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Azim Premji University

Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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Equal Parts of the Whole?

PULLOUT
PARALLEL LINES

As the world goes through lock down and social distancing, it is easy to feel less and less a part of the whole and more and more a small, isolated part. But that does not make us or anybody else, any less a part of the whole and we need to realise that every part of the whole matters.

But are all the parts of the whole equal?

If not, what can we do to redress the balance?

Think mathematically!

From the Editor's Desk . . .

Negative is the new positive!
Flat is the new high!
The spread should be contained!

It certainly sounds as if mathematical terms are being used in strange and new contexts in 2020. Thankfully, mathematics remains true to its nature and working on some lovely articles for AtRiA has kept the editorial team on track during these troubled times.

Our two articles on Fractions (one on Misconceptions by Arddhendu Dash and the other, a book review by Rajat Sharma) made us re-think equal parts of a whole as our cover illustrates. How equitable mathematics is…. far more than reality, unfortunately. And as we think of rebuilding, we also have several articles on constructions – constraints place an edge of challenge in articles by Shailesh Shirali (Compass and Ruler Constructions) and A.K. Malik (A Note on Geometric Constructions).

For the captive audience, entertainment has been arranged - Circles, parallelograms, triangles, tetrahedrons, rectangles,…….the line-up of geometrical celebrities is pretty impressive and like true reporters we dig deep to give you hitherto undiscovered stories about them along with strategies to push up their ratings with that discerning audience, the student population. Problem Corner gives you strategies to solve problems that you thought were insoluble, add to your armoury of heuristics with this section.

And for those who are engaging with online classes and never got round to dynamic geometry software, here is the introduction to GeoGebra –free software which gets you experimenting with geometry, with graphs, with spreadsheets, with animation….

The review section is rich with suggestions – a book, a website and a manipulative to teach numbers and number operations, what more would you want for those long periods of solitary splendour?

And the PullOut teaches you how to keep safe, by maintaining a constant distance between paths.

The teaching of mathematics often devolves to the teaching of algorithms and two things are very often ignored. The attitude of the teacher and the teacher herself. We feel this important enough to move the section Closing Bracket to Opening Bracket and to feature inspiring examples and actions of math teachers in the Closing Bracket section.

Make maths the new viral – with the July 2020 issue. As always, send in your feedback to AtRiA.editor@apu.edu.in and don't forget to check our FaceBook page AtRiuM.

We'd like to direct your attention to a Call for Articles for our new TeachSpace section on page 118.

Hoping to hear from you on all these platforms!

Lockdown with mathematics- stay safe, stay positive in the best possible way.

Sneha Titus
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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

The Opening Bracket . . .

In the context of a sudden switch to online teaching, forced on all of us by the pandemic, a colleague asks: “With children having access to the internet and to every bit of the subject matter that I have – problem sets, solution to problems, concept explainer videos, online tests – what is my role as a teacher?”

The question is profoundly important, and I hope that more teachers are asking it. It is actually a stand-alone question, relevant at all times. *What is the role of a teacher?* The current situation has merely focused a spotlight on it, thus demanding an immediate answer.

As a subject teacher, I must take my work seriously and ask, what is the best way of teaching this subject? What tasks and readings should I assign? What kinds of lectures should I give? What kinds of group activities should I design? What questions should I ask the student to ponder? How should I engage the student in discussion? How can I help the student go beyond the book? To what extent should I be ‘present’ in the classroom? In what way should I help the student who is struggling with the subject? What studies should I undertake so that I do not stagnate in my subject? As a subject teacher, I must ask all these questions and more – and I must ask them with feeling, in a spirit of honest inquiry. If I do not do this, then I am not doing justice to my work.

Is this all? It cannot be.

Surely, my work cannot be limited just to the teaching of the subject – no matter how important that subject, no matter how beautiful. My work must go beyond that. It must be that of educating the whole human being – helping the child to learn how to think, how to look, how to listen, and how to understand what one feels. It must be to help the child become aware of the hidden layers within oneself – the prejudices, the inherited identities, the tribal instincts and nationalistic urges, the competitive instincts, the violence, the addiction to power and aggression. It must also be to question tradition, to question existing norms, to question the book.

Surely, my work must be to help the child to grow up to be human, and to nurture a love of simplicity, of kindness, of beauty, and of truth.

The work of a teacher spans a vast area. Whether we are aware of it or not, we are shaping the future. Very few appreciate the vital importance of this work. Even if we do, very few of us give of our time and energy to explore the full nature of this work.

Carl Sagan once wrote, “We can judge our progress by the courage of our questions and the depth of our answers, our willingness to embrace what is true, rather than what feels good.” It is part of our work to ask difficult questions, to which we may not have any ready answers; questions that may bring about discomfort; questions that invite us to engage with one another, that encourage children to delve deeper. It is more relevant than ever to do so now, when there are so many falsehoods around us, in our country and elsewhere.

“Come, my friends, 'tis not too late to seek a newer world…” In that spirit, come, let us work together.

Shailesh Shirali
Chief Editor

Azim Premji University
At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Student Corner

Azim Premji University

A Publication of Azim Premji University
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TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

VIEWPOINT

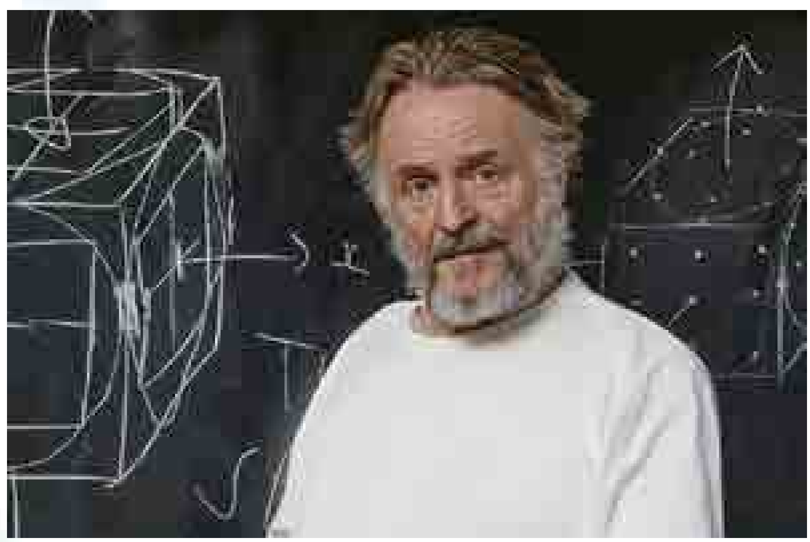
The Viewpoint section re-examines familiar mathematical concepts and pedagogical practices through different viewpoints.

Features

The Mathematical Artist of Play: A TRIBUTE TO JOHN HORTON CONWAY

26 DECEMBER 1937 – 11 APRIL 2020

R RAMANUJAM



John Horton Conway at Princeton University in 2009.
Credit: Princeton University, Office of Communications, Denise Applewhite.

A Maverick Mathematician

In these strange and troubled times, when the world is reeling under an infectious disease and a quarter million have died from Covid-19 (as of May 1, 2020), it is perhaps inappropriate to mourn the death of a single individual who died of complications caused by the coronavirus. However, for the world of mathematics, this individual was very special indeed. **John Conway** stood out among the community of mathematicians, a legend in his lifetime.

Keywords: Nim, surreal numbers, combinatorial games, Game of Life, sphere packing, knot theory, Leech lattice, Conway Group, mathematician

Stephen Miller, a mathematician at Rutgers University, said: "Every top mathematician was in awe of his strength. People said he was the only mathematician who could do things with his own bare hands." Conway liked to start from first principles, with very few concepts and build mathematical edifices. He loved to play games, preferably silly children's games, all day long. He would spend whole summers going from one mathematics camp to another, one for middle school children here and one for teenage students, and in all of them, he would play games with children, pose and solve puzzles. He would carry all sorts of things with him: decks of cards, dice, ropes, coins, coat hangers, sometimes a Slinky, even a miniature toy bicycle. These were all props he would use for explaining ideas, though Conway insisted that they were more for his own amusement.

Unlike most other mathematicians, Conway was always obsessed with seemingly trivial concerns. He would be constantly factoring large numbers in his head. He could recite more than a 1000 digits of π from memory. He developed algorithms you could use to calculate the day of the week for any given date (in your head!), to count the number of steps while you climb stairs without actually counting, . . . But importantly, Conway claimed that it was such thinking that led to his mathematical research, and his colleagues at Princeton agree.

If anyone has ever approached all mathematics in a spirit of play, it is Conway. He also led the world in the mathematics of playing games.

Games as Mathematics

Consider *Nim*, a two player game, where there are a number of heaps of sticks between them. It is a turn based game. A player chooses one of the heaps, removes some non-zero number of sticks from it, then the other player gets her turn. When it is a player's turn to move, and there are no sticks left, he loses; that is, the last one who makes a move wins. Here is the question.

Starting from any initial configuration of heaps, does one of the players have a *winning strategy*: a way to play in such a way that no matter what moves the other player makes, she is assured of a win?

How do we analyse such games? Suppose you are faced with the situation (1,1), when there are two heaps, each with one stick. This is a losing position for you, since no matter which heap you reduce (to empty), the opponent can move and then you have no move. Now, (2,2) is also losing since no matter what move you make, the opponent can copy that move and bring you either to (1,1) or to the empty configuration. Thus, in general, we can argue that the Nim position (m, n) is winning to a player iff $m \neq n$.

But this means that (m, m, n) is winning since you can remove the third heap entirely and present a losing position to the opponent. Similarly (m, m, n, n) is losing: we can consider it as $(m, m) + (n, n)$ or $(m, n) + (m, n)$, consisting of two subgames. Note that player II has a *copycat* strategy in the other subgame and hence cannot lose.

As it turns out, this operation + yields a **group** structure on these games, where every element is its own inverse. (Comment. It is known that such a group is necessarily abelian, i.e., commutative.) The mathematical study of such bipartisan games (where both players have identical moves at any configuration) leads to very interesting combinatorics and algebra. The game of Nim was solved by Bouton in 1902 [2]. For tasting the mathematical adventure of both playing such games and analysing them, there is no better introduction than the book *WinningWays for yourMathematical Plays* by Conway, Berlekamp and Guy [1].

Conway is credited with founding the area of *Combinatorial Game Theory*, a rich and beautiful subject of mathematical study. To give you a teaser, I have already mentioned that you can place an abelian group structure on games. There is a distinguished subgroup of games

John Horton Conway.
Credit: Thane Plambeck, https://www.flickr.com/ photos/thane/20366806/, CC BY 2.0, https://commons.wikimedia. org/w/index.php?curid=13076802

called *numbers* which can also be multiplied and which form a field: this field contains both the real numbers and the ordinal numbers. In fact, Conway's definition generalizes both Dedekind cuts and von Neumann ordinals. All Conway numbers can be interpreted as games which can actually be played in a natural way; in a sense, if a game is identified as a number, then it is "so well understood that it would be boring to actually play it!" Conway's theory is deeply satisfying from a theoretical point of view, and at the same time it has useful applications to specific games such as Go. There is a beautiful microcosmos of numbers and games which are infinitesimally close to zero, and ones which are infinitely large. The theory also contains the classical and complete Sprague-Grundy theory of impartial games.

