placed alternately one above the other respectively

Figure 6: The cubes and the cuboids placed one next to the other horizontally

Material 3: Long Rods

This material consists of 10 rods, all of the same thickness but varying in lengths, to help children get the idea of length and build vocabulary around "long" and "short" (Figure 7). The familiarity with this material allows them to understand "Number rods" going forward. The child is asked to place one rod at a time on the mat, starting from the longest to the shortest. The teacher allows them to observe how the length of the rod decreases as one moves from left to right. The vocabulary of longer, longest and shorter, shortest is introduced by placing a pair of rods separately and asking for the longer rod, or placing three rods and asking for the shortest and so on.

Other activities to try: Additional activities can include jumbling the rods and asking the child to arrange them in order, removing a rod from the arrangement and asking the child to place it correctly, or getting the child to arrange pairs of rods that are equal in length to the longest rod (or any rod of a chosen length) (Figure 8).

Figure 7

Figure 8: Rods paired to match lengths

Material 4: Number rods

Number rods are typically introduced to children in the age group of 3 ½ to 4 ½ years, after continuous repeated work with pink tower, brown stairs, and long rods. Children would have sensorially experienced the qualities such as big/small, long/short and are now beginning to wonder "by how much" something is longer or shorter. This is when they are ready to understand quantity. There is a mathematical awakening of the mind that is happening.

Number rods are similar to long rods in dimension, but with two colours (typically red and blue) on the rods (except for the rod of length 1) (Figure 9).

The child arranges the rods from the longest to the shortest, starting from the left and ensuring that the red side of the rod is at the bottom each time. In this activity the teacher calls out each rod by its name as it is being placed- rod of 10, rod of 9 and so on. The child eventually observes that the rod of 2 is twice the length of rod of 1, the rod of 3 is thrice the length of rod of 1 (or rod of 1 taken three times), and so on.

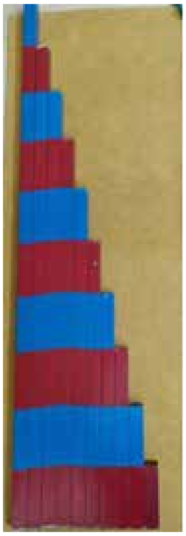

Figure 9

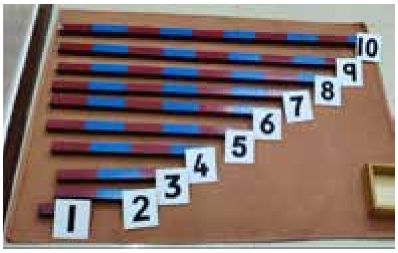

Figure 10

Other activities to try: The teacher makes number cards from 1 to 10. They are mixed up and the child is asked to place each card against the correct rod, after counting the length of the rod. This activity tests for the association of the quantity (number on the rod) with its symbol (Figure 10). Once they are comfortable doing this, the teacher mixes the number cards as well as the rods. The child is asked to pick a rod, identify it and place the right number card against it. Since the rods are not in order, memorisation of the number sequence will not help the child in this activity! Another activity is to find a pair of rods that add up to 10 (length of rod of 10). They may pick 9 and 1, or 8 and 2, or 7 and 3, or 6 and 4. Notice that the same activity was performed with long rods, without associating the quantity with the rods. Here they are expected to say the number names of the rod pairs. To make it more challenging the teacher may pick a random rod and ask them to find pairs of rods. And just like that, they are being introduced to some Addition Facts! Interestingly, this material initiates a number line using a continuum. Most other materials including the Ganitmala are discrete. The fact that each numerical symbol can be matched to a quantity in the form of a single object makes the association between the numerical symbol and the quantity clear and easy (Montessori, M., 2016). A detailed description of the various additional activities for each material can be accessed at (3).

Thus, there are 3 series of objects, the pink tower, the brown stairs, and the long rods which are made in such a way to define and isolate in each series, the possibilities in differences in 3 dimensions. These objects prepared in a series of successive dimensions, help to prepare and build up the mathematical mind of the child sensorially. This develops in the child a certain kind of reasoning, ability to judge quantity which composes the mathematical mind (Montessori, M., 2007).

Are you excited to try these materials in class? Well, although Montessori materials may seem appealing to incorporate, as you may have rightly guessed, there are several limitations to using them fully in a classroom setting. Firstly, the entire philosophy of Montessori education is imparted through a formal training program designed for teachers. Without this training, it may be challenging to implement the comprehensive approach envisioned by Dr. Maria Montessori. Nevertheless, one can strive to utilise these materials to the best of one's ability and share this experience with as many children as feasible. However, it's important to acknowledge that even with this intention, limitations still exist.

Montessori materials, though fantastic tools for discovery and learning, tend to be quite expensive. Additionally, they are challenging to replicate due to the importance of precision; even slight variations in dimensions can significantly affect their intended use. Typically, Montessori materials are crafted from natural materials such as wood, metal, or fabric, with those examined in this article being primarily wooden.

With the objective of making some of the materials affordable and easily accessible, Math Space at Azim Premji University has developed an innovative and efficient method for producing low-cost Montessori materials that maintain precision, enabling interested teachers to create and utilise these resources in their classrooms effectively. Here are some simple steps to make them yourself!

1. Make nets of cubes or cuboids (using thick chart paper or ivory sheets) of the desired dimensions to ensure precision
2. Fill the space within the net with square (for cubes) or rectangular (for cuboids) corrugated cardboard sheets to make it solid
3. Tape the entire solid to make it water-proof
4. Voila! Your Montessori material is ready

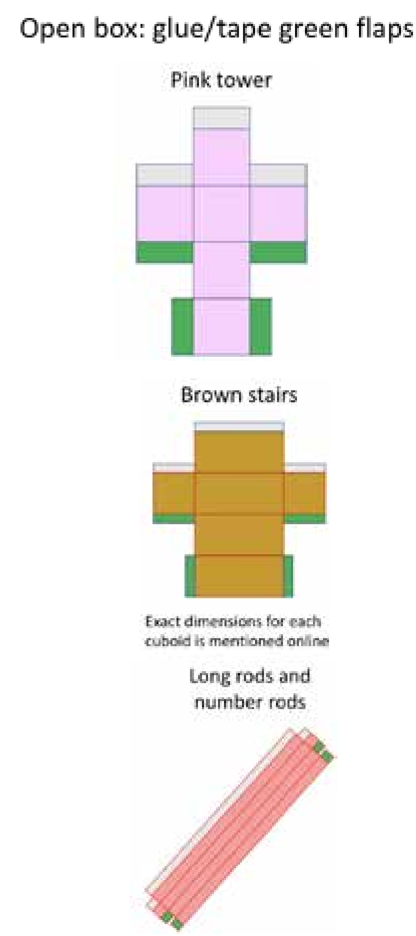

Figure 11: Nets for the various materials

Figure 12: Corrugated cardboard sheets for filling

Figure 13: The final products: Low-cost pink tower and brown stairs!

The step-by-step process to create these materials, and the dimensions required can be accessed at (6). Note that the original Montessori brown stairs are made of length 20 cm each, while the nets used here are of length 15 cm. Nevertheless, this difference in length does not affect the usability or effectiveness of the material.

Embracing the principles of the Montessori method and making it accessible through low-cost materials could help democratise quality education for all children. By focusing on the natural development of each child, we can nurture well-rounded individuals who are not only equipped with essential knowledge but also possess the confidence and skills necessary to navigate the complexities of the world. Ultimately, the Montessori method is more than a pedagogical technique; it is a philosophy that champions the unique potential of each child, paving the way for a brighter and more inclusive future in education.

Acknowledgement: *This article has been compiled with inputs from Ms. Sudha Rao, Parijatha Montessori and Swati Sircar, Azim Premji University.*

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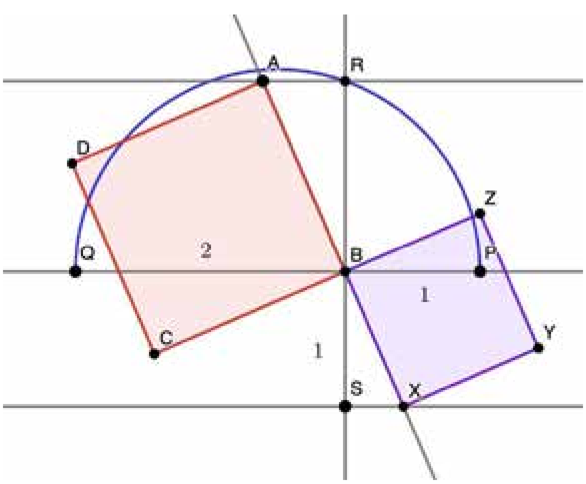
A construction to halve the area of a given square

In the following figure let $ABCD$ be a given square (of unknown side length). Let $PB = BS = 1$ unit, and $BQ = 2$ units. Let PRQ be a semicircle passing through points P and Q. Then the area of the square $XYZB$ must be half of that of $ABCD$.



QR Code for Online Article

This question and this picture were submitted by one of our authors and it made us think. Do spend some time observing this picture – Is the area of square $XYZB$ half the area of the given square $ABCD$? If so, why? To check your solution and to get more such problems, head forward to the online article "A construction to halve the area of a square without measuring"

Word Problems on Addition and Subtraction

Narayana Meher

What exactly are word problems? What challenges do children encounter when dealing with them? Are there various kinds of word problems? Continue reading to find answers to these inquiries and discover more, as this article delves into the topic of word problems related to addition and subtraction.

A word problem is simply a mathematical question embedded in a simulated real world situation described with a narrative or story. This is unlike a 'bare problem' which is a mathematical question expressed as mathematical symbols or equations. In word problems, the concepts, ideas or models are used to understand real-world phenomena. Word problems are an integral part of school mathematics, especially in the lower classes. Students begin to appreciate that mathematics is contained not only in symbols and abstractions, but may be situated in real world scenarios.

A well-crafted word problem can bring a meaningful real-world context to the learner. It should inspire children to solve problems which they perceive to be important and useful. Any word problem that does not have a suitable context connected to a student's immediate life may not inspire and invite students to solve it. Certainly, word problems can narrate contexts which can bring value to human lives. Narratives of children making diyas and selling in the market, students collecting money to help an orphanage, girls farming to break the gender stereotype, etc., are examples of the same. Children can also be inspired to look around them and create word problems based on their immediate environment. This article will focus on difficulties that students face when solving word problems and describe pedagogical strategies to address them.

Difficulties children face with word problems

Children's ability to solve word problems depends on their ability in mathematical literacy as revealed in PISA (2003, p 24). Mathematical literacy includes the ability to use mathematical knowledge and skills appropriately. It combines understanding mathematical concepts, interpreting data, recognising patterns, applying mathematical reasoning and using arithmetic operations.

When given a word problem to solve, children need to understand and interpret the situation, translate it into a bare sum and then solve it. Thus, solving arithmetic word problems demands two different abilities, the first is to translate between language and mathematical symbolism and the second is to

Keywords: Word problems, mathematics in context, application, reasoning, real-life problems

execute arithmetic operation(s). What should be presented first to students? A bare sum or a word problem? (It is a 'chicken or the egg – which came first' situation.)

Many children who are good at solving bare sums cannot solve word problems, i.e., given a mathematical expression or equation, they can simplify it and arrive at the correct answer. What happens when they encounter a word problem? If children cannot relate the problem to their day-to-day experiences, the point of using word problems is missed. They will see it as yet another hurdle in mathematics. Language too plays a key role in the framing of word problems. Students struggle with understanding what the problem statement is, translating it to the right mathematical language and with interpreting what mathematical operation is required to solve the problem.

It has been observed that in an attempt to help students attempt word problems confidently, undue emphasis has been placed on 'keywords', such as 'more', 'took away', 'altogether', 'difference', 'remaining', 'left', etc. This leads students to misinterpret the real-world situation and apply inappropriate operation(s) to solve the problem.

In addition, narratives may have details which are not necessary for the solution of the problem. Children find it difficult to identify and disregard irrelevant information.

Let's illustrate through an example how undue emphasis on the keywords could mislead students.

<table>
<tr><th>Problem 1</th><th>Problem 2</th></tr>
<tr><td>Habiba, a 10-year old girl has 9 guavas. Kalyani, a 11-year-old girl has 5 more guavas than Habiba. How many guavas does Kalyani have?</td><td>Habiba, a 10-year old girl has 9 guavas. Habiba has 5 more guavas than Kalyani, a 11-year-old girl. How many guavas does Kalyani have?</td></tr>
<tr><td colspan="2">Additional or Irrelevant information- Habiba is 10 years old, and Kalyani is 11 years old.
Key word- More (Student has been taught to do addition on identifying this keyword)</td></tr>
<tr><td colspan="2">Translated to expression: 9 + 5
Operation: 9+5 = 14</td></tr>
<tr><td colspan="2">Answer: Kalyani has 14 guavas</td></tr>
<tr><td>Check: Habiba has 9 guavas, and Kalyani has 5 more- 14 is 5 more than 9.
Correct: The keyword strategy has worked.</td><td>Habiba has 5 more guavas than Kalyani-
Habiba has 9 and Kalyani has 14 guavas.
Error: The keyword strategy has not worked.</td></tr>
</table>

Table 1: Illustration of errors due to focus on keywords.

Thus, the major challenges students face when attempting word problems include:

1. Relating to the context described by the problem.
2. Comprehending the word problem, especially if unfamiliar vocabulary or complicated sentence structure is used.
3. Choice of the correct arithmetic operation.
4. Undue emphasis on keywords.
5. Inability to identify unnecessary or additional information, and to abstract useful information.
6. Difficulty in framing mathematical expressions and equations.
7. Difficulty in using arithmetic operations correctly.

In order for the teacher to help students attempt word problems confidently, they must teach them how to deal with each of these challenges explicitly. But first, the teacher must understand the importance of framing mathematical expressions and equations. In addition, they must know the different types of word problems that the child may encounter at this stage.

Expressions and Equations

Mathematical equations and expressions are an intermediary step between a narrative of the word problem and its solution.

A mathematical expression is a combination of numbers, variables and operators which represents some mathematical value.

For example: 3 + 2 (numbers and operation of addition)
$3x + 5$ (numbers, variable and operations of addition and multiplication)

A mathematical equation is a statement that represents the equality of two different expressions. For example: 4 + ___ = ___ + 6 (The expression 4 + ___ is equal to the expression ___ + 6)

$3x + 2 = 11$ (The expression $3x + 2$ is equal to 11)

Each addition or subtraction expression can be converted to two equations.