To my taste, the slim volume *Games and Numbers* by Conway [3] is nothing less than a masterpiece of mathematics. I recall seeing it in the library during my graduate student days, spending an entire day reading it right there, and rushing out to get it photocopied. Donald Knuth based a mathematical novel on these numbers [7]. Knuth called these numbers *surreal*, because in this number field, every real number is surrounded by a whole lot of new numbers that lie closer to it than any other 'real' value does.

But to reiterate what was said earlier, playing games was intrinsically interesting to Conway, independent of all this algebraic structure. He created many games for people to play, especially for school children, and would spend enormous amounts of time on designing them. In fact, Conway is best known to the public for designing the **Game of Life**, a great boon to screen savers.

Game of Life

This is a very simple game. Take a sheet of grid paper, and keep a pencil and eraser with you. We have cells on the grid, each having eight neighbours. In this game, every cell can be *alive* or *dead* at any instant. There are (only!) two rules:

- A dead cell having exactly three live neighbours comes alive at the next instant; otherwise it stays dead.
- An alive cell that has two or three alive neighbours stays alive; else it dies.

Try out the evolution of the game starting from some random initial configurations. (You could do this using pencil and paper; or you could use the interactive website [9].)

Questions: If you start with any configuration of alive and dead cells, can we predict whether we would keep getting new configurations, or settle down to a specific configuration, or keep oscillating between some configurations? Pointing to a cell, can we figure out whether it will live for ever after some finite point in time? We can ask a variety of such questions. It turns out that there is no uniform algorithm to answer any of these questions. In fact, the Game of Life is exactly as powerful as the digital computer in a theoretical sense, and there is a variety of questions of this kind linking the game to computation theory and complexity theory.

In an interview in 2014 [4], Conway said he tinkered with the rules for "about 18 months of coffee times" before he arrived at such simplicity. He did not use any computers during this search either, he hand calculated the evolution of many configurations. This was typical of Conway, the search for extreme simplicity encapsulating almost universal capability, and doing it all 'in the head.'

Mathematics as Play

Conway was born in Liverpool, England and went to study in Cambridge on a scholarship. In the 1960s, he worked on *sphere packing*. Suppose that you want to fit as many circles as possible into a region of the Euclidean plane. How can one do this? Divide the plane into one big hexagonal grid and inscribe the largest possible circle inside each hexagon. The grid, called a hexagonal lattice, serves as an exact guide for the best way to pack circles in two-dimensional space. In the 1960s, John Leech came up with a similar lattice for the most efficient packing of 24-dimensional spheres in 24-dimensional space.

Conway decided to study the symmetry group of the Leech lattice. It is called the Conway Group now. This led him to study the properties of similar groups.

In a paper in 1979, Conway and Simon Norton conjectured a deep and surprising relationship between the properties of the so-called monster group and that of an object in number theory called the j-function. The paper was titled *Monstrous Moonshine*! The monster group is a collection of symmetries that appear in 196,883-dimensional space. A decade later, Borcherds proved the conjecture, which won him the Fields medal in 1998.

Another area of mathematics in which Conway made an amazing contribution was **knot theory**, a branch of topology. Knots can be thought of as closed loops of string. A fundamental problem in the area is that of knot equivalence: can one apply finitely many allowed operations to obtain one from another? Mathematicians have different kinds of tests they can apply that act as invariants: if applying them to a pair of knots leads to different knots, the pair was different. One such test is called the Alexander polynomial, which is effective but not unique: the same knot could give rise to different Alexander polynomials. Conway fixed this, leading to what is known as a *Conway polynomial*, a fundamental tool in knot theory now. Another interesting contribution of Conway was an arrangement of knots, akin to the periodic table, that makes their properties easy to study.

In collaboration with Kochen and Specker, Conway proved the so-called *Free Will Theorem* in quantum mechanics: in crude terms, it states that if you had the information about the states of every particle in the universe up to this point, you would not be able to predict what their states will be a second from now. (Stated in still cruder terms, it states that if human beings have free will, then so do all elementary particles.)

Overall, Conway was active in the theory of finite groups, knot theory, number theory, combinatorial game theory and coding theory, with some work also in geometry, geometric topology, algebra, analysis, algorithmics and theoretical physics. The vast range attests to Conway's ability to work on pretty much any mathematically posed question.

Conway was at Princeton University for the last quarter century, interacting intensively with students and colleagues. His biography by Siobhan Roberts [8] is a deeply inspiring story.

Extreme Elegance

Conway was known for his obsession for reducing proofs to the simplest terms. In 2014, Karamzadeh even argued that Conway's proof of Morley's theorem is the 'simplest possible' proof. Conway and Shipman [5] developed the idea of **Extreme proofs**. They consider 'values' we attach to proofs: brevity, generality, constructiveness, visuality, nonvisuality, 'surprise,' elementarity, and so on.

Indeed, because at any given time there are only finitely many known proofs, we may think of them as lying in a polyhedron (in our pictures, a polygon), and the value functions as linear functionals, as in optimization theory, so that any value function must be maximized at some vertex. We shall call the proofs at the vertices of this polygon the extreme proofs.

They go on to study 7 extremal proofs of the assertion that $\sqrt{2}$ is irrational, tabulating them and explaining the associated value functions.

Last Words

For me, a cherished memory is a 45-minute journey from Rutgers to Princeton, when Conway gave me a car ride. He mentioned three interesting problems in combinatorics during the ride and when we reached, showed me a card trick before I left. I listened to a lecture by Conway, which was on graph theory, where he walked in with a structure he had built with magnetic rods, and used it to pose problems that led to enumerative combinatorics. What he talked about in the lecture was original research, but he never published any of it. This was typical of John Conway: doing mathematics was always in a spirit of play, occasionally some theorems might be for publication.

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[3] John H. Conway, *On Numbers and Games* Academic Press, New York, 1976, Series: L.M.S. monographs, 6.

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[7] Donald E. Knuth, *Surreal Numbers*, (Reading, Massachusetts: Addison-Wesley, 1974), vi+119pp.

[8] Siobhan Roberts, *Genius at Play*, Bloomsbury USA, 2015.

[9] John Conway's Game of Life, https://playgameoflife.com/

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'A Tribute to John Horton Conway' – Glossary of terms

[1] **Nim:** a two-person mathematical game; see https://en.wikipedia.org/wiki/Nim

[2] **Dedekind cuts:** a method of constructing the real numbers from the rational numbers; see https://en.wikipedia.org/wiki/Dedekind_cut

[3] **von Neumann ordinals:** a way of defining and extending the notion of natural number; see https://en.wikipedia.org/wiki/Ordinal_number#Von_Neumann_definition_of_ordinals

[4] **Surreal numbers:** an extension of the notion of number, created by John Conway; see https://www.cut-the-knot.org/WhatIs/Infinity/SurrealNumbers.shtml

[5] **Sprague-Grundy theory of impartial games:** a way of studying two-person combinatorial games with perfect information; see https://en.wikipedia.org/wiki/Sprague%E2%80%93Grundy_theorem

[6] **John Conway's Game of Life:** a zero-player game invented by John Conway, in which the entire future evolution of a game is decided by the initial state; see https://en.wikipedia.org/wiki/Conway%27s_Game_of_Life

[7] **Sphere packing:** a study of the problem of packing spheres of identical size into a finite region, in an optimal manner; see https://en.wikipedia.org/wiki/Sphere_packing

[8] **Knot theory:** a study of mathematical knots; see https://en.wikipedia.org/wiki/Knot_theory

[9] **Leech lattice:** see https://en.wikipedia.org/wiki/Leech_lattice

[10] **Conway Group:** see https://en.wikipedia.org/wiki/Conway_group

[11] **Monstrous Moonshine:** see https://en.wikipedia.org/wiki/Monstrous_moonshine

What Can We Construct? – Part 1

SHAILESH SHIRALI

The geometers of ancient Greece invented a peculiar game for themselves, a game called ***Construction***, whose objective is to draw various geometric figures of interest. We are permitted to use just two instruments: an *unmarked straightedge* (a 'ruler'), and a *compass*. Using these, we can draw a straight line through any given pair of points, and we can draw a circle with any given point as centre and passing through any other given point. (Oh yes, we also possess a pencil and an eraser, please do not feel worried about that!)

The familiar school geometry box includes more than just these two items; it also has a marked ruler (to measure lengths), a protractor (to measure angles), two kinds of 'set squares' (to draw a right angle, and to draw a 60° angle; using these, we can also draw a line parallel to a given line and passing through a given point), and a divider. But we are not permitted to use these items if we wish to play the game as per the rules laid down by the Greeks. That is, the only instruments available are the unmarked straightedge and the compass. (Note that in the subject of engineering drawing, we are permitted the use of all of these instruments, and more. But that is another matter altogether.)

Using just these two instruments, how far can we go? What can we construct, and what can we not construct? This now becomes a mathematical question of considerable interest.

You may ask why we call it a 'game.' But it is just that, isn't it? – a game between you (i.e., the geometer) and the subject of geometry itself, played according to a fixed set of rules. If you are able to construct the required figure, you win; else, you lose; geometry wins and keeps its secrets!

Keywords: Geometry, instrument box, construction, compass, straightedge, ruler, divider, set square, regular pentagon

There are other areas where games of a similar sort have been devised. For example, there is the charming and delightful game of origami. Here too, we have a fixed set of rules (e.g., we must not use scissors), and we are required to perform all our operations and make all kinds of intricate objects within the boundaries set by these rules. Another example is that of solving polynomial equations using only the elementary operations (addition, subtraction, multiplication, division, taking of powers and roots). We shall have occasion to say more about these 'games' in later articles.

The central question. Let us state the central question of the construction game more clearly. On the plane, mark two points O and A; take the distance between them to be 1 unit. Draw the infinite straight line through O and A. Think of this as the x-axis of the coordinate plane, with O as the origin and A as the unit point, with coordinates $(1, 0)$. Define a set **CR** as follows ('CR' for compass and ruler): **CR** consists of all real numbers x such that we can locate the point P with coordinates $(x, 0)$, using only a compass and an unmarked straightedge. (Another way of putting this: we can construct a segment of length $|x|$ using only a compass and an unmarked straightedge.) We refer to **CR** as 'the set of constructible numbers.' It is clearly a subset of the set of real numbers. But what is the nature of this set? Which numbers lie in it, and which numbers do not? Is there a simple way of deciding membership of **CR**? We shall explore these questions in this article.