For example: -

A. The expression 27 + 54, can be converted into the equation 27 + 54 = 81 which can be viewed in two ways as follows:

  a. 27 + ___ = 81 and

  b. ___ + 54 = 81

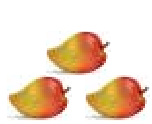
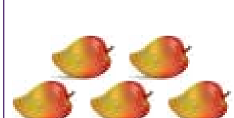
B. 81 – 54, this expression can be converted into the equation 81 – 54 = 27 and this can be viewed in two ways as follows:

  a. 81 – ___ = 27 and

  b. ___ – 54 = 27

Types of Word Problems

There is a pattern to all varieties of word problems that are created. Carpenter et. al. (1983) proposed four kinds of word problems for the operations of addition and subtraction. They are: *Combine, Compare, Change and Equalize*. Nesher, P., Greeno, J. G., & Riley, M. S. (1982) further classified combine, compare and change into 14 subcategories in their work. Based on this classification, the following tables have been drawn up

<table>
<tr><td colspan="2">Problem 1: Sita has 5 mangoes and Rahim has 3 mangoes. How many mangoes do they have in total?
Problem 2: Sita has 5 mangoes. Sita and Rahim have 8 mangoes in total. How many mangoes does Rahim have?
Suggested Strategy: Use modelling to represent the number of mangoes that each of them have and then ask students to frame the problem statement.</td></tr>
<tr><td>Type: Combine 1
General Description: Questions about the final set (whole)</td><td>Type: Combine 2
General Description: Questions about one subset (part)</td></tr>
<tr><td>Example: Sita has 5 mangoes and Rahim has 3 mangoes. How many mangoes do they have in total?</td><td>Example: Sita has 5 mangoes. Sita and Rahim have 8 mangoes in total. How many mangoes does Rahim have?</td></tr>
<tr><td>Representation
Sita  Rahim </td><td>Representation
Sita  Sita & Rahim </td></tr>
</table>

Framing the Problem Statement: **Question for discussion:** What is required to be found? **Answer:** The total number of mangoes	**Framing the Problem Statement:** **Question for discussion:** What is required to be found? **Answer:** The number of mangoes that Rahim has.
Expression: 5 + 3	**Equation:** ___ + 5 = 8 OR 8 – 5 =___
Answer: Sita and Rahim have 8 mangoes in total.	**Answer:** Rahim has 3 mangoes.
Check: Check answer by confirmation with problem statement.	

Table 2: Examples of Combine Problems and suggested pedagogical strategies for solution

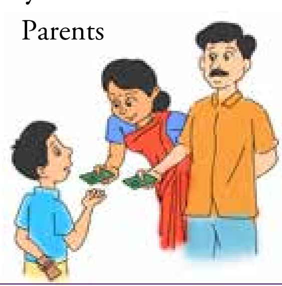
Problem 3: Rudra has ₹10, and he got ₹20 more from his parents for the upcoming festival. How much money does he have now? **Problem 4:** Rudra has ₹10 with him. His parents gave him some money for the upcoming festival. Now he has ₹30 with him. How much money did his parents give him? **Suggested Strategy:** Use role play to show the transactions and then ask students to frame the problem statement.	
Type: Change 1 **General Description:** Increasing, questions about the final set.	**Type:** Change 2 **General Description:** Increasing, questions about the change.
Example: Rudra has ₹10 with him. And he got ₹20 more from his parents for the upcoming festival. How much money does he have now?	**Example:** Rudra has ₹10 with him. His parents gave him some money for the upcoming festival. Now he has ₹30 with him. How much money did his parents give him?
Representation using role play Rudra  Parents 	**Representation using role play** Rudra = 10  Parents = ?  Total = 30 
Framing the Problem Statement: **Question for discussion:** What is required to be found? **Answer:** How much money does Rudra have for the festival?	**Framing the Problem Statement:** **Question for discussion:** What is required to be found? **Answer:** How much money did Rudra get from his parents?
Expression: 10 + 20	**Equation:** 10 + ___ = 30 OR 30 – 10 = ___
Answer: Rudra has ₹30 to spend at the festival.	**Answer:** Rudra's parents gave him ₹20.
Check: Check answer by confirmation with problem statement.	

Table 3: Examples of Change Problems and suggested pedagogical strategies for solution

Change 3	Increasing, questions about the initial set	*Rudra has some money with him. His parents gave him ₹20 for the upcoming festival. Now he has ₹30 with him. How much money did he have in the beginning?*
Change 4	Decreasing, questions about the final set	*Rudra has ₹30. He spent ₹20 buying a toy. How much money does he have now?*
Change 5	Decreasing, questions about the change	*Rudra has ₹30. He spent some money buying a toy. He has ₹10 now. What is the price of the toy?*
Change 6	Decreasing, questions about the initial set	*Rudra has some money with him. He bought a toy for ₹20. Now he has ₹10 with him. How much money did he have in the beginning?*

Table 4: More types of Change Problems with Examples

Title	General Description	Word Problems
Compare 1	Mentioning 'more', questions about the difference set	*Habiba has 9 guavas and Kalyani has 5 guavas. Who has more guavas and how many more?*
Compare 2	Mentioning 'more', questions about the 'compared set'	*Habiba has 9 guavas, and she has 4 guavas more than Kalyani. How many guavas does Kalyani have?*
Compare 3	Mentioning 'more', questions about the 'referent set'	*Kalyani has 5 guavas, and Habiba has 4 guavas more than Kalyani. How many guavas does Habiba have?*
Compare 4	Mentioning 'less', questions about the difference set	*Habiba has 9 guavas and Kalyani has 5 guavas. Who has less guavas and how many less?*
Compare 5	Mentioning 'less', questions about the 'compared set'	*Kalyani has 4 guavas less than Habiba. Habiba has 9 guavas. How many guavas does Kalyani have?*
Compare 6	Mentioning 'less', questions about the 'referent set'	*Kalyani has 5 guavas, and she has 4 guavas less than Habiba. How many guavas does Habiba have?*

Table 5: Types of Compare Problems with Examples

A fourth category i.e., Equalisation problems are also mentioned in the literature. Change and Compare Problems may be converted into Equalisation problems. For example, the Change 2 problem given above: *Rudra has ₹10 with him. His parents gave him some money for the upcoming festival. Now he has ₹30 with him. How much money did his parents give him?* may be changed to *Rudra wants ₹30 for the upcoming festival to buy a toy. He has ₹10 with him. How much does he need from his parents?*

Similarly, the Compare 1 example given above *Habiba has 9 guavas and Kalyani has 5 guavas. Who has more guavas and how many?* maybe rephrased as *Kalyani has 5 guavas. Habiba has 9 guavas. How many more guavas does Kalyani need to obtain to have the same number of guavas that Habiba has?*

In both problems, the expression that is formed:

Given quantity + __ is set equal to another given quantity.

While students should not be burdened with understanding the different categories of word problems, it certainly helps a teacher to know these categories. Firstly, by knowing the category (s)he can generate multiple problems for practice. Secondly, by giving students two problems in two categories to solve, (s)he can use the representations to familiarise students with the difference between the two problems. Here is where the steps of framing the problem statement, forming the expression and solving the problem become much more meaningful to the child. Finally, identifying categories in which students have difficulties as well as vocabulary which students cannot comprehend helps in formative assessment allowing the teacher to design remedial work which can help with specific difficulties.

To summarise the steps to solve a word problem, the teacher should train students to:

1. Read and comprehend the word problem as a whole. (Avoid focusing too much on keywords)

   The student should be very clear as to what the problem is and what is required to be found. Writing the problem statement helps students to do this.
2. Draw a diagram or a model which can depict the situation. Teachers need to use and demonstrate an effective model using sketches or manipulatives to solve the word problem. Role play can also help students understand the problem.
3. Segregate the information which is required to solve the problem.
4. A word problem might have information which is not important to solve problems. Students should be able to identify and ignore this.
5. Translate the narrative to mathematical equations or expressions

   When students understand what is given and what is missing, they are able to do this easily.
6. Use arithmetic operations to solve the problem.

Conclusion

To empower students to solve word problems a teacher needs to explicitly model how to comprehend the word problem as a whole, how to choose an effective model or diagram or representation, sense the number quantitatively, identify type of word problems, convert the situation to mathematical expression or equation and do operations.

It is important to introduce all operations in a real world situation and then help them to figure out how to solve them, giving both understanding and procedure. Children have an intuitive sense of addition and subtraction before they come to school although they will not be able to articulate as they have already experienced adding (Jodna) and subtracting (nikalna) in their day to day play activities. It is important to bring students' familiar context into mathematics through word problems just to help them to understand that it is not a different world but an extended world with symbols, notations and procedures.

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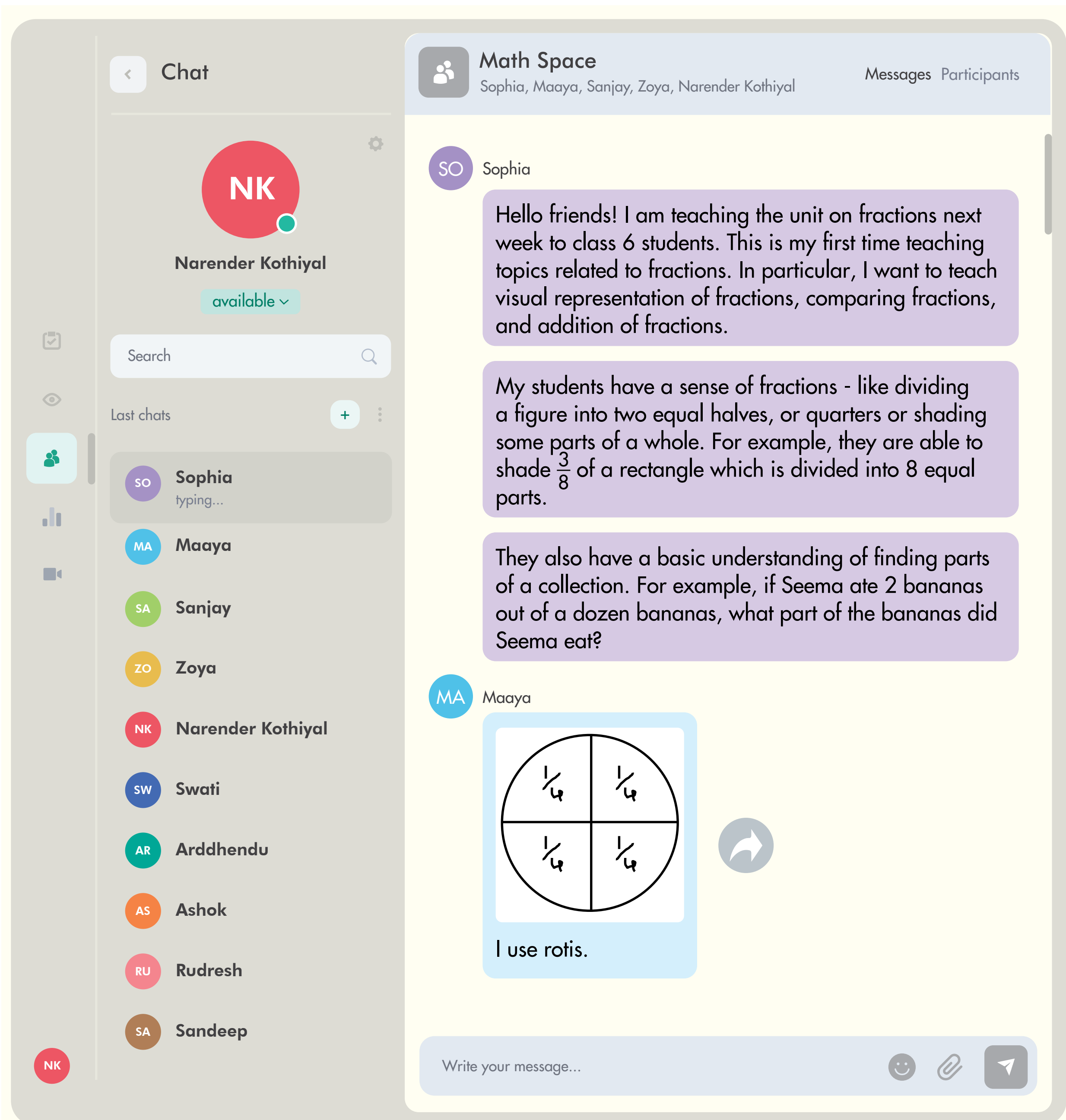
Fractions in Bottles!

A Conversation on Teaching Fractions...

Narender Kothiyal and Math Space

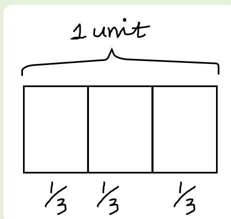


Math Space

Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

SA Sanjay



I used rectangles which are split equally into smaller rectangles along the length. If we split that way, the area is being split, as well as the length.

SO Sophia

How do children visualise fractions? What do they see getting split? Do they see parts as one dimensional quantities (length, circumference) or two dimensional quantities (area)?

SA Sanjay

Yes, one-dimensional…

MA Maaya

I think two dimensional.

SO Sophia

Has anyone tried three dimensional quantities... volume? capacity?

ZO Zoya

I used a different approach: fractions used in cooking.

The measuring spoons have the fractions they measure printed on them. So it is easier for the students to see what numerators and denominators really mean here.

SO Sophia

Interesting! I understand how this helps with comparing fractions, But could this be used for understanding the operations on fractions?

NK

Write your message...

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

SA Sanjay
I doubt if measuring spoons can help them understand the addition or subtraction of fractions.
👍 1

You
Sorry, I missed this interesting discussion. I had used bottles instead of the above TLMs, and it worked.

SA Sanjay
Bottles?

Do you mean measuring jars?

You
No! I used the usual plastic water bottles. In fact this helps the students 'see' the addition of two fractions.

MA Maaya
I'm curious! Please explain a bit more.

You
I also used paper folding, fraction walls and bottles.

Using the fraction wall they already knew how $\frac{1}{3} = \frac{2}{6}$, and how the unit fraction is getting smaller as the denominator increases. They could physically verify these things.

MA Maaya
I can do this with paper folding… But, how is this done with a bottle?

You
Bottles! - plural 😉

SO Sophia
Sorry, Just now saw your messages. I am a bit lost. Can you show us the bottles you used?

NK

Write your message...

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

You

Sure. Here they are.

SO Sophia

Before you explain, let me decode your idea 🤔

ZO Zoya

Aha!... I think I understand what you did here.

SO Sophia

By bottles you mean only the long cylindrical parts right? The top portion must be excluded. In this case, the height or the volume can be considered the whole.

👍 2

ZO Zoya

Right! Then a strip of paper is attached to the cylindrical part. Dividing the paper strip into $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ … is easy - just fold and mark. But what about $\frac{1}{3}, \frac{1}{5}$ etc.?

SO Sophia

Simply take printouts?

You

Actually, I used simple constructions to divide the height into n equal parts - using compass and ruler - nothing else.

SA Sanjay

But @Narender ji, how did you use these bottles?

You

I used them to teach the following concepts visually and hands-on.

- Equality of fractions such as $\frac{1}{3} = \frac{2}{6}$.
- Comparing fractions such as $\frac{2}{3} < \frac{3}{4}$.
- Addition and subtraction of fractions, such as $\frac{2}{3} + \frac{1}{4}$.

NK

Write your message...

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

SO Sophia
Using 2 different bottles..?

You
Yes. In fact, we need 3 bottles of the same size.

See, when we consider $\frac{2}{3}+\frac{1}{4}$, it seems too abstract and pointless for most students. But with the bottles we could get $\frac{2}{3}$ bottle of water and $\frac{1}{4}$ bottle of water. Then, when we poured the water from these two into one bottle, that was $\frac{2}{3}+\frac{1}{4}$. The sum made sense to them!

MA Maaya
Ah! Very interesting!

SO Sophia
And they could see that the sum was a proper fraction…

MA Maaya
Because it didn't overflow…the water level was below the 1 mark.

SO Sophia
@Narender ji, did you try $\frac{3}{4}+\frac{1}{3}$?

You
No, thankfully, I had selected fractions such that the sum was less than one.

ZO Zoya
Would be fun to consider $\frac{3}{4}+\frac{1}{3}$ 😉

SO Sophia
Won't the paper get wet?

You
Pencil marking won't bleed.

MA Maaya
That should work! Did you get the sum as a fraction?

NK
Write your message...

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

You

Actually yes! We discussed the common denominator for $\frac{2}{3}$ and $\frac{1}{4}$ and got 12. So, the 3rd bottle was split in 12 equal parts. When we poured both $\frac{2}{3}$ and $\frac{1}{4}$ in it, the water level was at $\frac{11}{12}$!

👍 2

500ml bottle showing $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}$

500ml bottle showing $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{15}$

SO Sophia

In fact, after $\frac{2}{3}$ is poured, you can see that $\frac{2}{3} = \frac{8}{12}$.

SA Sanjay

Quite fascinating!

SO Sophia

Yes! We can have multiple scales along each bottle, say three strips split in 4, 8 and 12 equal parts respectively…

ZO Zoya

That'll show $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$ and $\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$

SO Sophia

Also, $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{6}{12}$

ZO Zoya

Right!

SO Sophia

And three strips in the other bottle for $\frac{1}{3}, \frac{1}{6}$ and $\frac{1}{12}$

NK

Write your message...

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

ZO Zoya

So, $\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$ as well.

SO Sophia

Yes, and now when you add $\frac{1}{4} = \frac{3}{12}$ to the $\frac{2}{3} = \frac{8}{12}$, you see it rise to $\frac{11}{12}$

MA Maaya

Wow! How did the students respond, @Narender ji?

You

They got more clarity thanks to this hands-on experience of actually filling bottles to various levels. $\frac{2}{3}$ or $\frac{3}{5}$ were no longer abstract symbols, but actual quantities w.r.t. a given whole. There was a purpose to add two fractions. And their curiosity went up.

SO Sophia

Cool! I never experienced fractions as capacity or volume of liquid… How would you guide a new teacher like me who wants to use it?

You

Rule number 1 is to start collecting bottles of the same size and shape… mostly cylindrical from the base…

I experimented with the following bottles. The good ones are these.

SO Sophia

So, good.

You

And these won't work.

NK

Write your message…

Math Space
Sophia, Maaya, Sanjay, Zoya, Narender Kothiyal

Messages Participants

You

Then mark the top edge of the cylindrical part and find the height… Now, choose denominators of 2, 3, 4, 5, 6, 8, 10, 12, and possibly 15. Make paper strips for each denominator, 9 strips in total and split them into equal parts – halves, thirds, quarters, fifths, sixths, eighths, tenth, in twelve and in fifteen respectively… Now, paste the strips split in 5, 10 and 15 equal parts on one bottle, 3-6-12-15 on the second bottle, 2-4-8-12 on the third…

SA Sanjay

@Narender ji, with which grade did you do this?

You

Grade 7

SO Sophia

Cool! I never experienced fractions as capacity or volume of liquid… How would you guide a new teacher like me who wants to use it?

You

And having a fraction wall hanging in the class helps…

MA Maaya

Yes, to identify the equivalent fractions.

SO Sophia

Did you use this also to teach subtraction of fractions?

SA Sanjay

Why not? Try $\frac{3}{5} - \frac{1}{2}$

SO Sophia

Hm… You fill $\frac{3}{5}$ in the first bottle, then pour out $\frac{1}{2}$ to the second bottle. The remaining volume of water represents $\frac{3}{5} - \frac{1}{2}$.

SA Sanjay

Should it end there?

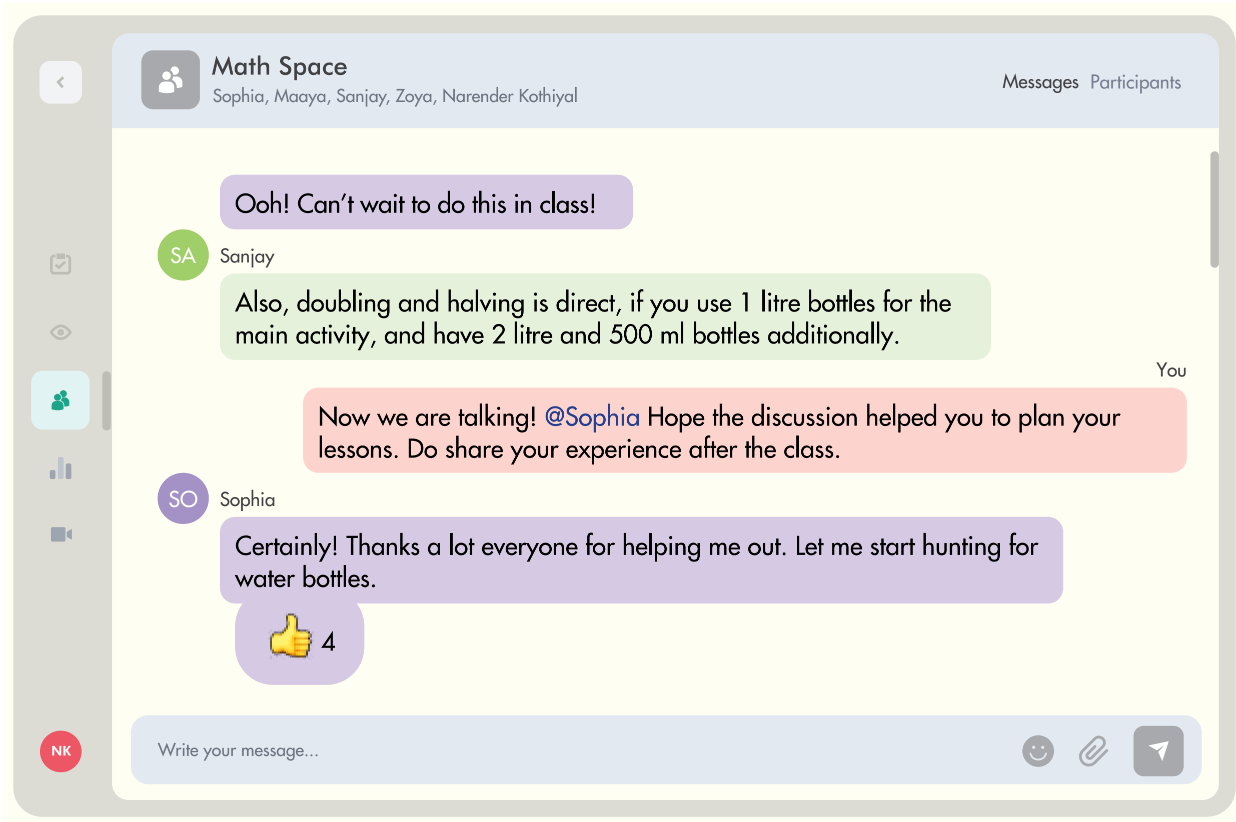
SO Sophia

Of course, not. The students would already see that $\frac{3}{5} = \frac{6}{10}$ and $\frac{1}{2} = \frac{5}{10}$. So I can ask them to predict the answer before pouring and they can verify that the answer is in fact $\frac{1}{10}$.

👍 4

NK

Write your message…



Editor's note

The above discussion is a fictionalised version of a conversation that happened in the Math Space at Azim Premji University. We modified the discussion to make it appear like a chat in a messaging app.

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MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Fun with Fractions

Tejas Sriram

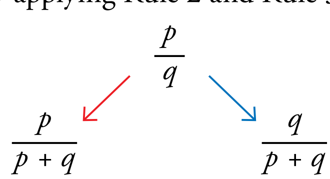
This article is inspired by a problem originally presented on NRICH, an online mathematics resource. While participating in a course named 'Ganit Manthan' under the auspices of Vichar Vatika, a problem caught my attention. In what follows, I answer the original problem(s) and explore a natural extension of that problem, where I found a few patterns and proved them.

Let us define "Fun Fractions" using the following simple rules:

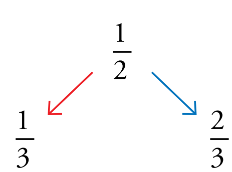
- Rule 1: $\frac{1}{2}$ is a fun fraction.
- Rule 2: If $\frac{p}{q}$ is a fun fraction, then $\frac{p}{p+q}$ is also a fun fraction.
- Rule 3: If $\frac{p}{q}$ is a fun fraction, then $\frac{q}{p+q}$ is also a fun fraction.

This means we can start with the fun fraction $\frac{1}{2}$, and generate all other fun fractions by repeatedly applying Rule 2 and Rule 3. We can represent the processes of applying rules 2 and 3 visually by drawing a red arrow whenever Rule 2 is applied and by drawing a blue arrow whenever Rule 3 is applied.

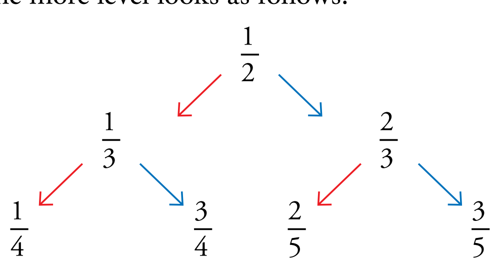
Assume that $\frac{p}{q}$ is a fun fraction. By applying Rule 2 and Rule 3 on $\frac{p}{q}$, we get the following.



For example, applying Rule 2 and Rule 3 to $\frac{1}{2}$ can be represented as



Extending this branching one more level looks as follows.



Keywords: Problem solving, Problem posing, Patterns, Fibonacci

This shows that each fun fraction branches out to create two new fun fractions.

Activity: Extend the branching to the next two levels. What are your observations?

Next, let us explore some natural questions that are also mentioned on the NRICH webpage. The reader is urged to try them before checking the solutions.

1. What is the biggest/smallest fun fraction?
2. What is the biggest/smallest numerator?
3. Is it true that numerators are not in decreasing order?
4. It seems that the numerator and denominator of a fun fraction share no common divisors except 1. Is this always true?
5. Is it possible to create a closed loop of fractions, where a sequence of transformations brings you back to the starting point?

Now, I would like to take the reader through my journey of solving the above problems. First let us introduce a notation: Since there are only two possibilities at each step, we can represent every fun fraction using the notation described below.

- The fun fraction generated by Rule 1 is represented by the symbol A.
- A fun fraction generated by Rule 2 is represented by the symbol B, and
- A fun fraction generated by Rule 3 is represented by the symbol C.

This means $\frac{1}{2}$ is represented as A, and any other fun fraction can be represented as a string starting with A and followed by a binary combination of Bs and/or Cs.

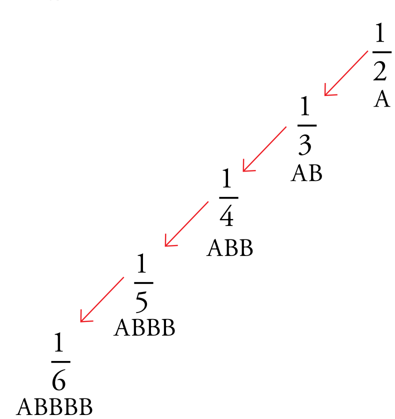
For example, ABCB represents the fun fraction $\frac{3}{7}$, because

$$\underset{\text{A}}{\frac{1}{2}} \xrightarrow[\text{B}]{} \frac{1}{3} \xrightarrow[\text{C}]{} \frac{3}{4} \xrightarrow[\text{B}]{} \frac{3}{7}$$

Solutions

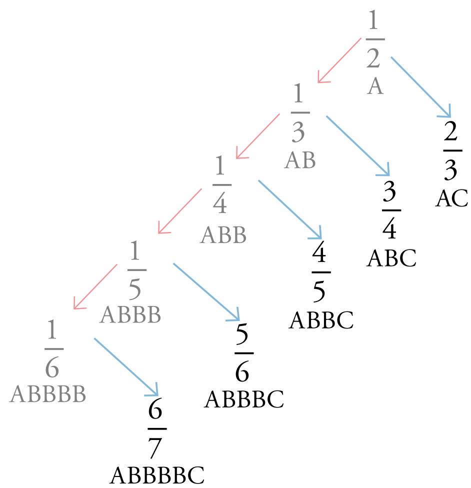
1. Since $\frac{1}{2}$ is positive, every other fun fraction has to be positive. Also, since the denominator of each fun fraction is more than the numerator, each fun fraction has to be smaller than 1.

   Now let us look at the fun fractions of the form A, AB, ABB, ABBB, ABBBB, and so on. This means we are applying Rule 2 repeatedly on $\frac{1}{2}$. Visually this means the left most branch:



Given any natural number n, its reciprocal $\frac{1}{n}$ should be there in this branch. Thus, the entries in this branch keep getting smaller and they never end. This means that there is no smallest fun fraction possible.

Now let us look at the fun fractions of the form AC, ABC, ABBC, ABBBC, ABBBBC and so on. This means repeatedly applying Rule 2 on $\frac{1}{2}$, and then applying Rule 3 *once* at the end. Visually, this means the 'second' left most branch:



This branch consists of fun fractions of the form $\frac{n}{n+1}$ for every natural number $n > 1$. So this is an increasing sequence of fun fractions that never ends. This means that there is no biggest fun fraction possible.

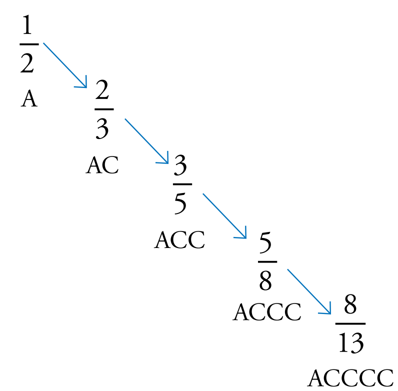
2. The smallest numerator is clearly 1, which is the numerator of $\frac{1}{2}$. However, as we observed in Solution 1, $\frac{n}{n+1}$ is a fun fraction for every natural number n. So there cannot be any biggest numerator possible.
3. Yes. Since Rule 2 retains numerators and Rule 3 increases numerators, the numerators can never decrease. But what if we can cancel out a common factor between numerator and denominator at some stage? The next solution says that is not possible.
4. Let $\frac{p}{q}$ be a fun fraction. By applying Rule 2 or Rule 3, we get fractions $\frac{p}{p+q}$ or $\frac{q}{p+q}$ respectively. Let us try to compare the common factors of numerators and denominators among these three fractions.

   Suppose that d is a common factor of both p and q. Then clearly d should divide $p + q$. Similarly, suppose d is a common factor of both p and $p + q$, then d should also divide $q = (p + q) - p$. This means that the list of common factors of p, q, and $p + q$ are exactly the same.

   Clearly, for the first fun fraction $\frac{1}{2}$, there is only one common factor between the numerator and the denominator, namely 1. So for every other fun fraction $\frac{p}{q}$, 1 is the only common factor of numerator p and the denominator q.

5. Both Rules 2 and 3 increase the denominators. So the only way in which we can obtain a closed loop is by cancelling out common factors in a fun fraction to get smaller numerators and denominators. However, by Solution 4, this is not possible. So there is no loop.

I also observed something interesting. Let us look at the right most branch, whose entries are in the form A, AC, ACC, ACCC, ACCCC and so on.



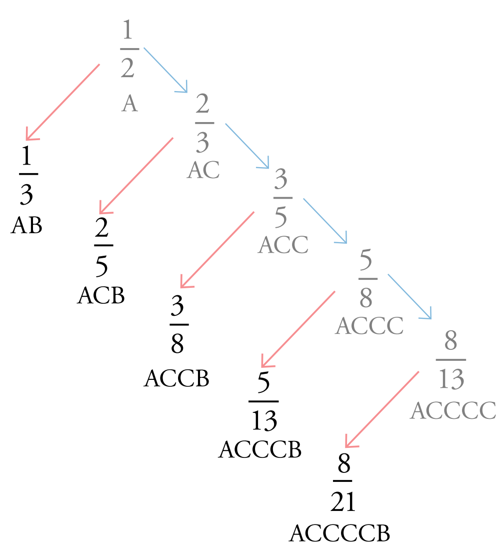
Notice that these fractions may be defined *recursively* as follows, and should remind us about the Fibonacci numbers. Let

$$F_1 = 1$$
$$F_2 = 2$$
$$F_{m+2} = F_{m+1} + F_m$$

for every natural number m. Clearly, any fun fraction of the form ACCC…CCC is nothing but $\frac{F_{m+1}}{F_{m+2}}$, where m is the number of C's appearing at the end. As we have already shown, this means that there are no common factors between consecutive Fibonacci numbers other than 1.