Properties of the Set of Constructible Numbers

We can use the compass repeatedly to lay out multiples of the unit length on the x-axis, as many as we wish. This tells us that every integer is part of **CR**.

There is a simple ruler-and-compass construction by means of which we can divide any given line segment into any given number (a positive integer) of equal parts. (It uses the properties of parallel lines and similar figures.) From this, it follows that every rational number is part of **CR**.

We now prove a series of properties of **CR**. The first result is obvious, but the others may come as a surprise.

Theorem 1 (Closure under addition and subtraction)**. CR** *is closed under both addition and subtraction. That is, if a and b are elements of* **CR**, *then so are $a + b$ and $a - b$.*

Proof. To streamline the writing, let us suppose that $a > 0$, $b > 0$ and $a > b$. It should be clear that if we can separately construct line segments having lengths a and b, then we can also construct line segments having lengths $a + b$ and $a - b$ (see Figure 1). Here, we use the fact that we can transfer distances using a compass and an unmarked straightedge. □

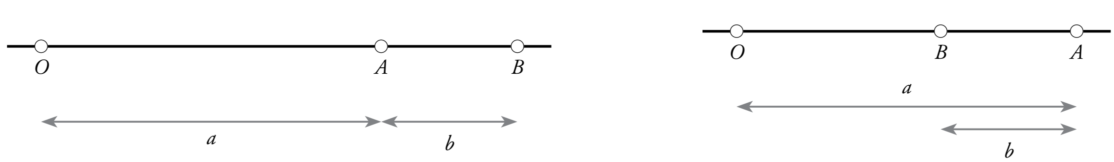
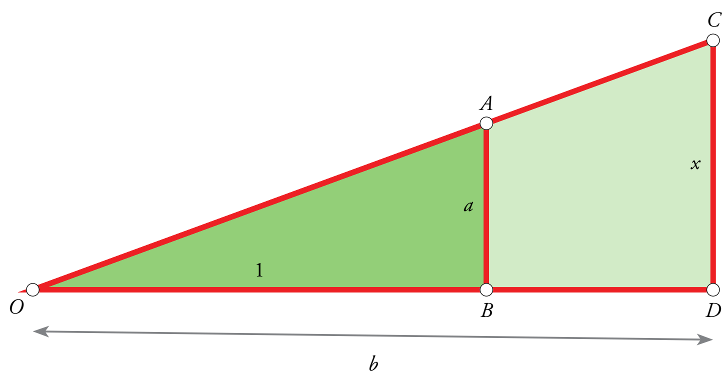


Figure 1.

Theorem 2 (Closure under multiplication). **CR** *is closed under multiplication. That is, if a and b are elements of* **CR**, *then so is ab.*

Proof. Let a, b be given positive numbers. We show two ways in which we can construct a line segment of length ab, starting with line segments of lengths a and b respectively.



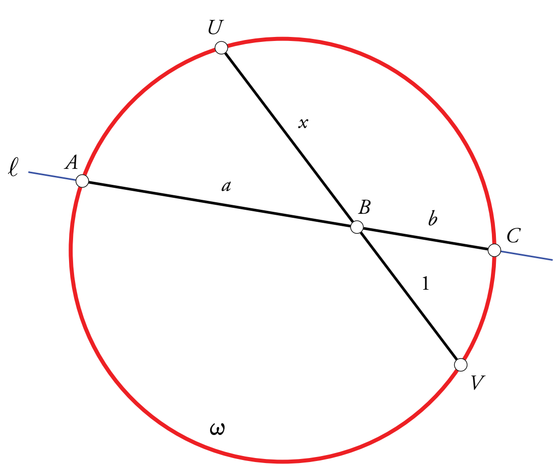
$a : 1 = x : b,$

$\therefore\ x = ab$

Figure 2. Here O, A, C are collinear, as are O, B, D, and $CD \parallel AB$.

The first method (see Figure 2) uses the fact that the number x satisfying the equation $a : 1 = x : b$ is $x = ab$. We use a pair of similar triangles to obtain the solution. Figure 2, which should be self-explanatory, gives the details. (In the figure, we suppose that $b > 1$; but that is obviously not a restriction. We have merely taken it so for convenience.)

The second method is based on the intersecting chords theorem of circle geometry. (Of course, here too we draw upon the properties of similar triangles.) Figure 3 gives the details.



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = b$. Locate a third point V, not on ℓ, such that $BV = 1$ unit (as defined previously). Draw the circle ω passing through points A, C, V. Extend segment VB beyond B till it meets the circle again at U. Let $UB = x$ units.

Figure 3. Using the intersecting chords theorem to do multiplication

Using the intersecting chords theorem, we get $a \cdot b = x \cdot 1$, and hence $x = ab$. □

Theorem 3 (Closure under division). **CR** *is closed under division by nonzero numbers. That is, if a and b are elements of* **CR**, *where $b \neq 0$, then a/b is an element of* **CR**.

Proof. We show two different ways (analogous to the above) by which to do the division.

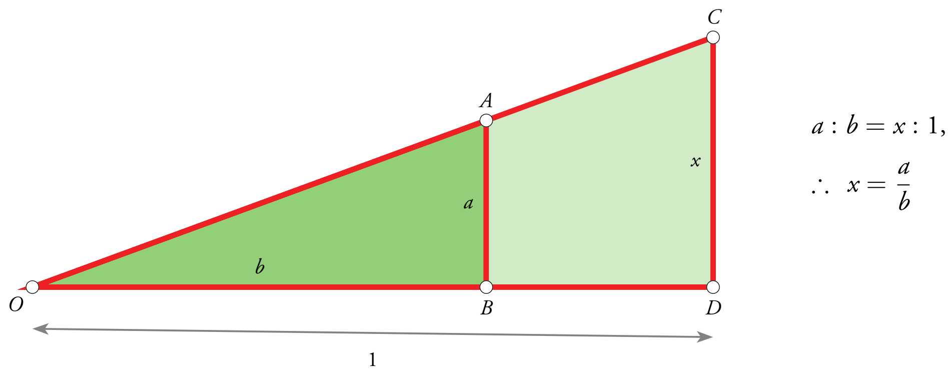
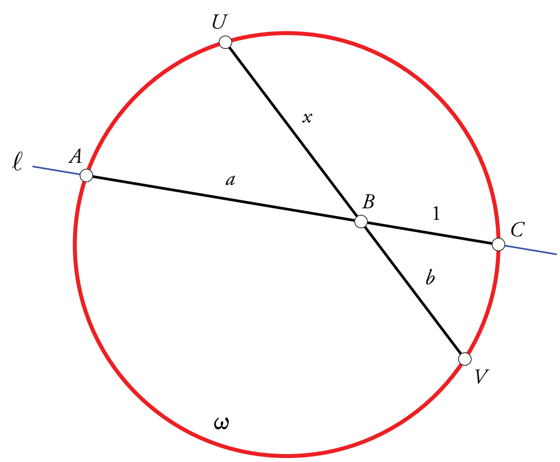


Figure 4. Here O, A, C are collinear, as are O, B, D, and $CD \parallel AB$.

Figure 4 shows the first way, using a pair of similar triangles. The figure has been drawn under the supposition that $b < 1$; but that is only for convenience and is not a restriction.

Figure 5 shows the second way, using the intersecting chords theorem.



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = 1$ unit (as defined previously). Locate a third point V, not on ℓ, such that $BV = b$. Draw the circle ω passing through points A, C, V. Extend segment VB beyond B till it meets the circle again at U. Let $UB = x$ units.

Figure 5. Using the intersecting chords theorem to do division

Using the intersecting chords theorem, we get $a \cdot 1 = x \cdot b$, and hence $x = \dfrac{a}{b}$. □

Field structure. The fact that the set of constructible numbers is closed under addition, subtraction, multiplication and division by non-zero numbers should alert us to something highly significant. Any subset of the set of real numbers $\mathbb{R}$ that has at least one nonzero number and is closed under addition, subtraction, multiplication and division by nonzero numbers is an example of a **field**. Obviously, $\mathbb{R}$ itself is a field; but $\mathbb{R}$ has very many proper subsets of great interest which are also fields.

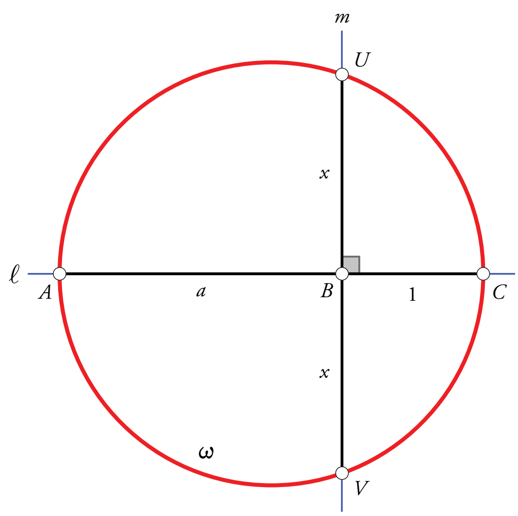
(Before proceeding, we must note that fields are abstract structures that are defined much more generally. We can have fields that have nothing to do with the real numbers. In this brief note, however, we only consider fields that are subsets of the set of real numbers.)

The simplest and most obvious example of a subset of $\mathbb{R}$ which is a field is the set of rational numbers. It is generally denoted by the symbol $\mathbb{Q}$. This field has the interesting property that it has no proper subset which is also a field. So, $\mathbb{Q}$ is the smallest possible field which is a subset of the set of real numbers $\mathbb{R}$. Or, stated more concisely: $\mathbb{Q}$ is the smallest possible subfield of $\mathbb{R}$.

This is easy to show. Let S be a subfield of $\mathbb{R}$; then S contains at least one non-zero number k. Since S is closed under division by nonzero numbers, it contains the number $k \div k$, i.e., it contains the number 1. Since S is closed under addition and subtraction, it contains every possible integer. Invoking again the fact that S is closed under division by nonzero numbers, we see that S contains every rational number. This means that $\mathbb{Q}$ is a subset of S. So there cannot be a subfield of $\mathbb{R}$ which is smaller than $\mathbb{Q}$.

Back to constructibility. We return to the study of the set of constructible numbers, **CR**. Using the above terminology, we see that **CR** is a subfield of $\mathbb{R}$. Is **CR** identical to $\mathbb{Q}$? The next result, which may come as a surprise, shows that **CR** is much larger than $\mathbb{Q}$.