I also noticed that if we divide the denominator F_{m+2} by the numerator F_{m+1}, we always get the remainder F_m. This is simply because $F_{m+2} = F_{m+1} + F_m$. Then I noticed something more intriguing. This is also true for fun fractions of the form ACCC…CCCBBB…BBB, where C appears in the middle m times and B appears at the end k times.

For illustration, here is how fun fractions of the form AB, ACB, ACCB, ACCCB, ACCCCB and so on look like.



Let us see why this is true. Let $\frac{p}{q}$ be a fraction of the form ACCC…CCCBBB…BBB, where C in the middle appears m times and B at the end appears k times. In other words, $\frac{p}{q}$ is obtained from $\frac{1}{2}$ by applying Rule 3 repeatedly m times first, and then applying Rule 2 repeatedly next. So $\frac{p}{q}$ would been obtained from $\frac{F_{m+1}}{F_{m+2}}$ by applying Rule 2 repeatedly k times. So we should have

$$\frac{p}{q} = \frac{F_{m+1}}{F_{m+2} + k(F_{m+1})}.$$

So if we divide the denominator q by numerator p, we should get $k + 1$ as quotient and F_m as the remainder.

So what we have proved is the following theorem.

Theorem. Let $\frac{p}{q}$ be a fun fraction obtained from $\frac{1}{2}$ by repeatedly applying Rule 3 m times, and then repeatedly applying Rule 2 k times, then if we divide q by p, the remainder is always the F_m defined as

$$F_1 = 1$$
$$F_2 = 2$$
$$F_{m+2} = F_m + F_{m+1}$$

Editor's note

The problems presented in this article naturally lead to numerous variations and interesting new questions. For example, what if the initial fun fraction were different from $\frac{1}{2}$? Does every fraction in its reduced form appear in the above tree? These problems can be adapted for students aged 9 to 16, offering opportunities to practise fraction operations, identify patterns, and justify their findings. Sivaraman (2021) explored a related variation in the July 2021 issue of *At Right Angles*, which we encourage readers to explore.

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1. Sivaraman, R. (2021, July). Tremendous tree. *At Right Angles*, (10), 17-22. http://publications.azimpremjifoundation.org/2786/1/02_Sivaraman_TremendousTree.pdf

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Explorations on Symmetric Polygons

Ajaykumar

This article examines different methods for making the study of symmetric polygons more engaging. It emphasizes hands-on activities that encourage observation and generalisation. Through carefully designed activities, mathematics can transform from an abstract discipline into a vivid, interactive, tactile experience. The latest edition of the NCERT Class VI Mathematics textbook (NCERT, 2024) introduces the concept of symmetry with a range of hands-on activities that can captivate students' imaginations. A particularly noteworthy activity on paper folding and cutting (NCERT, 2024, pp.223) involves students predicting the shapes of cutouts made from folded paper, a process that not only enhances their understanding of symmetry but also fosters an appreciation for geometric patterns. Building on the foundation laid by these exercises, this article explores a series of activities that may supplement the textbook and improve the classroom experience.

Polygons are defined as simple closed figures formed by straight line segments (that do not meet at 180 degrees). We typically name the polygon based on the number of sides it has. A polygon with three sides is called a 3-gon or a triangle, and one with four sides is called a 4-gon or a quadrilateral. In general, a polygon with n edges and n vertices is called an n-gon. The line segments or sides of a polygon are called edges and the points at which the edges meet are called vertices.

In the following, we discuss how different cuts in folded paper can lead to polygonal shapes. Students can gain a deeper insight into the relationship between symmetry and polygons by experimenting with various sequences of cuts and observing the resulting shapes.

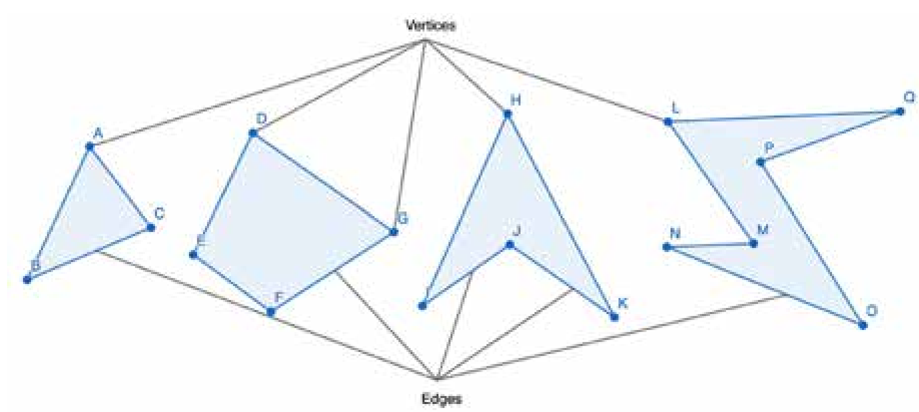

Figure 1

Keywords: Line symmetry, Symmetric quadrilaterals, Symmetric triangles

Let us understand a 'cut', a 'sequence of cuts' and a 'cutout', for the purpose of this article, as follows. A 'cut' is a single stroke of cutting the paper along a straight line (as shown in Figure 2). A second cut in a 'sequence of two cuts' would begin from the endpoint of the first cut, along a straight line that does not align with the first cut. That is, there is some angle (other than 180 degrees) between the two cuts at the common point they share. Figure 3 below shows a sequence of 3 cuts. By a 'cutout', we mean the piece of folded paper that comes out on completing a sequence of cuts that start at a point along the fold line and end at another point along the fold line. We call this a multiple cut. The picture on the right of Figure 3 shows the cutout obtained using the sequence of 3 cuts (as shown in the adjacent figure there).

Some General Observations

1. To obtain a polygon as a cutout, one has to start at a point on the fold line, make cuts along straight lines (as the cutout should be made of only line segments), and end at another point on the fold line. This constraint ensures that the cutout will be a simple, closed polygonal shape.
2. As illustrated in Figure 2, a single cut will not produce a polygon due to the need for more vertices and edges that mirror themselves along the fold line. Instead, multiple cuts (as described above) are necessary.

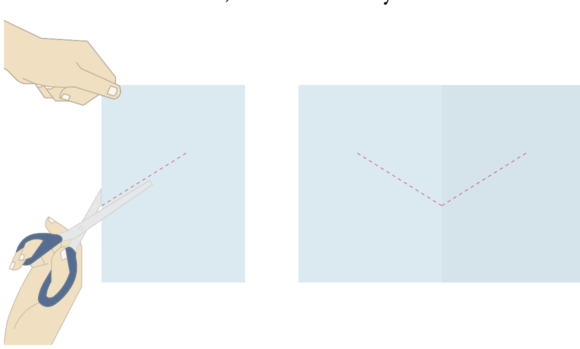

Figure 2

3. If a point on the fold line is on a cut that is not at right angles (90 degrees) to the fold, then the point is a vertex of the polygonal cutout (see Figure 3).

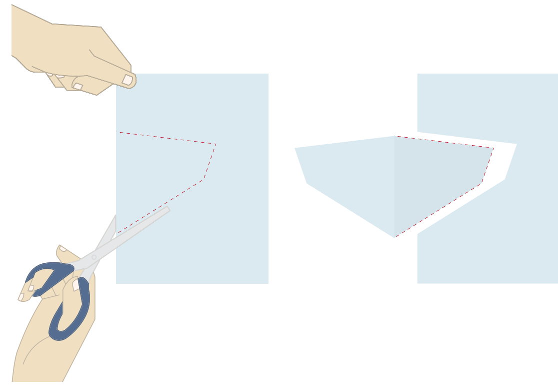

Figure 3

4. If a point on the fold line is on a cut that is at right angles to the fold, then the point is not a vertex of the polygonal cutout (see Figure 4). Furthermore, such a point will be the midpoint of an edge of the polygonal cutout.

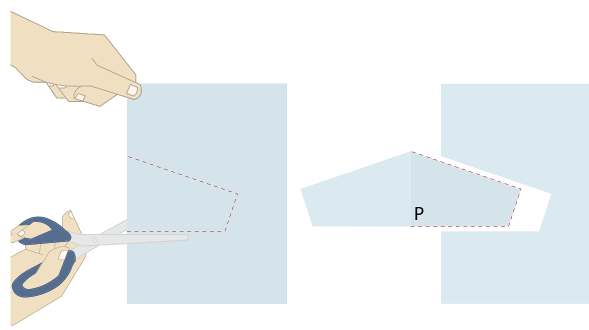

Figure 4

5. Any two cuts meeting at a point (this point cannot lie on the fold line) results in two vertices of the polygonal cutout (See Figure 5).

Exploration 1: Creating Symmetric Triangles (3-gons)

Let us begin by investigating how to create a triangle that is symmetric along the line of the fold.

Question: What are the possible ways to make the cuts to obtain a symmetric triangular cutout from a folded paper?

Let us fold a piece of paper in half as shown in the figures above. Observe that we cannot use three (or more) cuts as they result in two (or more) distinct points formed by two cuts meeting and hence by Observation 5, the cut-out will have four vertices as a result. So, to obtain a triangle we should have exactly two cuts.

If neither of the two cuts are at right angles to the fold line, then by Observation 3, the two points on the fold line result in two vertices of the polygonal cut-out. The point where these two cuts meet, by Observation 5, results in two more vertices of the polygonal cut-out. Thus, in such a case the polygon has four sides and hence cannot produce a triangle.

So, the only way we can strategically make two cuts to obtain a triangle is by having exactly one of these cuts at right angles to the fold line. The below figures demonstrate the kind of strategic cuts leading to a triangle. Clearly, the fold line is a line of symmetry for the triangle obtained.

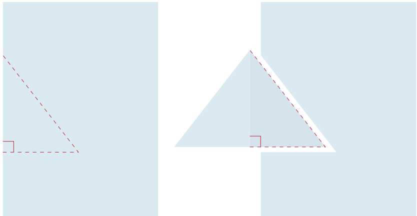

Figure 6a

Moreover, observe that on folding these triangular cutouts along this line of symmetry, the edges on either side of it coincide, meaning these two edges are of the same length. So, we can conclude that if a triangle has a line of symmetry then the two edges on either side of the line are of equal length. Such a triangle is called an *isosceles triangle*. Furthermore, we see that the two vertices on either side coincide indicating that the edge joining them is bisected by the fold line.

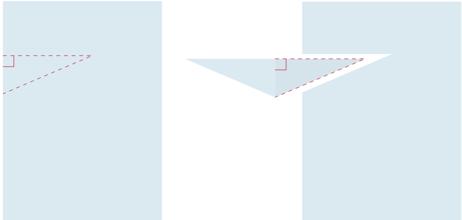

Figure 6b

Further exploration: How do we obtain a triangle with all three sides equal (called an equilateral triangle) by strategically making two cuts?

Exploration 2: Creating Symmetric Quadrilaterals (4-gons)

Next, we turn our attention to creating a 4-sided polygon, quadrilateral, that is symmetric along the fold.

Question: What are the possible ways to make the cuts to obtain a symmetric quadrilateral cutout from a folded paper?

First, let us fold a piece of paper in half. Observe that we cannot use four (or more) cuts as they result in three (or more) points where two cuts meet and hence by Observation 5, the cut-out will have six (or more) vertices as a result. So, to obtain a quadrilateral we should have either two cuts or three cuts.

Case 1 - With two cuts: First, let us use just two cuts to obtain a quadrilateral cut-out. As described in Observation 3, when we make two cuts neither of which is at right angles to the fold line, we obtain a quadrilateral.

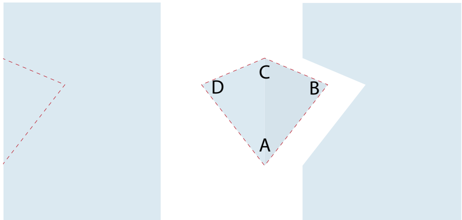

Figure 7

In the quadrilateral ABCD (see Figure 7), observe that the fold line is a line of symmetry. Clearly, B and D are vertices of the quadrilateral that are symmetric to each other about the fold line, while A and C are vertices that lie on the line of symmetry. Let us connect B and D through a line segment, and let it cut the line of symmetry at O as shown in Figure 8.

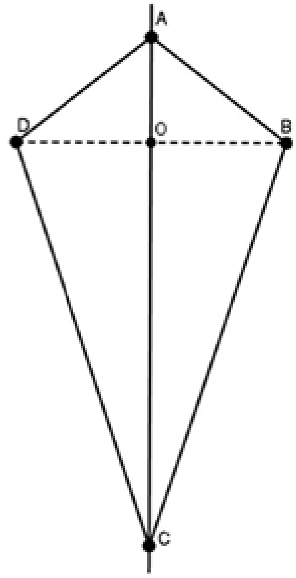

Figure 8

Observe that when we fold ABCD along the line of symmetry, OB and OD coincide. Also, the angles AOB & AOD, and COB & COD coincide with each other respectively. Thus, OB and OD are of the same length. Furthermore, each of the angles AOB, AOD, COB & COD is a right angle.

Hence, we can conclude that in such a symmetrical quadrilateral obtained using exactly two cuts, two vertices lie on the line of symmetry and the diagonal joining them is the perpendicular bisector of the other diagonal. There are two kinds of quadrilaterals with this property: Kite and Dart. (Note that there could be a special case when a Rhombus is obtained.)

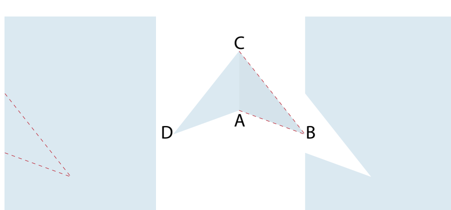

Figure 9

The one in Figure 7 and Figure 8 is called a *Kite*, and the one in Figure 9 is called a *Dart*. The difference between the two is that in the case of the former, lies completely inside the quadrilateral while the diagonal BD in the case of the latter, the diagonal BD lies outside the quadrilateral.

Case 2 - With three cuts: If none of the three cuts is at right angles to the fold line, then by Observation 3, the two points on the fold line result in two vertices of the polygonal cut-out. The two points where two pairs of these cuts meet, by Observation 5, result in four more vertices of the polygonal cut-out. Thus, in such a case the polygon has six sides and hence cannot be a quadrilateral.

Also, we see that if exactly one of the three cuts is at right angles to the fold line, then by Observation 4, the point on such a cut will not form a vertex of the polygonal cutout. However, the other point on the fold line that is on one of the other two cuts, forms a vertex. Also, the two points where two pairs of these cuts meet, by Observation 5, result in four more vertices of the polygonal cut-out. Thus, in such a case the polygon has five sides and therefore cannot be a quadrilateral.

Hence, the only way we can use three cuts to obtain a quadrilateral is by making two of the cuts that are at right angles to the fold line. In that case, the two points where two pairs of these cuts meet, by Observation 5, result in four vertices of the polygonal cut-out, and there are no other vertices. The figures below (Figure 10) show a couple of such scenarios.

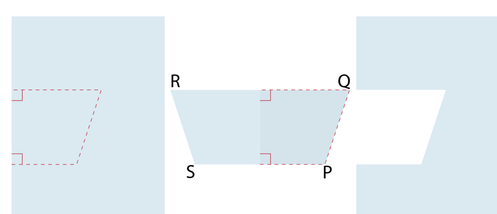

Figure 10

In the quadrilateral PQRS (Figure 10), the fold line is a line of symmetry. Clearly, P and S, and Q and R are pairs of vertices of this quadrilateral that are correspondingly symmetric about the fold line. Furthermore, PS and QR are parallel to each other as they are both perpendicular to the line of symmetry. Also, on folding PQRS along the line of symmetry we observe that PQ and RS coincide meaning they are of the same length.

Hence, we can conclude that in a symmetrical quadrilateral obtained using exactly three cuts, no vertex lies on the line of symmetry, and there is a pair of opposite sides that are parallel while the other pair of opposite sides have the same length. We call a quadrilateral with such a property an *Isosceles Trapezium*.

Further Exploration: Under what conditions, can we guarantee that the symmetric quadrilaterals obtained through the above processes have at least one more line of symmetry?

Generalising our observations: What are the possible ways to make the cuts to obtain a symmetric n-gonal cutout from a folded paper?

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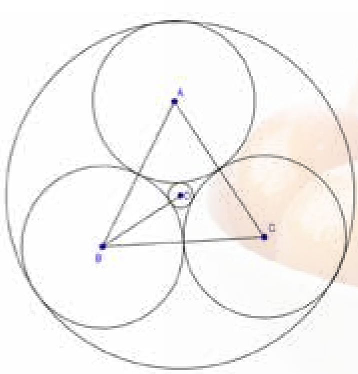
Sweet Stuff

Here is reader **TEJASH PATEL**'s solution to the problem on page **40** of the March 2024 issue, available at https://publications.azimpremjiuniversity.edu.in/5562/1/07_Division%20with%20Multi-Digit-Divisors.pdf



Yes, Arjun can find the radius of each of the gulab jamuns and he can also find the radius of the bowl.

Let **r** be the radius of the gulab jamun and **R** be the radius of the bowl. We are given that the radius of the straw is 1 unit. As shown in the figure, ΔABC is equilateral, let O be the centre of ΔABC.

Now $OB = 1 + r = \dfrac{r}{cos30°} \Rightarrow r = \dfrac{\sqrt{3}}{2 - \sqrt{3}} = 2\sqrt{3} + 3$

Now $R = 2r + 1 = 2(2\sqrt{3} + 3) + 1 = 4\sqrt{3} + 7$.

∴ The radius **r** of the gulab jamun is $2\sqrt{3}+3$ and the radius **R** of the bowl is $4\sqrt{3} + 7$.

Tejash is a teacher at Chanasma Primary School No.2, Gujarat. He has proposed a second solution using Descartes' Circle Theorem available at https://en.wikipedia.org/wiki/Descartes%27_theorem#:~:text=In%20geometry%2C%20Descartes'%20theorem%20states,satisfy%20a%20certain%20quadratic%20equation. Readers familiar with these ideas can see if the solution can be arrived at using this theorem.

My Pedagogical Experience with Arrow Cards

Reviewed by Mokhtar Zaman

The author shares his experience of using Arrow Cards with Grade 3 students, illustrating how this TLM progressively enhanced their comprehension of place value. The effectiveness of this material is evident in the article through a range of examples.

Mathematical concepts, such as numbers and their operations, or patterns and shapes, can be hard to grasp because they're not always something you can see or touch. As Jean Piaget's research suggests, children learn concepts through three levels of knowledge - concrete, pictorial, and abstract (Wadsworth, 1976). That's why it's important for children to begin their learning with hands-on activities. They should start with real objects that they can interact with, then move on to pictorial representations and finally arrive at abstract representations of these concepts on paper. The National Curriculum Framework for Foundational Stage (NCERT, 2022, pp 118-119) also suggests using the ELPS approach in education. ELPS stands for E for experience, L for spoken language, P for pictures and S for symbols.

Procedures or skills may be introduced to children by explorations with concrete materials. In division, for example, this might be done by sharing 12 sticks amongst 6 children. As students handle objects, they take the necessary first steps toward building understanding and internalising mathematical processes and procedures. Working with physical objects allows students to explore concepts at first, which is the concrete level of understanding. Strategies and algorithms can be developed over time.

We, the teachers of Azim Premji School at Dhamtari, were engaged in the process of creating various Teaching and Learning Materials (TLMs) specifically for mathematics. As part of this activity, we made arrow cards. Arrow cards are TLMs used to teach concepts such as place value visually and interactively. Arrow cards also help children understand Partitioning and Recombining. Partitioning involves splitting numbers into smaller manageable parts and Recombining refers to the process of regrouping numbers in different place values.

Challenges faced in teaching place values

Moving from counting objects to understanding place value can be challenging for children because it involves shifting from something they can see and touch (concrete) to something more abstract that exists in their minds.

There are limited practical examples of place value that relate to students' everyday experiences.

Keywords: Place value, pedagogy, challenges, TLMs, arrow cards.

Understanding the role of zero as a placeholder can be confusing for students.

Complex vocabulary, for example, terms such as "units," "tens," "hundreds," and "thousands" are very difficult to understand and communicate about.

Working with Arrow Cards

Initially, I wasn't familiar with using arrow cards effectively in the classroom. To learn more, I consulted a mentor who explained how they help students intuitively grasp place value. With this insight, I felt confident to implement them. Previously, I used bundles of straws for teaching place value, but as my students progressed to larger numbers, I found the arrow card method to be more helpful. I'll narrate my experience of using the arrow cards in the classroom below.

As soon as I entered the Grade 3 classroom, I took out the box of arrow cards. The children looked at the box with great curiosity, and when colourful cards emerged from inside, they were delighted to see them. Each arrow card shows hundreds, tens or ones of a number. For example: 500, 100, 50, 20, 5, 2. They can be placed on top of one another to make 2-digit, 3-digit numbers and so on. I gave a set of arrow cards to each bench, and the children quickly opened them and started arranging them on their benches. They eagerly awaited further instructions, curious about what would happen next.

First, I called out various single-digit numbers from 0-9 and asked the children to pick up the corresponding arrow cards. Next, I explained the concept of tens and hundreds and called out several numbers for them to pick up the correct arrow cards. For example, to make 25, they will need to select the arrow card for '20' and '5' and then put them together so that the slanted lines align. This teaches students that two-digit numbers are made of tens and ones.

Again, when I asked for the number 234, many children picked up the number 200 card, but instead of picking up the number 30 card, they picked up the number 3 and for the one's place picked up the card 4.

Then I intentionally formed incorrect numbers with the arrow cards and asked the students to identify and correct the mistakes.

I placed the cards for 345 as 3, 40, and 5 instead of 300, 40, and 5 and asked them to hold the corners of the arrow cards. As soon as they did this, the middle card, which was the number 3 card, fell out. This provided a perfect opportunity to explain the concept of tens and ones, and how to correctly form the number 345 using the arrow cards. By using arrow card expansion of a multi-digit number for e.g., 234 = 200 + 30 + 4 becomes automatic and effortless.

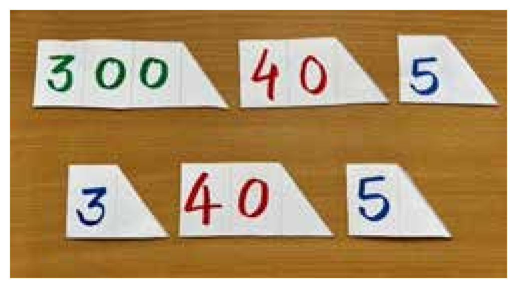


Figure 1

I asked again by using an arrow card to make a number 9,383 whose expansion is 9383 = 9000 + 300 + 80 + 3.

Figure 2

I turned the activity into an interactive quiz game. I called out numbers randomly, and the first student to correctly form the number with the arrow cards got a point. This game made the activity more exciting and motivated the students to think quickly and accurately about place values.

Using arrow cards in addition and subtraction

Arrow cards can be used to break down the addition process into smaller, manageable steps. For example, to add 23 + 15, use cards to represent 2 tens and 3 units as well as 1 ten and 5 units separately, then combine them to show how we get 8 units and 5 tens which gives us 58 as the total.

We can also visualize subtraction with arrow cards by arranging the cards in descending order to represent the process of subtraction. For example, to subtract 5 from 15, you might show the number 15 and then visually represent decreasing it by 1 each step until reaching 10.

Overall, arrow cards have significantly enhanced my teaching of place value. They make learning more interactive and enjoyable. Students were highly engaged, and I have observed a noticeable improvement in my students' comprehension and retention of numerical concepts as it provides a hands-on, visual approach to understanding place value. I highly recommend arrow cards to other educators looking for effective ways to teach place value and related mathematical concepts. and a great learning experience. In the coming days, I will make further progress in creating more TLMs for mathematics.

Acknowledgements

1. Swati Sircar, Assistant Professor at the School of Continuing Education and University Resource Centre of Azim Premji University, India.
2. Arddhendu Shekhar Dash, Principal of Azim Premji School, Dhamtari (Chhattisgarh).

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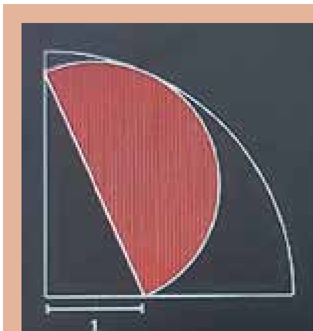

Can you find the area of the red region?

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Dienes Blocks and Static Beads: A Comparative Analysis

Reviewed by Math Space

This article reviews two widely used TLMs - Dienes blocks and static beads, and compares them to provide to the reader a complete understanding of how each of them works, along with their pros and cons.

When a child is introduced to numbers, it is very important to establish a 3-way connection among

(i) The quantity represented

(ii) The number name and

(iii) The numeral or the symbolic representation (Figure 1).

Base-10 blocks play an important role in grasping the idea behind place-value or how we write numbers by making bundles of tens. While the most useful ones are the two-dimensional base-10 blocks, popularly known as the flats-longs-units (FLU), there are two versions of three-dimensional blocks conceived by different people. We have already discussed FLU [1] and how it generalizes into algebra tiles [2] in the March 2024 and the July 2024 issues respectively of At Right Angles. This time, we will look closely at the 3D avatars.

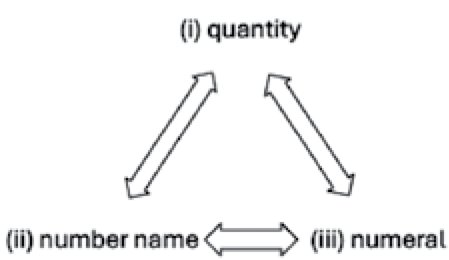


Figure 1

Dienes Blocks

The Hungarian mathematician, Zoltan Dienes (1916-2014), popularized the 3D base-10 blocks. The unit is a small cube, usually 1cm × 1cm × 1cm. The ten is a long cuboid (sometimes called a rod) 10cm × 1cm × 1cm with grooves so that one can easily see that it is 10 units lined up. The hundred is a flat cuboid (often called a plate) 10cm × 10cm × 1cm, with grooves indicating that it is both 10 tens and 100 units. These three blocks are essentially the same as FLU, but with unit thickness (Figure 2). The thousand is a bigger cube 10cm × 10cm × 10cm with grooves on all six faces. One can stack up 10 hundreds and see that their combined volume is the same as the thousand cube (Figure 3). Since each block (except the unit) can be exchanged with 10 of the smaller blocks, i.e., 1 thousand = 10 hundreds, 1 hundred = 10 tens, 1 ten = 10 units, all blocks should have the same colour irrespective of size.

Figure 2

Keywords: Dienes blocks, Static beads, Montessori, comparison, Place value blocks, FLU, FRB

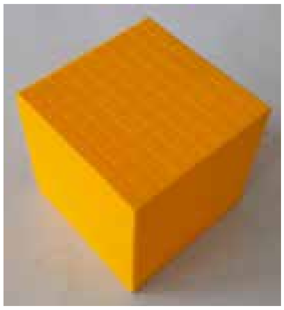


Figure 3

Online version

Many online versions including the ones in Mathigon Polypad: Number Cubes, have different colours for blocks of different size. This can be very confusing when a purple thousand splits into 10 green hundreds or 10 orange units merge to a blue ten (Figure 4).

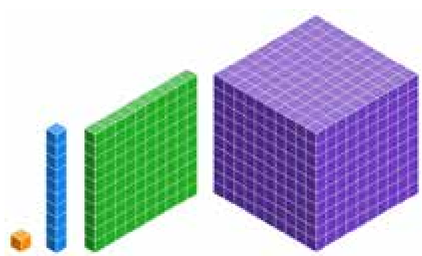


Figure 4

Thankfully, the user can change the colour (Figure 5). But whenever a block is split further or 10 blocks of a kind are merged together, the resulting block(s) resume their assigned colours. So, while these can be very useful for generating pictures for worksheets etc., young learners may raise legitimate questions about the colour changes if they play with these online blocks themselves.

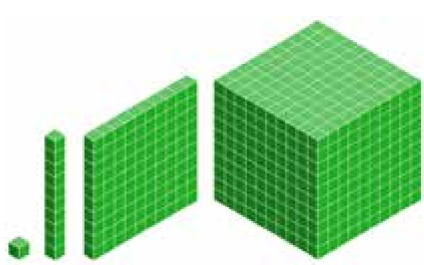


Figure 5

Another interesting aspect of the polypad version is that the orientation of each type of block is fixed, i.e., the ten always stands and does not lie down, the hundred always stands facing right and never faces left! But it also allows one to create a new block by choosing the dimensions (1-10). So, 1-10-10 generates a plate facing left; 10-10-1 is a plate lying down; 10-1-1 and 1-10-1 are rods lying in different orientations (Figure 6).

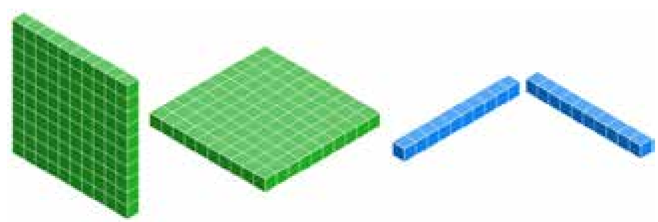


Figure 6

Static Beads

Maria Montessori (1870-1952), an Italian physician and educator, developed a whole range of materials for teaching children as well as pedagogy and philosophy of education known as the Montessori method. One such set is the static beads or golden beads (Figure 7). Unit or one is a single (golden) bead, the wire prevents it from rolling away and provides two handles on either side. Ten is 10 such beads strung in a line (called a string). Hundred is 10 such strings joined to form a flat structure (called a square). So, there are actually $10 \times 10 = 100$ beads in the hundred. Finally, thousand is 10 such squares joined to form a cube. So, there are actually $10 \times 100 = 1000$ beads. One can see that the thousand is clearly 10 layers of beads, each layer being a hundred. Also, a learner can feel how heavy the thousand is compared to a hundred, or a ten, or a unit. So, the static beads set is not only visual but also a tactile material. These have been used by learners at the preprimary stage (3-5yrs) for decades.