Theorem 4 (Closure under the square root operation)**.** *The set of constructible numbers* **CR** *is closed under the square root operation. That is, if a is constructible, and $a > 0$, then $\sqrt{a}$ is constructible.*



Explanation. On a line ℓ, mark three points A, B, C, such that $AB = a$, $BC = 1$ unit. Draw the circle ω on AC as diameter. Draw a line m perpendicular to ℓ at B. Let m intersect the circle at points U, V. Then we have $UB = BV$. Let $UB = x$ units.

Figure 6.

Proof. See Figure 6. Using the intersecting chords theorem, we get $a \cdot 1 = x \cdot x$, and therefore that $x = \sqrt{a}$. □

All kinds of numbers .... Using Theorems 1, 2, 3, 4 in combination, we find that the set **CR** contains all kinds of numbers! Here is a small sample:

$$\sqrt{2}, \quad \sqrt{2+\sqrt{3}}, \quad \frac{\sqrt{3}+1}{\sqrt{5}+1}, \quad \frac{\sqrt{2+\sqrt{3+\sqrt{5+\sqrt{11}}}}}{1+\sqrt{5-\sqrt{2-\sqrt{3}}}},$$

and so on. It is hard to imagine any circumstance under which one may want to construct a segment whose length is the fourth number listed above! But the important point is that we **can** construct a segment with this length, if we wish to.

In a follow-up article, we shall consider some very famous construction problems which the Greeks had posed for themselves and which remained unsolved for an extraordinarily long period of time.

Appendix: Some Subfields of the Set of Real Numbers

Purely for the sake of completeness, we give here in the appendix some examples of subfields of $\mathbb{R}$.

Example 1: For our first example, we make use of the square root of 2. Let the set S be defined as follows

$$S = \left\{ a + b\sqrt{2} \ : \ a, b \in \mathbb{Q} \right\}.$$

That is, S consists of all numbers of the following form: a rational number + a rational multiple of $\sqrt{2}$. It is easy to see that S is closed under addition, subtraction, and multiplication. To see that it is also closed under division by nonzero numbers takes a little more effort. For this, we need to verify that if a and b are rational numbers, not both 0, then the following number

$$\frac{1}{a + b\sqrt{2}}$$

lies in S. To verify this, we use the old-fashioned technique of rationalisation:

$$\frac{1}{a + b\sqrt{2}} = \frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

The crucial observation now is that the denominator of the last expression cannot be 0. This is true because $\sqrt{2}$ is an irrational number. (If $a^2 - 2b^2 = 0$, then it would mean that $\sqrt{2} = a/b$, which is not possible since $\sqrt{2}$ is irrational.) So we have managed to express $1/(a + b\sqrt{2})$ in the form 'a rational number + a rational multiple of $\sqrt{2}$.' The claim that S is closed under division by nonzero numbers is thus proved. Hence S is a subfield of the field of real numbers, as claimed.

Example 2: This is similar to the first example, except that we use the square root of 3 rather than the square root of 2. Let the set T be defined as follows

$$T = \left\{ a + b\sqrt{3} \ : \ a, b \in \mathbb{Q} \right\}.$$

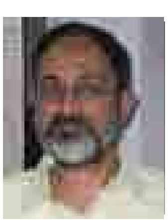
By going through the same steps as earlier, we may check that T is a subfield of the field of real numbers. The crucial point is that $\sqrt{3}$ is an irrational number. This ensures that we can express $1/(a + b\sqrt{3})$ in the form 'a rational number + a rational multiple of $\sqrt{3}$.'

Infinitely many such subfields can be constructed, using the square roots of other positive integers which are not perfect squares.

We will say more about this topic in a follow-up article.

References

1. Wikipedia, "Straightedge and compass construction," https://en.wikipedia.org/wiki/Straightedge_and_compass_construction.



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Continuing VIEWPOINT, where we re-examine familiar mathematics concepts and practices through different viewpoints. We encourage you to write in with your thoughts on the viewpoints expressed here. Send in your mail to AtRiA.editor@apu.edu.in.

VIEWPOINT

Working Together

JAMES METZ

"Manveen can paint a room in 3 hours and Allen can paint it in 6 hours. How much time will it take them to paint the room if they work together?"

Students often find such problems challenging. While using the standard computation $(3 \times 6)/(3 + 6) = 2$ does produce the correct answer immediately, it is unaccompanied by understanding. Perhaps we can lead students to understand such problems if we introduce the concept with whole numbers. To this end, suppose we have a machine that makes 2 items (for example, sweaters) in an hour and a second machine that makes 4 items in an hour and we ask, "How many items are made in one hour when both machines are running?" We can write the solution as: 2 items/hour + 4 items/hour = 6 items/hour. We are really only adding 2 items and 4 items to obtain 6 items and expressing this as an hourly output. We can then easily calculate how long it will take to make, say, 24 items (24 ÷ 6, or 4 hours).

Returning to the original problem, Manveen has an output of 1 room/3 hours, which can be expressed as (1/3 room)/hour, and likewise Allen has an output of (1/6 room)/hour, and just as we did with items, we add (1/3 room)/hour and (1/6 room)/hour to obtain a combined hourly output of (½ room)/hour. We then express this final fraction using a numerator of 1, i.e., as 1 room/2 hours, which means that 1 room can be painted in 2 hours when they are working together. We want a numerator of 1 because we want 1 complete task. Looking first at an example with an hourly output more than 1 can help students understand problems in which the hourly output is less than one.

What do you think?

Have you encountered challenges that students face in solving these problems through the traditional way? Is this better: mathematically and pedagogically?

Share your thoughts at AtRiA.editor@apu.edu.in

Keywords: Time and work, rate, fractions, completing a task

Area of Rectangle = Length × Breadth: Conversations with a 9 year old

RAKHI BANERJEE

Children are taught many kinds of measures in the primary grades, like length, weight, volume, money, time and finally area and perimeter (which is nothing but measurement of length). While for teaching length, weight and volume, efforts are made to give students an exposure to informal/ non-standard units of measurement (and thus some implicit understanding of what we may use to measure something), the other measurement ideas start with the standard units, conversions and formulae. Even when non-standard units are introduced, rarely is emphasis laid on explicating the principles which are routinely followed in measuring – like choosing a unit and iterating it repeatedly without leaving any gaps or overlaps. It may be assumed that since children already know about informal/ non-standard units for some aspects/ dimensions of measurement, they will now be ready for dealing with standard units for all kinds of measurement, including time and area. Also, measurement of time and area are more difficult ideas to teach and learn. "Time" is often reduced to reading and understanding the clock, without a focus on what is meant by time and measuring time. Similarly, area and perimeter are more about formulae than about understanding what area and perimeter are, how they are different or connected. The following piece describes my attempt to engage my 9-year old daughter (M) in thinking about area and perimeter. My goals are to illustrate some challenges we face while attempting to teach children these ideas, suggest some instructional ideas as well as give a peek into a child's thinking.

Keywords: Measurement, units, standard, non-standard, area, perimeter, formulas, mathematical conversations

The context

It all started with a visit of my family to a newly constructed house and a conversation about the carpet area of the house. As the adults discussed the carpet area, M asked us what square-feet meant? She was aware of quite a few measuring units, including units for length, weight, volume. She had recently studied perimeter of rectilinear and circular shapes also (whose unit was the same as of length) but could not guess what such a unit as "square-feet" could mean.

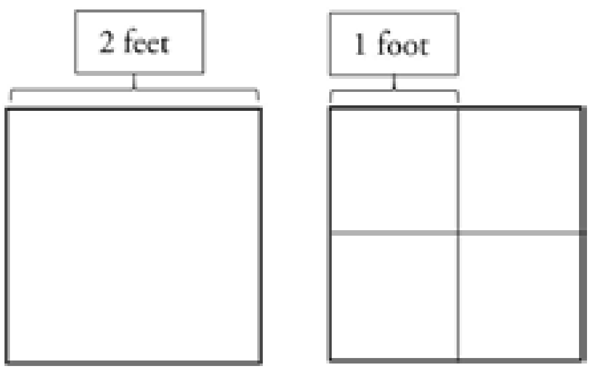


Figure 1: Schematic figure of tiles and partitioning

I quickly explained that each of the tiles on the floor was 2 feet long on each side and the space occupied by the entire square is 4 square feet. Further, I went on to say that if we were to make 4 equal parts of the square tile, so that each part still remained a square (which meant that the equal parts have to be created by one horizontal line and one vertical line as shown in Figure 1), then the area of the smaller square would be 1 square foot. In effect, I defined "square-foot" as the area of a square whose sides were 1 foot each. I told her that if she were to count the total number of squares of dimension 1 foot each on the floor of the house, she would find the carpet area of the house. We did not have the time to do that, so we moved on.

A couple of weeks later, she was introduced to area in school. She understood that area is the space inside the boundary of the shape. Rectangle and square were the only two shapes they were introduced to in this context. Also, they wrote the formulae for finding the area of a rectangle and a square. They were kept separate and she did not believe that there is any relation between rectangle and square; one had opposite sides equal and parallel while the other had all sides equal. On different occasions, I have tried to push her to think why square is also a rectangle and why the same formula can be used in the contexts of perimeter and area. I must admit that I have not been very successful in helping her see the square as a special case of rectangle, where all properties of rectangle remain, with an additional property that all sides are equal. She in fact counter-argues that if they were the same, then why were there two different formulae in both the contexts. Though the relation between square and rectangle and many other quadrilaterals will be explored in the years to come in the context of geometry, we start seeing the limitations of teaching concepts like area and perimeter using formulae as the basis.

Being a mathematics teacher, I got my act together and decided to teach M what area and perimeter of such rectilinear figures are. It was simple to deal with perimeter as she understood that perimeter is the "total boundary" which could be found by adding the lengths of the sides for a rectilinear figure; and in the case of a circular shape, one could use a thread around the boundary and measure the length of the thread using the ruler. Therefore, it was area that became my focus.

Introduction to measuring area

I started by revising how length, weight and volume are measured by using a variety of non-standard units, that she had tried at school or home. This helped us see the kinds of measure that we were measuring, the choice of unit and the principles underlying measurement. For example, when we measure length, we use any of the following non-standard units: digit, cubit, hand-span, foot-span, strips of paper repeatedly placed along the object till we have exhausted the expanse of the length. M remembered that repeated iteration of a strip of paper, without leaving any gaps, either completely fitted the object or sometimes needed to be folded in some ways (some fractional part) to quantify the measure (length in this case). We had done this task quite exhaustively and therefore she vividly remembered the piece of paper that she had used to measure different objects in her room – the bed, the study table, the chair, etc. She could connect this folding process to fractions also, so this was a good way to introduce her to the idea of scale. Sometimes these revelations lead to a spark and surprise in her eyes and I am thrilled with it as a teacher. This is not to say that these insights stay with her always.