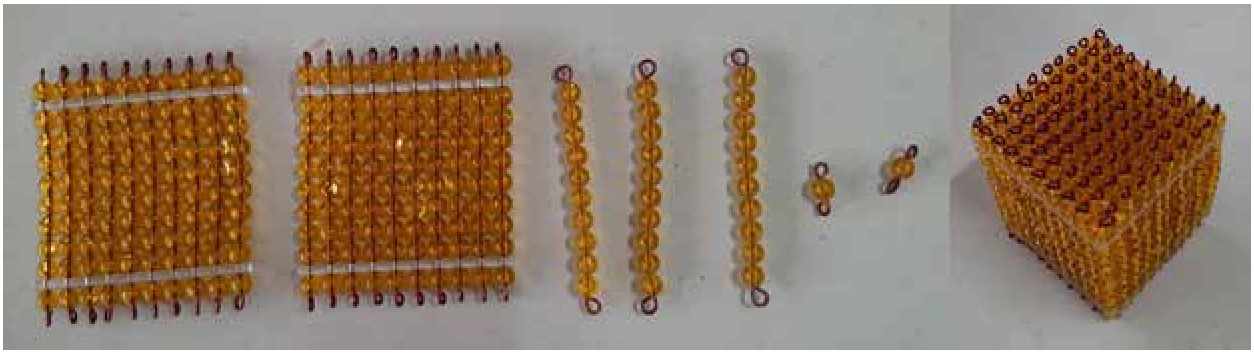


Figure 7

However, it is more expensive and difficult to make. So, after the introduction, static beads are sometimes replaced by wooden blocks. There are circles drawn on the blocks to represent the beads (Figure 8).

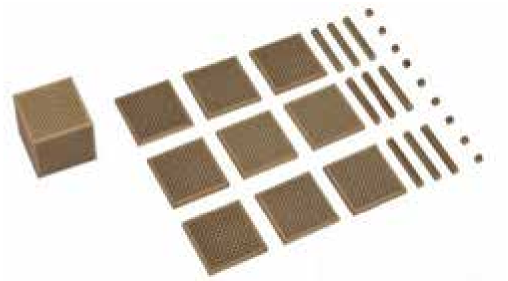


Figure 8
Source: https://www.kidkenmontessori.com/product/static-decimal-beads-and-cards/

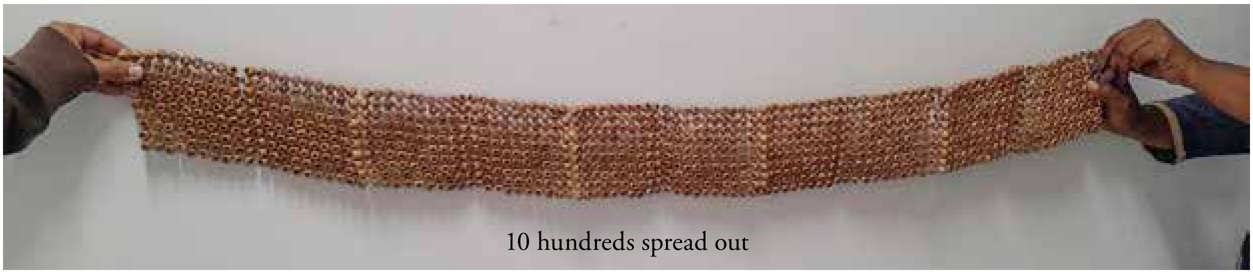
The manufacturing of static beads can be made easier if plastic threads are used instead of metal wires (Figure 9). The tens, the hundreds and the thousand would be less rigid, but serve the same purpose. And it is possible to make the thousand in a way such that the 10 layers are very clear (Figure 10) thanks to the idea of Anupama S M, Azim Premji University.

Figure 9

Possible extensions

The smaller three Dienes blocks have all the advantages of FLU. But these are more tedious to make because of the third dimension. The thousand, which really uses the third dimension, however, does not help young learners get a sense of 1000. Many see it as 600 since each face is a hundred. While adults and older learners can think in terms of cuboid volume as length × width × height, i.e. 10 × 10 × 10 = 1000, the young learners find it too difficult to grasp. Moreover, the blocks usually available in the market are hollow. So, weight-wise the thousand is not the same as 10 hundreds or 100 tens, since 10 hundreds and 100 tens have more partitions inside the big 10 × 10 × 10 cube.

However, 3D base-10 blocks can be useful in giving a sense of how the quantity increases with each digit, i.e., a sense of the exponential growth: 1(cube) → 10 (rod) → 100 (plate) → 1000 (bigger cube) → 10,000 (bigger rod) → 1,00,000 (bigger plate) → 10,00,000 (even bigger cube). Such models can be made with wood or other material and can explain why the comma comes twice in writing a million. So, 3D base-10 blocks or cuboids are quite helpful at the middle school stage (Class 6-8) but not at the Foundational one (pre-primary and Class 1-2) or even in Class 3. Even if models can't be made, visuals such as Figure 11 can provide a similar sense to most learners.



10 hundreds spread out

Star-like arrangement

Figure 10: the layers of the usual thousand can't be spread out.

Layers separated

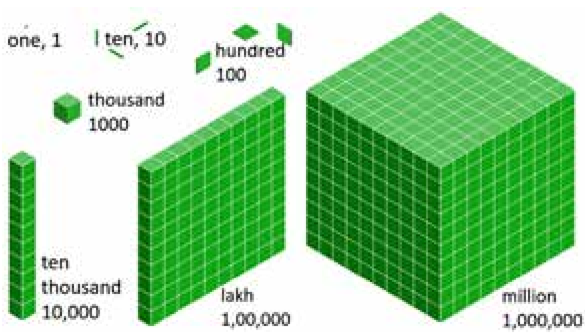


Figure 11

These can also be extended for decimals, which was used in Decimal Division [3] published in the March 2024 issue of At Right Angles.

In contrast, static beads can be extended (in theory) for bigger numbers. But it would be a very tedious job. More importantly, since the learners are much older by then, they are expected to abstract out and use volume formula and measurement instead of depending on counting. So, there is hardly a need to make similar models for 5- or 6-digit or bigger numbers. And since it is practically impossible to split a single bead, this model cannot be extended for decimals.

	Dienes Blocks	**Static Beads**
Credited to	Zoltan Dienes (1916-2014)	Maria Montessori (1870-1952)
Chronological order	Came later	Came earlier
Conceptually	Based on volume	Based on count
Makes sense to	Older learners, Class 4-5, 9+yrs	Pre-primary stage learners, 3+yrs
Manufacturing	Easier	Labour intensive and material intensive
Cost	Less	More
Extension(s)	Can be extended to bigger numbers and decimals	Can be extended to bigger numbers (in theory), not possible to extend to decimals

In short, Dienes block is more extendable and meaningful in the long run. But it does not provide a sense of 1000 adequately to young learners. Static beads do a much better job of communicating to them. So, one should choose between these two based on the age of the learner.

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3. Division with Decimals: https://publications.azimpremjiuniversity.edu.in/5563/1/08_Division%20with%20Decimals.pdf

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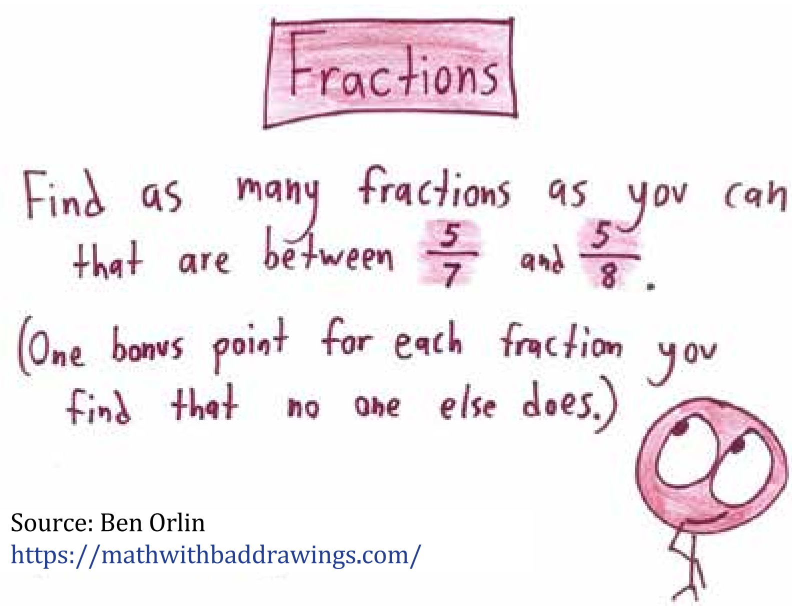

Source: Ben Orlin
https://mathwithbaddrawings.com/

Constructing a Square with the Area 1/n of a Given Square

Dikshha Sinha

We present here a solution to the problem posed on page **26**

A construction to halve the area of a given square.

In the following figure let *ABCD* be a given square (of unknown side length). Let *PB* = *BS* = 1 unit, and *BQ* = 2 units. Let *PRQ* be a semicircle passing through points *P* and *Q*. Then the area of the square *XYZB* must be half of that of *ABCD*.

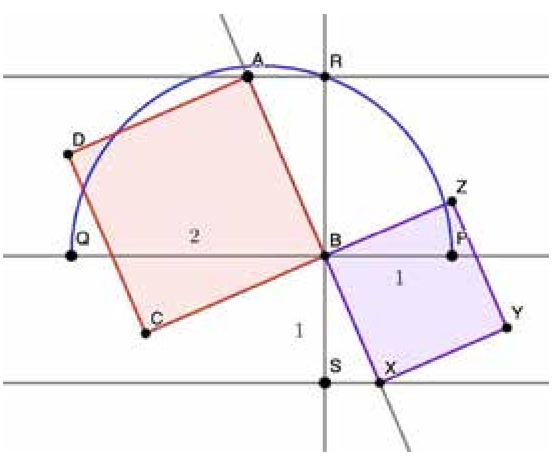

Figure 1

This question and this picture were submitted by one of our authors and it made us think. Do spend some time observing this picture – Is the area of square *XYZB* half the area of the given square *ABCD*? If so, why? We explain below.

If we apply computational thinking to solve this problem, this is the first step:

Statement of the Problem: Is the area of square *XYZB* half the area of the given square *ABCD*? If so, why?

Let us try to decompose the steps: We are given that *QB* = 2 units, and *PB* = *BS* = 1 unit.

Given a square whose area is x square units, we want to draw a square whose area is $\frac{x}{2}$ square units. A quick observation reveals that this is the same as given a line segment of $\sqrt{x}$ units, we want to construct a line segment of $\sqrt{\frac{x}{2}}$ units.

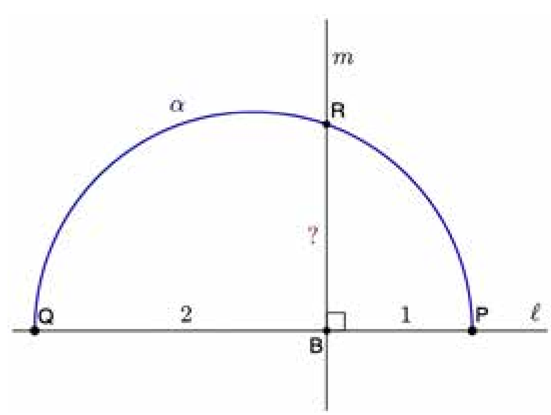

Figure 2

Keywords: Construction, fractional areas, reasoning

Step 1. Mark three points P, B and Q on a line l, such that B lies in between P and Q. $QB = 2$ units and $PB = 1$ unit.

Step 2. We draw a semicircle α with PQ as the diameter.

Step 3. Let the line m perpendicular to l and passing through B intersect α at R. (See Figure 2)

What will BR be?

From the right-angled ΔPBR, $PR^2 = PB^2 + RB^2$.

From the right-angled ΔRBQ, $QR^2 = RB^2 + BQ^2$.

From the right-angled ΔPRQ, $PQ^2 = PR^2 + RQ^2$ (note that $\angle PRQ$ is a right angle in the semicircle).

By using these three equations, we arrive at the fact that $BR = \sqrt{2}$.

Step 4. Mark the point S on the line m at the other side of l such that $BS = 1$ unit (see Figure 3)

Step 5. Draw the line α parallel to l passing through R, and the line b parallel to l passing through S.

Step 6. Choose any point A on the line α and join the points B and A via the line c. Let c intersect b at X.

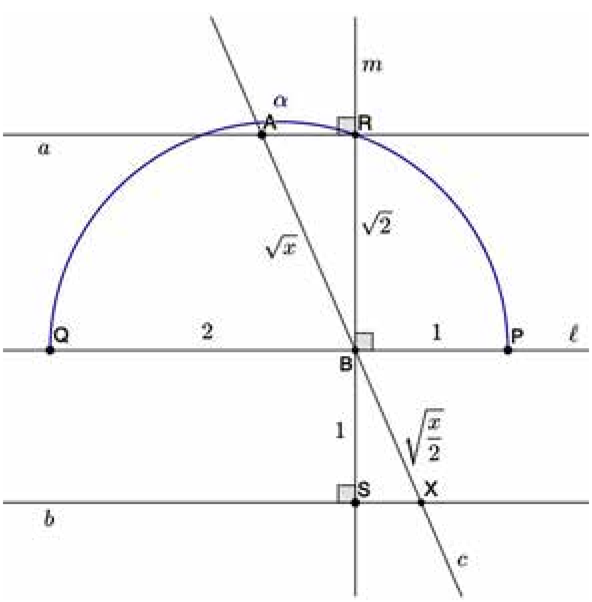

Figure 3

Now ΔARB and ΔXBS are similar. So, if $AB = \sqrt{x}$ units, then $BX = \sqrt{\frac{x}{2}}$ units.

Thus, if we construct a square with AB and XB as respective side lengths, then necessarily we should have that the square with AB as a side, must be two times the area of the square with XB as a side. Thus, by decomposing the solution steps, we conclude that the area of square $XYZB$ is indeed half the area of the given square $ABCD$.

The reader might have already observed that there are simpler constructions of squares that can halve the area of a given square, such as in Figure 4. Here $ABCD$ is a given square, and P, Q, R and S are midpoints of sides AB, BC, CD and DA respectively. However, the above construction can be extended to construct a square with area $\frac{1}{n}$ of a given square, by taking $BQ = n$ instead of 2 (See Problem 1 below).

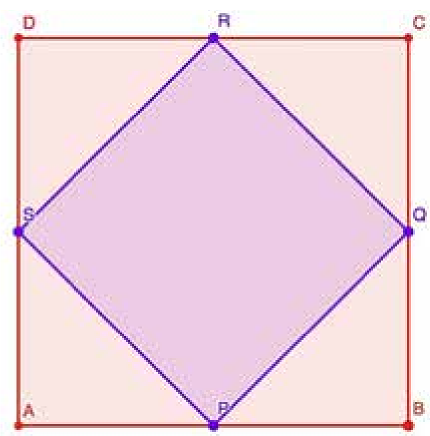

Figure 4

Why stop at halving the area of a square? We claim that this construction can do more! We provide a set of problems for the reader to try their hands on.

Problems

1. If we assume that $BQ = n$ units, what would be the length of BR? What would be the length of BX? What is the ratio between the areas of $ABCD$ and $XYZB$?
2. The above construction works only if the side AB is more than or equal to BR. What if $AB<BR$? The above construction can be

slightly modified to accommodate this possibility (Figure 5). Argue why the following construction would work. Here β is a semicircle joining the points B and R, and γ is the semicircle joining the points B and S.

3. Do the above constructions work if we want to halve the area of an equilateral triangle? A regular hexagon? A regular 13-gon? A circle?
4. Given a two dimensional shape, can you argue if the above constructions give a shape which is similar to the given shape and has area $1/n$ of that of the given shape?

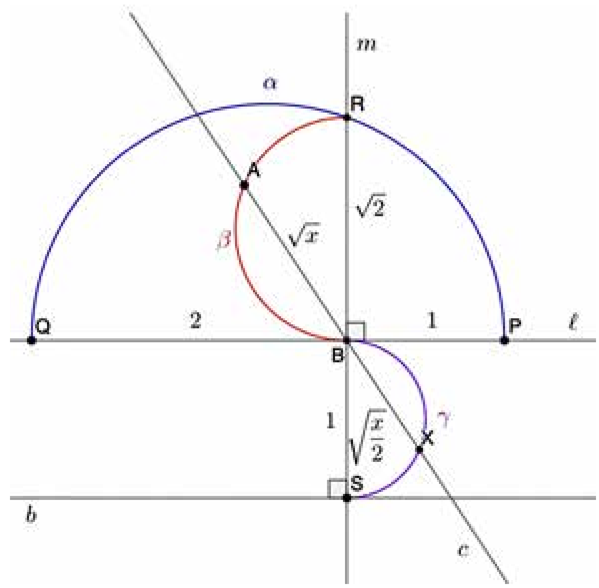

Figure 5

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Generalising a Square Root Problem

Gauri Ghormade

The author describes her experience of observing a class where a visual way of solving a problem involving square roots was used. She takes it a step further and attempts to generalise the method in this article.

At Azim Premji School in Dhamtari, Chhattisgarh, I observed a grade 8 mathematics class where the teacher used a novel method to introduce square roots. He employed a visual way of solving a problem involving square roots which was different from traditional methods. In this article, I attempt to generalise the method, developing a connection between this visual approach and the traditional method.

The teacher began the lesson with the following question:

I bought 1000 saplings to plant them in a rectangular array such that the number of rows are the same as that of columns. If I were to do so, how many more saplings would I require to fulfil my condition?

Let us see how we would have solved the problem traditionally. This would suggest that we are looking for the next square number after the largest 3-digit one. The requirement for equal rows and columns indicates that we are looking at square numbers. Why the largest 3-digit one? While solving this question by long division method, we can find that the result will be the square root of the largest perfect square less than 1000, which is the greatest 3-digit perfect square. Following this logic, we need to find the next square number after the largest 3-digit one which is nothing but the smallest 4-digit square number. If 1000 is subtracted from this number we get the number of saplings needed.

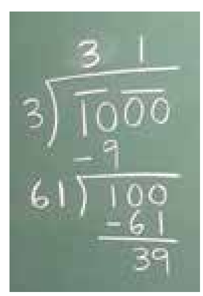

Figure 1

Figure 1 shows the long division method to find the square root of 1000. Here if we take a square of 32 and subtract 1000 from it we have the answer.

$32^2 - 1000 = 24$ Hence, 24 is the answer.

Now, if we try to generalise this method by taking 1000 as N and 31 as m.

The equation can be reformulated as $(m + 1)^2 - N$ = number of saplings needed.

Note that here one needs to do a 2-digit by 2-digit multiplication after performing the long division.

Keywords: Squares, square roots, word problem, visual problem solving

As a prerequisite, the class had enough practice in solving questions such as identifying the largest or smallest 3- or 4-digit square numbers. Most of the students in the class attempted to solve it by using a long division method of finding square roots, and some were successful in solving it.

However, the teacher did not use exactly this way of explanation which involves the largest 3-digit square number and smallest 4-digit square number.

Instead, he drew a diagram similar to Figure 2 and said that using 1000 saplings only the square of 31 by 31 can be formed with few saplings remaining, precisely equal to the remainder, which can be arranged on the sides. This gave a hint to students to find the square of 32 which will be the total number of saplings and subtracting 1000 from this will give the number of saplings to be bought. Now this visual way can be extended to find the answer to the question where we don't need to find the square of 32.

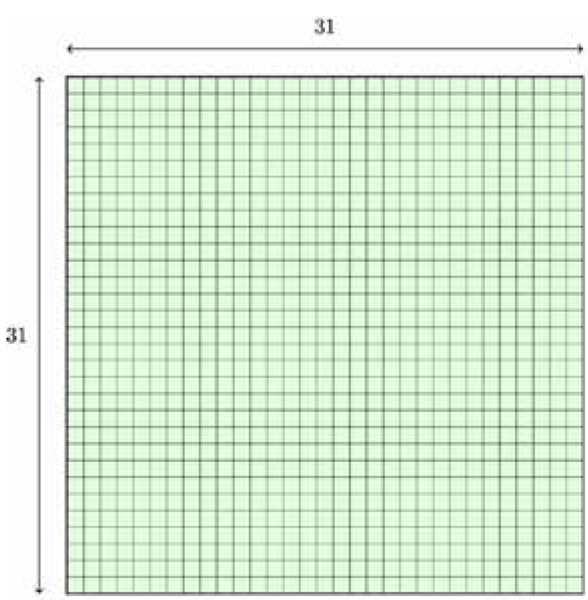

Figure 2

Let us assume that the number of saplings with us is N, which is not a perfect square, and m^2 is the largest perfect square less than N. Now after arranging the saplings in m by m square, the remaining saplings are $N - m^2$. Remember how it was done in the previous question, $1000 - 31^2 = 39$, the remainder in the long division method.

Now exactly m saplings out of these remaining saplings can be arranged on one of the sides. So after arranging that, we have now $(N - m^2) - m$ saplings left, i.e., $(1000 - 31^2) - 31 = 39 - 31 = 8$ in the previous question. Recall that $39 - 31 = 8$ were the plants planted on the other side of the square. See Figure 3.

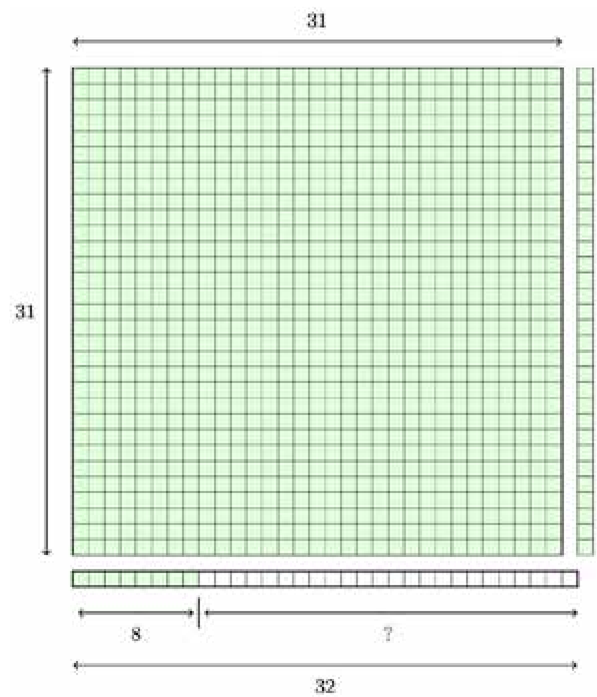

Figure 3

But note that the other side has $m + 1$ units in total. So we need to buy $(m + 1) - [(N - m^2) - m]$ saplings. In the previous example, we should have bought $(31 + 1) - 8 = 24$ saplings.

However, if we began with 990 saplings instead of 1000, our reasoning will not work. This is because we would then have $(N - m^2) - m < m$, which means we would not have filled either of the additional sides. This can be addressed as follows. Instead of planting all the plants first on one side of the square, we can plant saplings one by one on both sides (See Figure 4).

In general, this means we have $m + m + 1 = 2m + 1$ slots on the sides of square. If we plant them using the $(N - m^2)$of the remaining saplings, then the number of required saplings to be bought should be $2m + 1 - (N - m^2)$.

Note here that students only need to perform addition and subtraction instead of multiplication. As these are simpler operations compared to multiplication there are fewer chances of mistakes. This also provides them with an opportunity to visually verify that $(m + 1)^2 - N$ is same as $2m + 1 - (N - m^2)$.

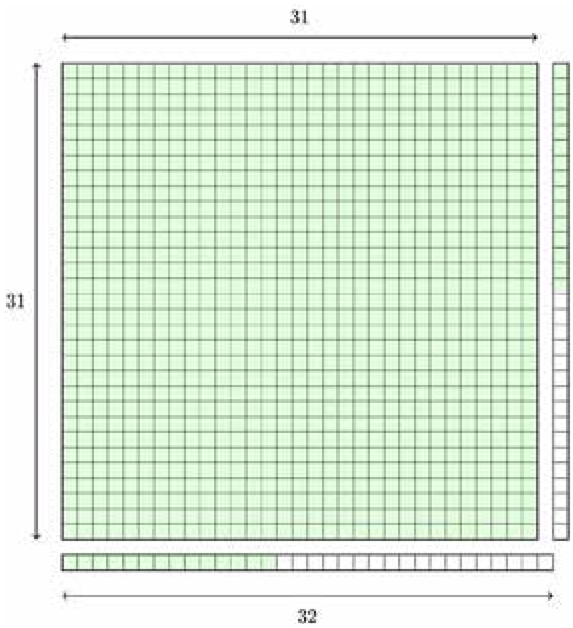

Figure 4

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Call for Articles!

At Right Angles is a quality resource dedicated to mathematics education in India's public education system. It is specifically designed for teachers and teacher educators at the foundational, preparatory, and middle school levels.

We invite articles from mathematics teachers, educators, practitioners, parents, and students. If you are looking for a platform to contribute articles that support and enhance the learning experience of mathematics particularly for students approximately in the age group 6-14 years , we welcome your submissions.

Suggested Topics and Themes

Submitted articles should focus on curricular content applicable to Classes 1-8 and could:

- Explain and illustrate themes and topics outlined in the National Curriculum Framework for School Education 2023 (NCF-SE 2023).
- Specifically address challenges discussed in the NCF-SE 2023.
- Be substantiated accounts of the history of mathematics or the history of mathematical thinking.
- Include innovative worksheets or methods to engage students in drill and practice.
- Describe real-life applications of mathematics relevant to the child's context.
- Describe interdisciplinary activities or projects.
- Review puzzles or games with a practical connection to the syllabus.
- Offer guidance on selecting relevant content, including online resources.
- Develop pedagogical strategies for foundational numeracy as well as computational thinking.
- Assist teachers in implementing differentiated teaching practices.
- Review of Teaching Learning Material (TLM) or describe how to use local context, and local TLM in the math class.
- Provide material to help students bridge gaps in conceptual understanding.
- Address issues in assessment.
- Suggest ways to identify and address misconceptions in mathematics learning.
- Offer a list of problems along with discussions on their solutions and problem-solving strategies that are not commonly found in textbooks.

In addition to full-length articles, we also welcome shorter pieces that can include a variety of engaging content. These could be reviews of books, mathematics software, or YouTube clips that explore mathematical themes. Other contributions can be 'proofs without words', mathematical paradoxes, 'false proofs', or creative expressions such as poetry, cartoons, or photographs with a mathematical theme. We also welcome anecdotes about a mathematician or interesting examples of 'maths in craft, movies, etc'.

Articles may be sent to atria.editor@apu.edu.in

Please refer to specific editorial policies and guidelines on the inside back cover.

Policy for Accepting Articles

At Right Angles is an in-depth, magazine on matters of consequence to early mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions, and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be indicated in the article.

At Right Angles brings out translations of the magazine in other Indian languages. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published elsewhere, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to **At Right Angles**.

While **At Right Angles** welcomes a wide variety of articles, submissions that are found relevant but not suitable for publication in the magazine may be used in other avenues of publication within the University network, with the author's permission.

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. **Engaging Introduction:** Write in a readable and inviting style, aiming to capture the reader's attention from the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, a figure with an interesting question, or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. **Catchy Title:** Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. **Style:** Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. **Balance:** Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps that depend on hidden calculations.
5. **Accessible language:** Avoid specialized jargon and notation that will be familiar only to specialists. If technical terms are needed, please define them.
6. **Use visuals:** Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. **Concise References:** Provide a compact list of references, with short recommendations.
8. **Exercises and Questions:** Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. **Citation format:** Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. **Abbreviations and Acronyms:** Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. **Labelling visual elements:** Label and number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note: the minimum resolution for photos or scanned images should be 300 dpi).
12. **Precise references to visuals:** Refer to diagrams, photos, figures and tables by their numbers and avoid using references of these kinds: 'here', 'there', 'above', 'below', 'to the left', 'to the right'.
13. **Author Bio:** Include a high-resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. **British Spelling:** Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. **Format for submission:** Submit articles in MS Word format or in LaTeX.

Printed and Published by Sharad Sure, Registrar on behalf of Azim Premji University.
Printed at Lakshmi Mudranalaya, # 117, 5th Main Road, Chamrajpet, Bengaluru, Karnataka 560 018.
Published at Azim Premji University, Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562125
Editor: Sneha Titus

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Azim Premji University

MONEY

BY PADMAPRIYA SHIRALI

MONEY

Most children witness the usage of money during purchase transactions at grocery stores or vegetable vendor shops. While the usage of digital payments has increased across the country and in all types of transactions, currency (coins and notes) is still used for various payments such as services offered by maids, cobblers, pavement purchases, vending machines, etc. Also, older people continue to be more comfortable with money transactions.

Through exposure to household activities and talk, children implicitly grasp the concept of exchange of money for goods/services. Basic recognition of coins and notes of small denominations gets built up. They develop ideas of low and high costs though the value of different denominations and money in general will develop slowly over the years.

However, there are challenges that young children face, such as distinguishing between the value of a pile of coins and the number of coins in the pile. Also, the concept of the monetary value of an object is not familiar to all young children. The concept is abstract. If you were to take a 2-rupee coin, there is no intrinsic 'twoness' about it. It must be observed that notes and coins are non-proportional materials (while ganitmala, bundle-sticks, flats-longs-units (FLU) etc. are proportional). Size does not determine the value either, as notes of different denominations are often the same size.

Some children may handle currency of lower denominations when they have to obtain items from a shop. Some families may give piggy banks to their children to develop the idea of saving.

Some children may be given pocket money on a regular basis to plan their expenditure. Children overhear adults talk about prices of vegetables, vehicles, loans for land or houses, EMI payments, etc.

This exposure can be incorporated into classroom discussions.

Class projects with students of class 5 and above can involve planning for the purchase of books needed for the classroom, say, story books. Teachers and students can look at various price lists of puzzles or story books and select needed items. Is it necessary to buy all materials or can some items be bought second hand? Discuss needs and wants, emphasise the need for reuse and reduction of costs.

Plenty of opportunities arise for creating problems from school outings or a visit to a mela involving money.

A teacher can use all the experiences of children to take the topic forward to enhance the conceptual understanding of students about money. The topic is often taught in conjunction with number work involving hundreds and thousands.

Note for the teacher

Some of the currencies, e.g., ₹200 and ₹500 should be introduced after those numbers have been taught in the respective chapters.

Traditionally there used to be a separate chapter on Money, but in Class 3 it has been combined with the numbers chapter.

Activity 1 - Classification of real coins   Level: Class 2,3

It will be good to use real coins for this purpose as students can feel the texture and mass, and notice the colour differences. Ask students to classify the coins as per their choice. This can be done in multiple ways: size, shape, motif, colour or denomination.

Use pictures of the coins and ask students to match the coins with the pictures.

Encourage students to use words and phrases such as *alike, different, belong to the same group because* .... and to give reasons.

Ask them to order the coins. This could be in terms of size, number, imprinted year, value, etc.

Activity 2 - Description   Level: Class 2,3

Each student can select a coin (it is better to stick to Indian currency initially and to include a few old coins of different shapes). Encourage the student to describe the coin in terms of its colour, shape, round or straight edges, motif found on the two sides of the coin. Discuss the meaning of head and tail.

Students can do coin rubbings (by placing the coin under a paper and colouring it) and create a poster.

Note: It's important that students observe the following:

1. Some coins are not used anymore.
2. Older coins used to come in different shapes, but new ones are all circular. Is there any advantage to coins having different shapes? Discuss. Why do you think they are all circular now?
3. Coins of the same denomination may have different sizes based on the year. ₹2 & ₹1 are good examples.
4. Coins of different denominations can have the same size - new ₹2 and old ₹1.