This revision was useful in understanding that area is also a measure of the space inside the boundary of the shape and therefore we need to identify a unit for measuring it. What could be the unit was the next question. M did not seem to recollect the conversation we had had about the carpet area of the empty house a few days back. She started by saying that we could use strips of paper as a unit. Prima facie, agreeing to the suggestion, we discussed how the strip of paper would look. In her imagination the strip had the same length as of the rectangle which was iterated within a bigger rectangle (shown below, Fig 2). I asked her: what is the area of the strip? Guessing that my objection was to the size of the strip, she suggested that we can reduce the strip to half. The question of the area of this smaller strip bugged her as she realized no matter how small the strip was, we still needed to know the area of the strip. And she gave up briefly.

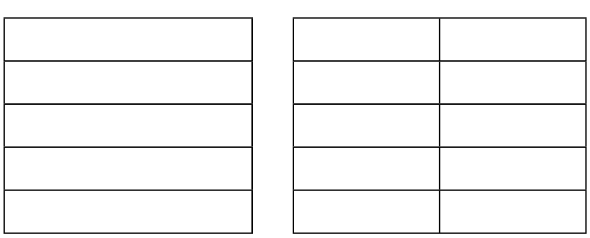


Figure 2

Leading to a choice of unit for measuring area

Before she could get out of this situation, I proposed what if instead of these strips, we use lines to fill the rectangle. She readily accepted this suggestion – she had found the thinnest possible strip of paper. I asked her how we would find the area then. She said "find the length of each of the line segments, and multiply by the number of lines!" This perhaps is closest to the formula of area of rectangle that we learn and in children's heads may be the rationale for why we multiply the length and the breadth of the rectangle to get the area. She of course realized that it was a daunting task, as there would be so many lines which will be required to fill the space inside a rectangle of a similar size. She was well aware of the way she had iterated the strip of paper to find the length of the object – there must not be any gaps.

This was an important point in this discussion and discovery, both for me and my daughter. The square unit that we use in the measurement of area is taken for granted. We need to identify and construct a unit for measurement. But before we could do that, I had to show the illegitimacy of using lines as the unit for measuring area. So, I asked her to tell me what a line is. She promptly gave me the definition that they had learnt in school – a line is straight which extends indefinitely in both directions. But the important Euclidean idea that line has no breadth but only one linear dimension is not emphasized in school so early. And therefore, the conflict between the concrete instantiation (physical manifestation) that we draw in notebooks and the ideal line is evident. The lines that we draw with our pencils/pens (however pointed) can fill the required space, a small breadth getting attributed to it as the tip of the graphite pencil scratches the surface of the paper but the "ideal" line will not. There was no possible way for me to proceed without explaining this aspect to her. She was amazed to realize this. I also pointed out in the same vein that the "ideal" point is dimensionless, no length, no depth, no height. But a mark on the paper, again with pen or pencil, gives the look that it has all of these.

So, we moved on from here to discuss if thread can be a unit of measure of area. Now she was quick to figure out the difficulty – that it indeed will fill the space but how are we to account for the thickness of the thread. I took her back to the strips of paper that she had initially suggested and we felt that was a better option, except for the fact that we needed some way to find the area of the strips as well. But in the meantime, we could place the strips inside the rectangle and state the area of the rectangle as (say in figure above) 10 rectangle units. (A bit weird!) we had some more conversations that day about what all shapes we could fill in the rectangle to measure the area, with the understanding that we will eventually need to find the area of each of these shapes also. Working with some materials like small split pulses, circular buttons, small squares and rectangles, she realized that iterations of some shapes (squares, rectangles) leave no space between them. Circular buttons left some space even when she used a smaller button to cover the gap between a few larger buttons. Split pulses could be arranged very close but still there were very small gaps. This was a good moment to say that the shapes must "tile", an idea that we explicitly explore in tessellation. Circles do not tile. Similarly, pentagons will not tile by themselves, unless we use some other shape together with it with specific dimensions. We can see lots of such examples on the footpaths. She quickly recalled a newly laid pavement with square tiles.

This conversation had moved us towards an understanding that not all shapes can be used as a unit of measuring area. It has to be rectangles, squares, and even right-angled triangles (if scalene, two of them will make a rectangle and if isosceles, two of them will make a square). By the end of that day, we had delimited the shapes that may serve as a unit of measuring area. I once again drew her attention to the square tile on the floor of the room and she measured to see how much of it was 1 foot. The tile was once again a square of side 2-feet, leading to the same situation of dividing the tile into four equal parts as discussed earlier. I now casually asked – what is the area of the small square with sides 1 foot? The response was baffling – "I know how to find the area with the formula but here how do I find?" With a little prodding and nudging she realized that the formula can be easily used here also, that what we learn in school is not just for mathematics notebooks but for the world outside as well. After she found the area of the square with unit length, she suddenly asked - "is it because we are arranging squares of unit length inside shapes that we call the unit of area square-foot/ square-centimetre?" And she also figured out that the number of squares laid inside the shape is the area of the shape. She had encountered a big revelation on that day but it required more substantive talk around squares now to conclude this learning of area. This session would have easily lasted more than an hour.

Connecting area and perimeter formulae to unit squares

After a few days, I started the conversation by making a rectangle with equal sized small square pieces. In the meantime, in school they had done lots of practice questions on applying the formulae for area and perimeter for rectangles and squares; there was also a firm belief that writing the correct formulae was very important before one solved the question. I had made a rectangle with dimension 6×8 with small square tiles in two colours (blue and orange) and her task was to find the perimeter and the area of the rectangle (Figure 3).

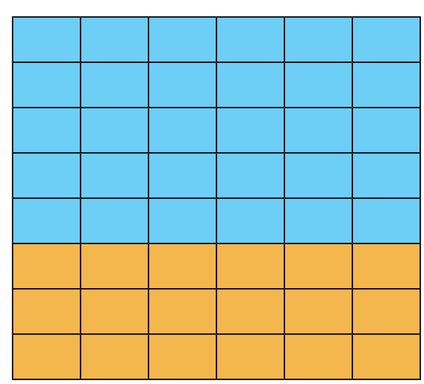


Figure 3

By now M had learnt, both through some work at school and at home, that for area one just needed to count the number of squares inside and for perimeter, she has to add-up the lengths of the sides of the given shape. We agreed to use the side of the squares as unit length and thereby the square itself having unit area. She found the area of the rectangle by counting the number of squares, in this case 48 unit squares. There was some hesitation in calling it square units and she would have preferred it as square metres, square centimetres or square feet. I intervened and told her that since we do not know the exact length of the squares, we can call it 1 unit, same as we had called the strip of paper for measuring length as 1 unit. She reluctantly agreed. Next, she found the perimeter by counting the sides of the unit squares, carefully counting each square in the boundary of the rectangle only once, which was found to be 24 units.

Having realized what M had done, I pushed her to use the formulae that she had learnt in school to see if she would get the same area and perimeter. She found the length and the breadth of the rectangle by counting the edges of the small squares that appeared along the length and breadth respectively. She correctly found these to be 6 and 8 respectively. The area formula for rectangle was l × b and putting these values gave her 48 unit squares. However, putting these values in the perimeter formula which was 2(l + b) led her to 28 units as the perimeter. Now she was confused. I left her with the task of figuring out which of the two results of perimeter was correct. She once again checked if she had counted the length and the breadth of the rectangle correctly by slowly moving her fingers over the squares in the boundary of the figure (see figure 3). Then she checked the perimeter by doing the same thing slowly so that she could understand the reason for the discrepancy. And of course in a little time she found that while she attempted to find the perimeter by counting the edges of the squares in the boundary for perimeter, she was counting the squares in the corner only once. However, for finding the length and the breadth of the rectangle, each square in the corner was counted twice, once towards the length and the other time towards the breadth. She resolved this conflict, indicating that the squares in the corner will need to be counted twice as they contributed to the length as well as to the breadth (See Figure 4 for the corner squares marked by a cross).

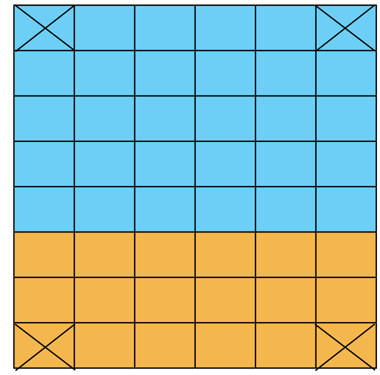


Figure 4

I further wanted to see if she could now see the reason for multiplying the length and the breadth of a rectangle to find the area of the rectangle. This was not easy. I kept prodding her to tell me some easier way of counting the number of squares in the rectangle. Unfortunately, she did not see the rectangle as being composed of 8 groups of 6. It took some time for her to recognize that she can count the squares by thinking it as 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6 or 6 × 8.

Exploring relationships between area and perimeter

Then we thought of rearranging the rectangle by removing some of the squares from the bottom to the side. We started doing this by trial and error but it was taking a lot of time and we were not sure how many rows we have to move to the columns to make it a rectangle with a different dimension. Eventually, we had to find some factors for 48. We found 12×4 as a good option and we arranged the new rectangle, as below (Figure 5). M once again found the area and perimeter of the new rectangle, this time without much difficulty. The area now remained the same but the perimeter changed to 32 units. She could see that since we have not changed the number of squares, but only rearranged them, the area should not change. But the fact that for the same area of the rectangle, its perimeter could change was surprising. She also thought that the difference between perimeter and area should lead to some pattern, probably a constant difference was in her minds, which was also falsified.

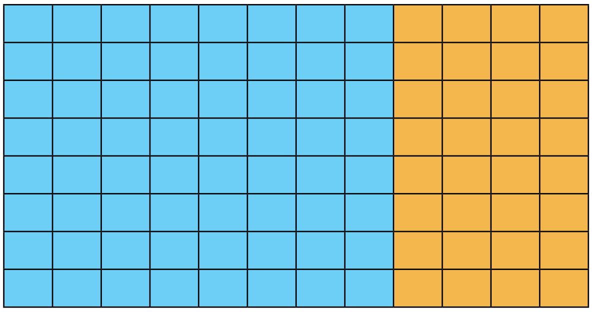


Figure 5

Then it was her turn to rearrange the squares to form a shape. Given the newly developing understanding of area and perimeter, she was no longer constrained to reorganize the small squares into a rectangle or square. She made an irregular shape, like the following (Figure 6), and she decided to remove one square, marked by Y. She was clear that the area had decreased by 1 as she had removed one square, but was bewildered by fact that the perimeter remained the same 32 units. Further, she removed another square marked by X, reducing the area by one more, but to her surprise the perimeter remained the same, once again. But this time, she was able to say why it was the case. She could articulate that the two side-lengths which were lost by removing X were compensated by the two new side-lengths which were opened-up by the removal.

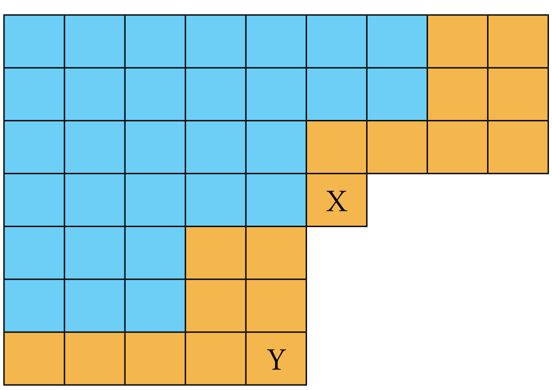
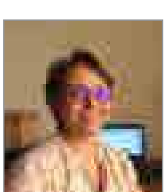


Figure 6

Final thoughts

This was the end of our conversation related to area and perimeter. Looking through the examples that were constructed during this conversation, we realized that we had constructed rectangles with the same area but different perimeters, irregular shapes with the same perimeter but different areas. M engaged and persevered with these tasks for a long time and tried to resolve the conflicts that emerged during the conversation.

However, there are still a lot of things which are unclear in the instructional sequence that I ended up developing as well as in M's understanding. Why did I not pursue seriously the idea of rectangular strips? Why did I distract her from using rectangles for measuring area and instead introduced my geometric ideas? What will happen if I filled the entire space using rectangles? What am I gaining by using a square of unit length? The fact is that we are looking for generalizable ideas, a core of mathematical thinking. When we use unit squares for measurement of area, the area of the square itself is 1 square-unit, much like 1 unit (or centimetre or metre) for measuring length, which has its own advantages. We need to find a unit which can help us measure length and breadth of the shape in a coordinated fashion and 1 square-unit helps us do that, and a rectangle-unit does not help us do that. We can now use this unit for measuring areas of different kinds of shapes and even in cases where the lengths are integers as well as fractions (will need to use ideas of fractions and decimals for this). I am not sure if M understands this and I did not give her enough time to explore this. I did try to talk to her about this after a few more days and showed her how the rectangle itself can be used to mark a square but it did not excite her much. So, I will perhaps postpone this to next year.



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An Angle Bisector in a Right Triangle

VICTOR OXMAN &
MOSHE STUPEL

A **Proof Without Words** is a way to get students to speak without the teacher verbalising. It is important to get them to describe what they see and to scaffold their reasoning with leading questions if they have trouble moving to a conclusion.

Given a right triangle ABC ($\angle B$ = 90°) with $\angle A$ = 60°. Point M lies on BC so that BM = 0.5 MC (Figure 1). Prove that AM is an angle bisector in triangle ABC.

Answer the following questions to reason why AM bisects angle A.

- What kind of triangle is ACA'?
- How is the line CB related to the line AA'?
- What is the point M called in relation to the triangle ACA'?
- How is M related to the vertices of the triangle ACA'?
- How is M related to the angles of the triangle ACA'?
- Would this hold if angle A was not equal to 60 degrees?

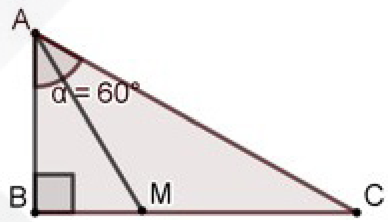


Figure 1

PWW

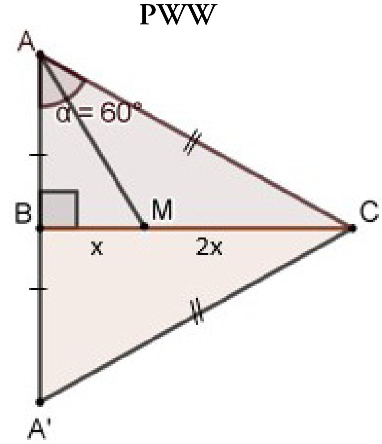


Figure 2

Victor Oxman, Western Galilee College, Israel

Moshe Stupel, Gordon Education College and Shaanan – The Religious Education College, Haifa, Israel.

The Approach to Teaching Fractions: Misconceptions and More

ARDDHENDU SHEKHAR DASH

Fractions have long been earmarked as a danger zone for both students and teachers – while one needs to tread carefully here, the topic should not be shied away from or treated with so much caution that students tend to handle it with reservation. Misconceptions are a natural stage of conceptual development and should not be viewed as an undesirable occurrence. What is important is that the teacher is aware of them and addresses them to the extent possible.

A word of caution: there is always scope for a student to over generalise or misapply a rule. A teacher may not be able to prevent all such misconceptions as they can be hidden from view. The focus needs to be on exposing children to carefully chosen examples. What is of value is helping children to explain their thinking and discussing their thinking with their peers and teachers, which in turn leads to a growth in understanding.

In the concept of fraction, there are some misconnects observed like - each part of the whole must be congruent, the shapes should be symmetrical in case of halves, the objects in the set model must be of equal shape and size. These misconceptions can be avoided if students are allowed to explore concepts through the use of objects/shapes, use of various models in classroom teaching, use of various types of examples other than examples of the textbook, connecting the concepts with real-life situations and building discussion among the students in the process of teaching.

The focus of this article is on identifying possible misconceptions that students can have in their understanding of fractions, analyzing the reason for such misconceptions and suggesting

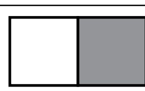
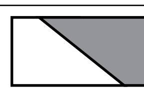
Keywords: Fractions, modeling, misconceptions, pedagogy

ways to plan teaching for a better conceptual understanding of fractions. The process of identifying misconceptions is based on the analysis of data from the work of the Azim Premji Foundation with teachers and students of class 5. Most of our primary school curriculum uses the examples, illustrations, and activities related to the 'Area Model' and 'Set Model' in teaching fraction (details of which are given in the latter part of the article). I had prepared a worksheet with some direct questions based on the textbook content and some higher order thinking questions. I shared these with the students and then discussed their responses with them as well as their teachers. In this article, I am highlighting the responses to some questions. I hope this will help teachers to know more about how children understand this topic and to revisit and, perhaps, bring to surface and address gaps in their classroom practice.

Area Model

Students were shown representations of different fractions using this model and asked if the fraction matched with the representation. The responses of children to the questions and discussion on these questions could be divided into four cases.

Case 1 (Symmetry):

	 **Fig 1**	 **Fig 2**
Children's response	The shaded part in figure 1 represents ½ of the whole.	The shaded part in figure 2 does not represent ½ of the whole.

During the interaction regarding Figure 2, the children were asked to cut the figure into two parts along the line and place one part on top of the other. The students then said that both parts have the same area but if we fold along the line then both parts will not coincide completely on each other, unlike in the case of Figure 1.

It was clear that students understood the importance of equal area of each part but, along the way, had developed the notion that the shape must be symmetrical along the line of division of the shape. This symmetry issue is only limited to the representation of ½.

During my interaction with teachers regarding this, it was observed that the figures used to teach the concept of ½ using the area model usually had a straight line segment dividing the whole into two equal parts horizontally or vertically (see Figure 3). Figures in which a shape could be divided into half using slanting or curved lines as shown in Figure 4 were hardly ever used.

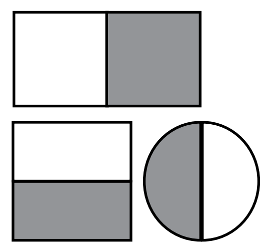


Fig 3

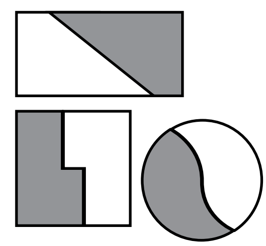


Fig 4

Another observation was that the state textbook has not focused on providing the opportunity for students to divide a shape into some given number of equal parts in different ways. They only allow children to identify the correct divisions in the given figure. Also, the paper folding activity in teaching equal division or teaching fractions could lead to such misconception, as usually, in paper folding, we use the concept of line symmetry. Note: The students are exposed to both line symmetry and rotational symmetry in class 5.

Case 2 (Congruent Shape): While students agreed that the shaded part in Figure 5 represented 1/8 of the whole, they felt that the shaded part in Figure 6 did not. When taken through the step by step process of dividing the shape of Figure 6 into equal parts they agreed that all parts have equal area but the shape of each part is different. It became apparent that they had developed the notion that each part of the whole would be of the same size and the same shape.

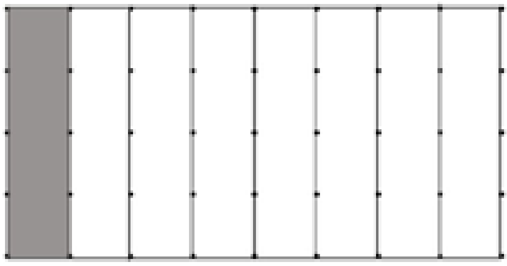


Fig 5

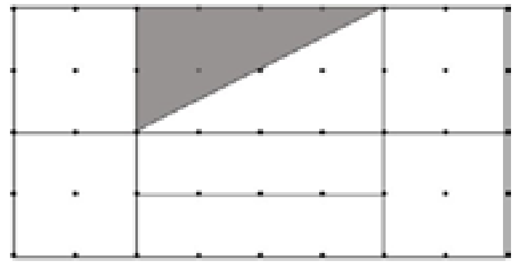


Fig 6

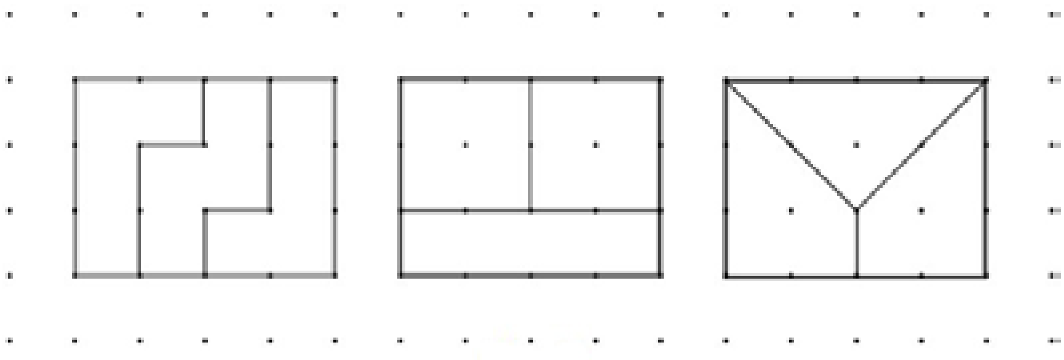


Fig 7

While interacting with teachers regarding this, it became clear that some of them also felt the same. Unfortunately, those who agreed that Figure 6 also showed 1/8 had neither used such examples in their classroom teaching nor come across them in the textbook.