Activity 3 - Free Play   Level: Class 2,3

Provide many opportunities for free play with coins. Most students of this age like to engage in pretend purchases at a shop. Let them set up a shop in the classroom using a few items displayed on a table (Pencils, crayons, sketch pens, erasers). Distribute the coins (real/plastic) to students and let them engage in free play using the coins for making their purchases. It is best to allow them to price the product and figure out the quantities of items for sale. Further discussions can happen about the price of a product and students' awareness of prices of certain products like biscuits, toffees, stationery, clothes, etc.

Design activity: Students can design coins and discuss which part of the coin indicates the denomination. It will increase their observation of variation in shapes, denominations in big numerals, etc.

Activity 4 - Objective: Recording data of sorted coins   Level: Class 2,3

Students can be encouraged to represent the data of sorted coins in the form of a graph using pictorial representations. They can use coin rubbings to make it look authentic. Discuss the various images that they see on the coins.

Multiple graphs can be created based on the different methods of sorting employed by the students.

Coin Graph

Activity 5 - Understand the value of a pile of coins   Level: Class 2,3

Students sort and stack coins, so that coins of the same denomination are in the same stack. Pose the question, 'If you are allowed to use these coins to buy some food, which stack would you choose?'. Let the students identify the stack they wish to take.

Raise questions:

1. Which stack of coins did you choose and why?
2. Did anyone else choose a different stack of coins? If so, why?
3. What problem solving strategies did you use to make your decision?

Through discussion let it emerge that the number of coins does not determine the value of a pile of coins.

Game 1: Who am I?

Ask one student to select any coin without the knowledge of others. Let the other students ask 3 questions which will be responded to with a Yes or No. For example, Is it a silver coin? Is it less than 5?

Students should try to get the answer after they hear the responses.

Activity 6 - Familiarity with the face value of the currency notes — Level: Class 2,3

Use coins and currency notes (play money) of all possible denominations that are currently in use in the country. Please note that the relationships between the various denominations in terms of equivalence of value will come in the following level.

Students should use the words rupee, rupees and describe the notes by making comparisons such as 'the ten-rupee note is smaller than the hundred-rupee note'. 'The two hundred-rupee note is orange in colour'.

Activity 7 - To build a sense of the value of 100, 200, 500 rupees — Level: Class 3

Children develop a sense of the value of higher denominations when it is translated in terms of their favourite objects or eatables.

The price of one Dairy Milk chocolate is ₹20.

- How many can you buy for ₹100?
- How many for ₹200?
- How many for ₹500?

Activity 8 - Exchange of coins and notes for coins and notes of lower denominations

Level: Class 4

It is assumed that students have worked with place value material that involves the exchange of 10 units for a ten and 10 tens for a hundred prior to this.

While money can be used to reinforce the exchange concept it is to be kept in mind that the hundred rupee units as an exchange for a hundred rupee currency note or ten 1-rupee coins as an exchange for a 10-rupee coin is an abstract idea. The ten or hundred is not visible as in the case of place value materials. Since money is a non-proportional material, it should be used for place value after exposure to proportional material.

Students need to gain confidence and be comfortable with the exchange process before being asked to record it in writing.

For example, 1 hundred = 10 ten rupees, etc. However, students can be encouraged to articulate their thought process 'I have 15 one-rupee coins. I can exchange ten of them for 1 ten-rupee coin'.

Ask different questions that develop their ability to find equivalent value using other denominations.

- I have a 10-rupee note. Which 2 coins will add up to the same value?
- How many 5-rupee coins are needed to match this 50-rupee note?
- How many 2-rupee coins equal a 10-rupee coin in value?

How many 2-rupee coins equal a 5-rupee coin in value!!!

Activity 9 - Finding equivalent sums   Level: Class 4

Plenty of exercise work can be created in an activity form.

Make a group of 4 students. One person picks at random 2 items, one note and a coin or two notes or two coins. The others have to find different ways of putting together currency of equal value.

Pose questions that have varied answers.

Leela bought a balloon at the mela. She paid for it using six coins. How much might the balloon have cost? What is the largest amount Leela could have paid? What is the smallest amount Leela could have paid?

Students may use their awareness of prices and also of available coin denominations to give reasonable answers. Such questions generate a good deal of discussion in the classroom.

Game 2: Who gets the 500-rupee note?

Make groups of 4 students. Students take turns to throw two dice and sum the numbers. They can collect ten times the sum. If the sum is 9 they collect 9 tens amounting to ₹90.

Each time the amount exceeds ₹100, they exchange the change for a 100-rupee note.

Whoever reaches ₹500 first is the winner.

Activity 10 - Money in operations, Paying the correct amount   Level: Class 4

Materials: Play money, pictures of objects with price tags/real objects like pencils, sketch pens, etc.

By setting up a classroom shop, with items displayed with appropriate price tags, students can be involved in the purchase of items, estimating the amount to be paid, making bills, figuring out change to be given/taken, etc. Each student gets a 500-rupee note.

Raise questions:

- How much have you spent so far?
- How much have you got left?
- How much more money do you need to buy ten pencils?
- If you buy those two pens, how much will you have spent?
- How much will you have left?
- How do you know?
- If pens are on an offer, 'buy one get one free', how many will you get for fifty rupees?

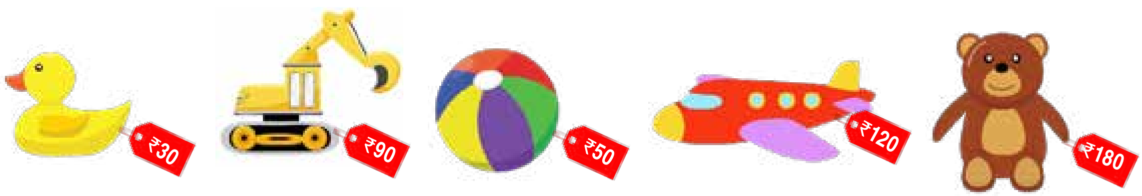
Here is an opportunity for mental calculations and the varied approaches used by students in addition and subtraction problems. Counting on approach is frequently used by shopkeepers. For example, if the item is priced at ₹78 and they receive a 100 rupee note, they may first give ₹2 to make it ₹80 and then 2 tens to make it ₹100.

Pose questions that offer options.

Here are the prices of stationery items. If you were given ₹100 and wished to spend all of it, what items can you buy?

Pose questions that involve comparisons.

Rahul says, 'I have bought 2 items at the mela' One item cost ₹90 more than the other'. What might Rahul have bought?



In higher classes, profit and loss concepts can be brought out.

Many concepts related to the maintenance of adequate stock, high price, cheapness and value for money can be discussed.

Activity 11 - Exposure to a real market

Level: Class 4,5

Students can be asked to accompany their parents for the purchase of groceries, vegetables and fruits. They can be required to maintain a list of the items bought on such a trip and the price paid.

Students may get exposed to the idea of bargains, discounts, gift items that come with some products, the advantage of buying large sized packs, etc.

All these aspects can be discussed in the classroom.

Discussions can bring out connections with other maths topics, for example, the usage of balance/weighing scales, weights, etc, ways of storing products and stacking.

Activity 12 - Understanding a note

Level: Class 5

Students can observe carefully a real note of 100 rupee denomination. Can they make a list of the various pictures and writing that they find on it?

- What picture is shown on the note? Is it a historical building? What do we know about it?
- Is there a picture of any person on it? Who is he/she?
- Are there any other historical artefacts?
- Are there any other designs on the note? Can they describe the designs?
- What type of writing is on the note? What language scripts can be found? Can the students identify all the scripts?
- Are there any numbers on the note? How many types of numbers are there? What do these numbers stand for?
- Does the note indicate the year it is printed?
- What is the promise written on the note? Who signs it?
- Are there any special marks on the note?
- Are there any embossed shapes on 100, 200, 500 rupee notes?

Find out: Does each note have a unique number? What is a watermark?

What are the currencies of other countries?

Activity 13 - Understanding substitutes for money, barter system

Level: Class 5

Students would have also noticed their parents make payments by other means, digital transfer of money.

Encourage them to share the various modes of payment they have observed. Discuss how these methods are used and the advantages of using such methods.

Discuss if there are communities that do not use currency and how they obtain goods and services.

Activity 14 - Understanding the history of currency   Level: Class 5

Bring some old coins that were in use a few years ago like (50, 25, and 10 P coins) and talk about them. It can be a good time to talk about what one could buy with a rupee in the olden days.

Help students to find out information about coins used in earlier times (say about 200 years ago)

Activity 15 - Understanding banks   Level: Class 5

Organise a trip to a local bank.

Discuss the trip with the students and get them to prepare a set of questions to ask the bank official. Students may wonder: 'Where does the bank get its money from?'

'Can children have a bank account?'

'How does an ATM operate?'

Speak to the manager beforehand so that they are prepared to explain in a simple manner the concept of saving, the purpose of keeping money in a bank, interest earned and how banks assist people in giving loans etc.

Discuss how money is earned by people in various ways and is saved.

Activity 16 - Budget planning   Level: Class 5

Most schools celebrate Independence Day or the school day by organising some decorations and purchasing sweets within a certain budget.

Involve the students in planning for such an event, the number of people expected, cost of the decorations like streamers or balloons etc., sweets, refreshments like tea or juice.

Encourage students to make lists of various items and find out the prices.

Does the cost exceed the budget/ If it does, how can they cut it down?

Activity 17 - A Birthday Celebration

Level: Class 5

Each student can work out a list of items to have for a birthday party (sweets, snacks, drinks, plates, tumblers, decorations), the number of guests and calculate the cost of such a party.

Students can share their lists in groups.

Discuss how one can celebrate such events in economical ways, without generating waste. Discuss how people can share resources.

Activity 18 - Making calculated choices

Level: Class 5

Refer to the students' experience of eating at a restaurant or buying food in the school canteen.

Discuss how the adults around them would have selected items to be ordered.

Customers choose between different options like a fixed plate meal or ordering individual items. Some items come in large quantities and are shared amongst 2 or 3 people.

Snacks	Price (₹)
Ginger Tea	15.00
Coffee	20.00
Milk	15.00
Dal Vada	08.00
Medu Vada	15.00
Chilli Bajji	07.00
Banana Bajji	07.00
Aloo Bonda	08.00

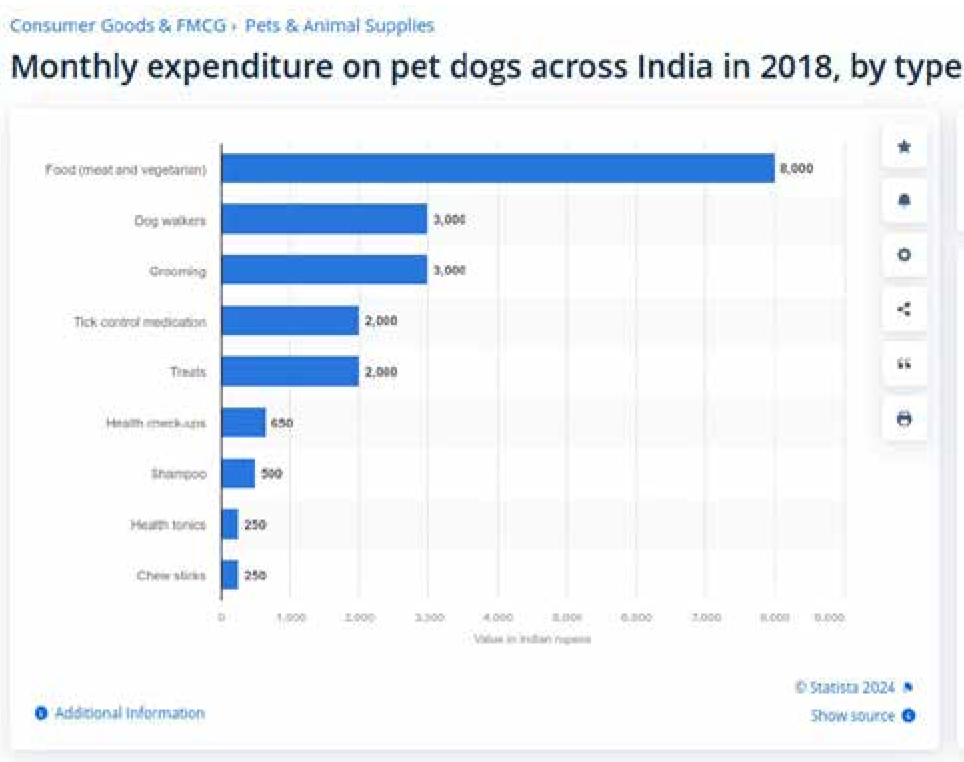
Problem solving: Create problems that develop problem solving skills.

- Shiv and Shravan ordered 4 different items for ₹50 exactly. What could they have ordered?
- Diva bought 2 different items from the snack shop, paid with 2 identical notes and got ₹4 as change. What did she order?
- Can we make snacks at home instead of buying them?

Project: Adopt a pet

Most children enjoy having a pet or would like to have a pet. Students can evaluate what it would cost for the school to adopt a pet.

Teachers can obtain information about the cost of keeping a pet.



https://www.statista.com/statistics/1031188/india-monthly-cost-pet-dogs/

Highest currency note ever printed by RBI is ₹10,000

short by Pragya Swastik / 07:51 pm on Wednesday 9 November, 2016

The highest denomination currency note ever printed by the Reserve Bank of India is a ₹10,000 note during the British Raj in 1938. While the note was demonetised in 1946, a new version of the note was introduced in 1954. However, the ₹10,000 note along with ₹1,000 and ₹5,000 notes were demonetised by the then PM Morarji Desai in 1978.

read more at Hindustan Times

Acknowledgement

To Swati Sircar and Sneha Titus for their valuable suggestions.

PADMAPRIYA SHIRALI

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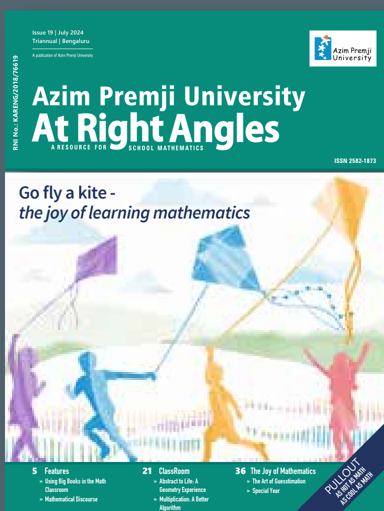

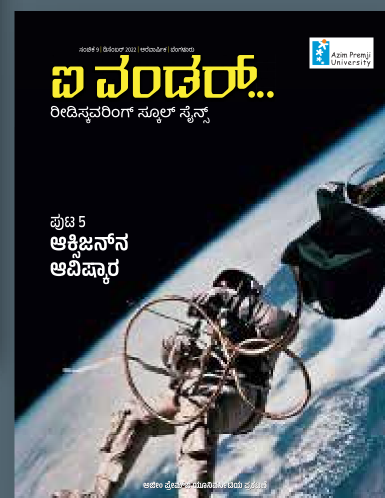

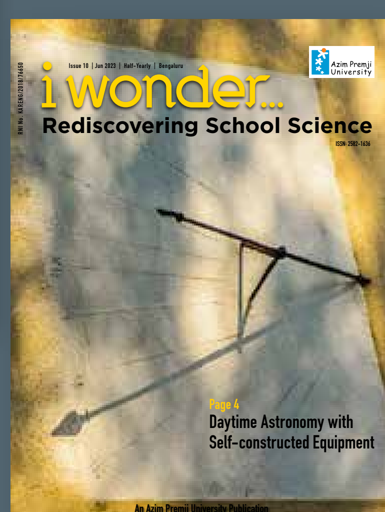

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