One of the best ways to deal with such issues is by using a square dot sheet for dividing a shape into an equal number of parts in different ways. In Figure 7, a rectangular shape is divided into three equal parts in different ways. We can provide such square dot sheets and ask students to explore different ways of dividing a shape into equal parts.

This square dot sheet helps in visualizing the dimension of each part without any measuring instrument and also in finding the area of each part by just counting the square blocks or parts of square blocks.

Case 3: When the figure representing the whole is not completely divided into equal parts, children get confused. In Figure 8, most of the students, when asked for the fraction representing the shaded portion, gave the answer 2/9. They directly counted the total number of parts as 9 with 2 of the 9 parts shaded. They knew that equal division is important in fraction. However, applying the concept of equal division in such cases was challenging for students. We observed that they did not find it easy to divide the whole into equal parts, even though the lower half of the whole was divided into six equal parts.

When we interacted with the teachers, we pointed out that their textbook had a few simple examples of such type but that they needed to practise such types of questions in the classroom. We recommended that they highlight that in all cases, before writing the fraction, students should first do a check if the whole had been divided into equal parts.

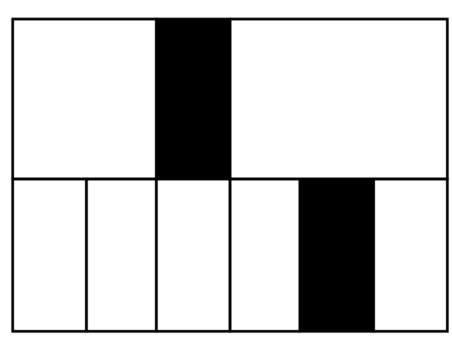


Fig 8

Case 4: At first sight, students did not agree that the shaded part in the given Figure 9 represented 4/6 of the whole. Their response was that it is 2/3.

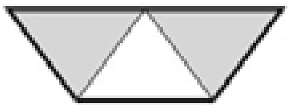


Fig 9

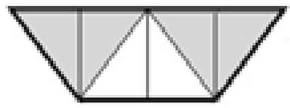


Fig 10

When, as shown in Figure 10, we divided each triangle further into two equal parts, they could now see the shaded part as 4/6 of the whole. But some of them were a little confused with the two values of the same representation as they faced difficulty in connecting the concept of equivalent fractions in this problem. The reason could be lack of exposure in the visualization of equivalent fractions in their regular classroom teaching and such types of problems could have been omitted from their practice.

Set Model

Again, students were shown representations of different fractions using this model and asked if the fraction matched with the representation. The responses of children to the questions and discussion on these questions could again be divided into four cases.

Case 1 (Scaling or Equivalent Fraction): Students do not agree that the encircled part represents 2/7 of the set; their answer was 4/14 (refer Figure 11), as the number of objects in the set is 14 and the objects encircled is 4.

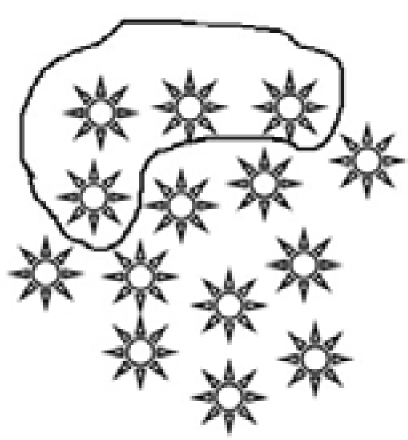


Fig 11

They faced difficulty in connecting the concept of equivalent fractions and the teachers also agreed that they hardly ever used the set model to teach equivalent fractions.

One way of addressing such types of questions is by asking the children how we can make the 14 objects in the set to be 7? One way could be by pairing objects in the set. (See Figure 12.) The total number of pairs is 7 and the number of pairs in the encircled part of the objects is 2. Now students can understand that the fraction of the encircled pair of objects is 2/7 of the pairs of objects in the set.

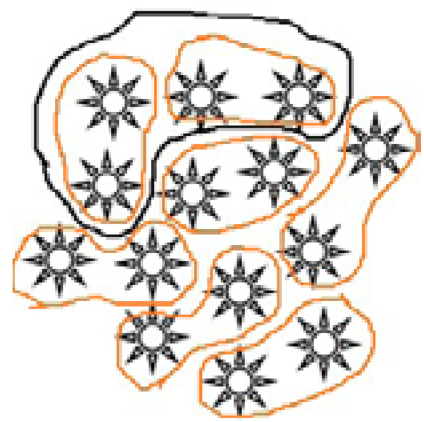


Fig 12

Case 2 (Size of objects in the set): Students do not agree that the encircled part of the marbles is 1/3 of the set of marbles (Figure 13). The reason was that the marbles in the set were of different sizes. They relate this question with the area model, where the whole is divided into parts of equal size. During the discussion, we used the example – suppose half of the students in your class are absent (the total number of students in that class was 24), then their response was 12 students were absent. When asked how they knew this, they replied that there are 24 students in the class and half of the students of the class are absent, so 24 ÷ 2 is 12. Then my question was whether all the children in their class were of equal height, weight, color. Their answer was that all students are not of the same height, weight and color. Then we discussed that in representing part of some objects from a collection, it is not necessary that all the objects in the set/collection be of the same shape, size, color. Only the number of objects is important in this case.

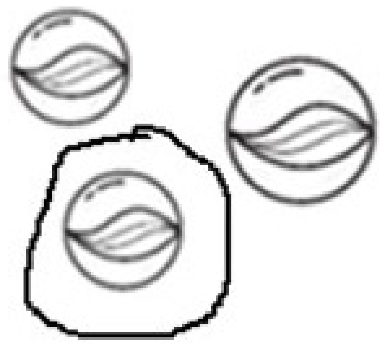


Fig 13

Another way could be to place 3 marbles (we can take more than 3) of different sizes as in Figure 13 and ask children how many marbles are there in the collection. Ask them to separate 1/3 of the marbles, then observe the response of students and if they could separate one marble, then discuss why they separated one marble and discuss the importance of number (not size) in this case. If they are confused with the size, then discuss the real-life situation as discussed above, of the absence of students in their classroom.

During the interaction with teachers, it was clear that they shared the same doubt regarding the size of the objects in the set. In the textbook, all the problems had sets of objects of the same shape and size.

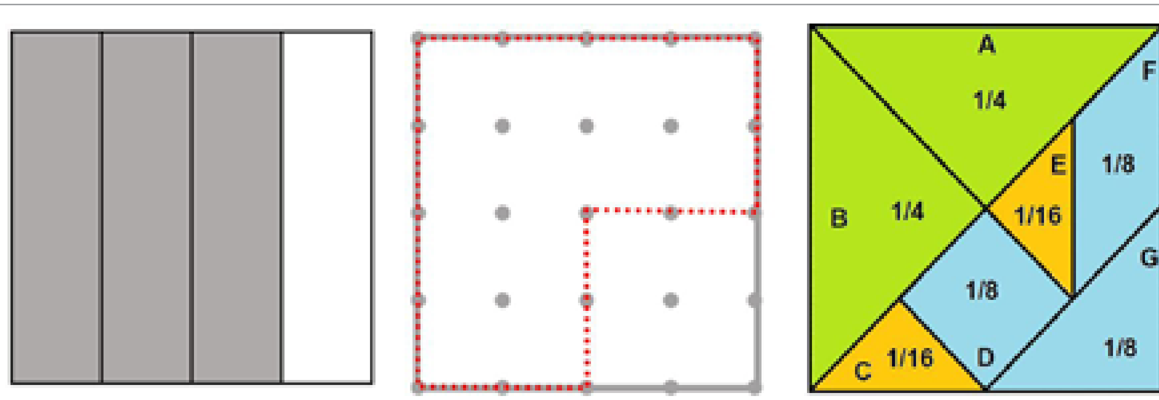
It is important to understand the characteristics of both the models and the suggested process to plan the use of these models in classroom teaching.

Characteristics of the Area Model and Set Model

Area Model: This is the simplest model of fractions and is widely used in textbooks and classroom teaching. As compared to the line model or volume model, the area model requires a two-dimensional plane to explore and most of the objects available with us such as the surface of the textbook, board, page of notebook, etc., are two dimensional and also easy to operate on. In this model of fraction, the whole is determined by the area of a defined region. Each part of the whole should be of the same area, but not necessarily the same shape. In this model, the fraction indicates the covered part of the whole unit of area.

For example, as in Figures 5 and 6 we can find that the whole is the area covered by a rectangular shape and the whole is divided into eight equal parts. Here equal parts mean the area of each part is the same. But, if we compare both the figures, we see that in the former, each part has the same shape whereas in the latter they are not all of the same. The shaded part in both the figures represents 1/8 of the whole unit.

While planning to teach, in the initial discussion, we should divide the whole into parts of equal area with the same shape and then we will work with students having figures with the same area and different shapes. As we know, before teaching fraction, we should allow students to explore different ways of dividing a shape into some number of equal parts.



Shared Part The area enclosed by red dots Area of (A + B + F + G)

Fig 14

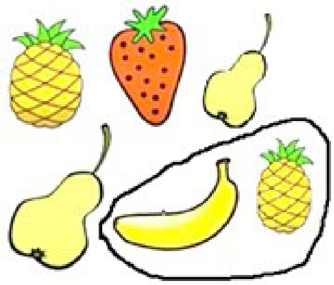


Fig 15

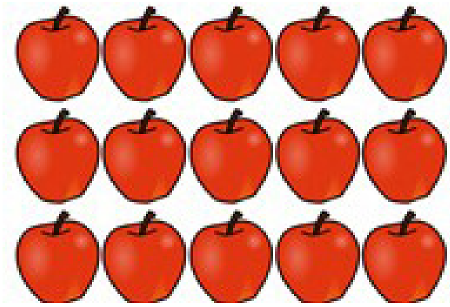


Fig 16

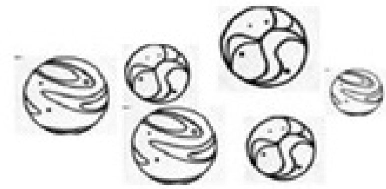


Fig 17

In this area model of teaching fraction, we should use figures, grid papers, paper folding activity, tangrams, etc. In Figure 14, 3/4 is represented by using a square, grid paper, and tangram.

Set Model: This is the model of fraction, in which a set is defined as a collection of well-defined objects, and a part is defined as a certain number of these objects, and the fraction indicates the number of objects in the subset of the defined set of objects.

For example, as in Figure 15, the set is the collection of six fruits, and the enclosed fruits are 2/6 or 1/3 of the set. Here we are considering the number of fruits only, not the size or color or shape of the objects.

In this model of fraction, the objects in the set can be arranged in an array (as in Figure 16) or arranged randomly (as in Figure 17).

Teaching Approach: Based on the above misconceptions observed among students we can plan our teaching as below –

- Before teaching fraction, we should work on equal divisions. With identifying the equal division in shapes, we should also ask students to divide the shapes into some number of equal parts in different ways.
- Use different types of examples: In teaching both the models use a variety of examples so that students can get exposure and develop the correct concepts. For example, to introduce fraction using the set model we can use different examples of sets having objects of a homogeneous group, different examples of sets having objects of a heterogeneous group and real-life examples: the fraction of children of class 5 dressed in red, etc. This will help students in developing the concepts correctly.
- Always initiate the discussion from simple examples and concepts based on the level of students. For example, we can start with the area model before set models. In the area model, we can start with parts having the same shape and then of different shapes.
- We can use shapes, dot sheet papers, paper folding activities, tangram activities and real-life examples in teaching fraction.

Mindful planning will yield rich results and it is hoped that this article will help you do just that.

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Mathematical Doodling using the what-if-not approach

HANEET GANDHI & NEHA VERMA

There are many ways of generating new problems. This paper proposes one of them, the "what-if-not" approach. The write-up shares how this approach was used to do explorations and create new problems. Not all the generated problems could be solved but the experience was indeed enriching and overwhelming.

Problem-solving, which has always been an important part of learning mathematics, received considerable attention after it was recognised as a route for promoting mathematical thinking in the Position Paper on Teaching of Mathematics (NCERT, 2005). For problem-solving to flourish in its true spirit, two ingredients are essential: adequate knowledge and skill to solve the problems and the acumen to generate good meaningful problems.

How do we pose new problems and what must be done to generate them? Often, teachers feel that problem solving and problem posing are "out-of-the-syllabus" activities and regard them as 'extra' work. We propose problem posing as an act of extension to the existing textbook problems so as to let children and teachers build a deeper connection with the textbook. In this article, we illustrate an approach that can be adopted for generating new problems from the existing textbook problems. We are suggesting a way through which students can be involved in making and solving many problems generated from the basic problem in the textbook. They get to identify the underlying conditions that define a particular problem and then change these conditions one-by-one to create more problems. We believe that in this way, students get tempted to challenge textbook problems and feel a desire to know new concepts.

We present the 'what-if-not' approach that can be used for generating problems from the existing ones. The 'what-if-not' strategy opens avenues for challenging, observing, creating new situations and delving into newer ideas. When people create their own problems they also get persuaded to solve them, thus begins

Keywords: Problem solving, problem generation, constraints, playing with math

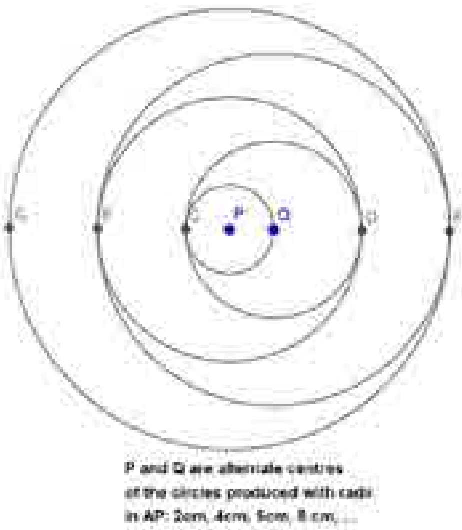


Figure 1a

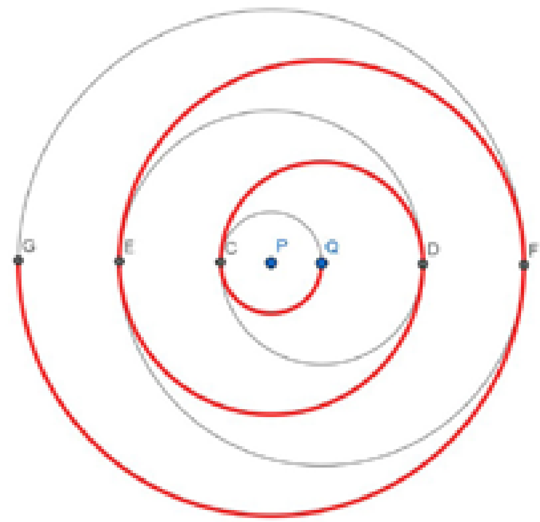


Figure 1b

a personal journey of mathematical thinking. People look out for patterns, make conjectures, deduce logic and may propose generalisations and even proofs.

The what-if-not approach

The 'what-if-not' approach proposed by Brown and Walter (1972) is based on identifying the key attributes of a problem, making a note of the attributes, modifying or altering each attribute one at a time to make new problems; in turn, also finding newer ways of solving them. Each alteration from the original problem offers scope for generation of a new problem.

It works like this. Every problem has some conditional statements. For example, the statement, *The product of two consecutive positive integers is …* rests on three conditions:

Condition 1: The numbers are positive integers

Condition 2: The numbers are consecutive

Condition 3: The numbers undergo an operation of multiplication.

In any mathematical learning act, recognising these conditions is an essential step. The mathematical journey begins when these conditions are challenged.

That is, What if:

- The first condition is changed? The selected numbers are not positive integers, will the result be any different?
- Condition 2 is changed? What would happen if the positive integers differ by two?
- Only Condition 3 is altered? Instead of multiplying, some other operation is performed?

How would the results get affected? We used the 'what-if-not' approach on a problem from the Class X, NCERT textbook.

A spiral is made up of successive semi-circles, with centres alternately at P and Q, starting with centre at P, of radii 2 cm, 4cm, 6 cm,… as shown in the figure. What is the total length of such a spiral made of 13 consecutive semi-circles? (Class X, NCERT, 2006)

The above NCERT problem was attempted on GeoGebra. Taking P as centre, a circle C1 of radius 2 cm was created. Then, the next circle, C2 was created taking point Q as centre and radius 4cm. Circles C3, C4 and henceforth were created by alternating the centres P and Q and increasing the radii each time by 2cm (Figure 1a).

From the network of the circles created, a spiral, marked in red, emerges on joining the points of contact of the circles. The first leg of the spiral is the semi-circle of the first circle C1, from Point Q to C. The next leg of the spiral emerges from the point of contact of Circle C1 and C2, at point C. We traced the semi-circle of circle C2 from point C to point D. Continuing this process, the points of contact of two consecutive circles serve as the emerging points for the on-shoot of the next leg of the spiral (Figure 1b).

There are three essential conditions which led to the semi-circle spiral:

Condition 1: The radii of the circles are in Arithmetic Progression. In the given problem, the radius of the first circle is 2 cm and that of each subsequent circle increases by 2 cm. In terms of the conventional nomenclature used to represent Arithmetic Progressions, we can say that the initial term ('a') and the common difference ('d') are the same.

Condition 2: The spiral is made up of successive semi-circles.

Condition 3: The centres of the circles alternate.

The "what-if-not" approach rests on altering the underlying conditions of a problem one by one. The conditions can be modified and/or negated one at a time. Then, based on each alteration, attempts to solve the new problem are done.

We demonstrate the work done by us using this approach, with the help of a dynamic software (GeoGebra). Each of the above conditions was altered to capture a new view. New situations were created and new patterns emerged. However, we confess, we were not successful in finding solutions to our created tasks. Nevertheless, the attempts gave us meaningful insights into what it really means to be a mathematician. Thus, we add a caveat. At some times, you may find it difficult to solve the newly generated problem, but you would be happy to have created another problem. There could also be times when you may not be able to pose another new problem but remember that that is learning in itself. The idea is to generate more problems in a connected way; and so we present our journey.

Altering only Condition 1: What if, the initial radius and the difference between the consecutive radii are not the same (if we have an AP in which 'a' is not equal to 'd') Is it possible to get semicircular spirals?

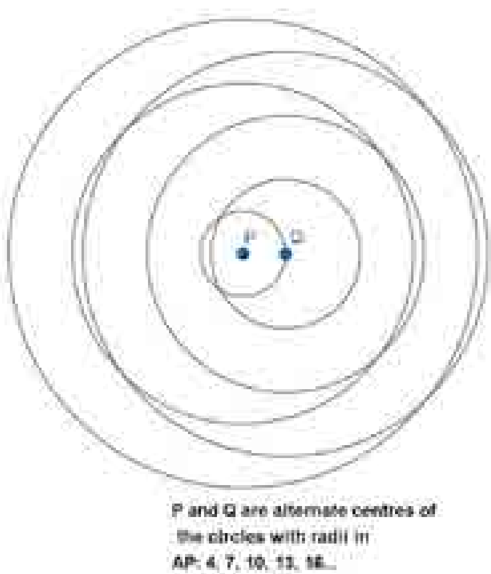


Figure 2a

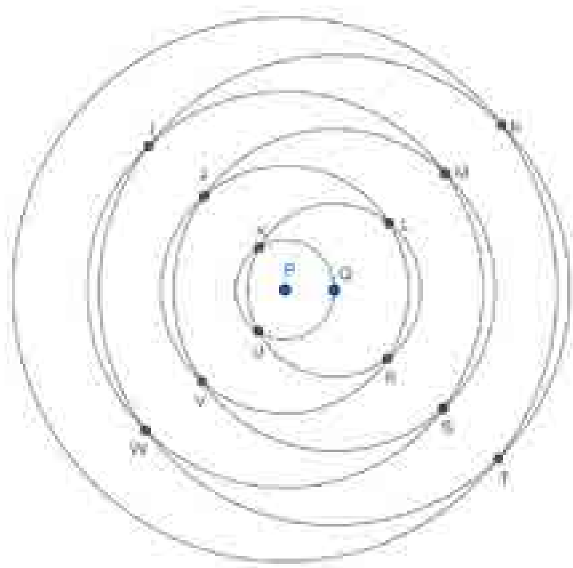


Figure 2b

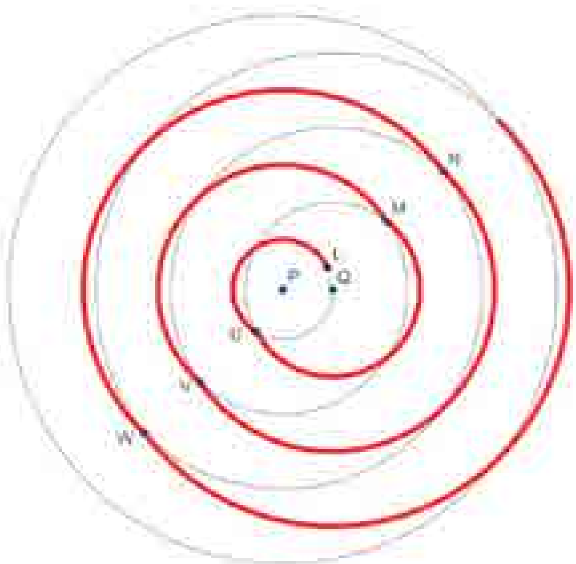


Figure 3a

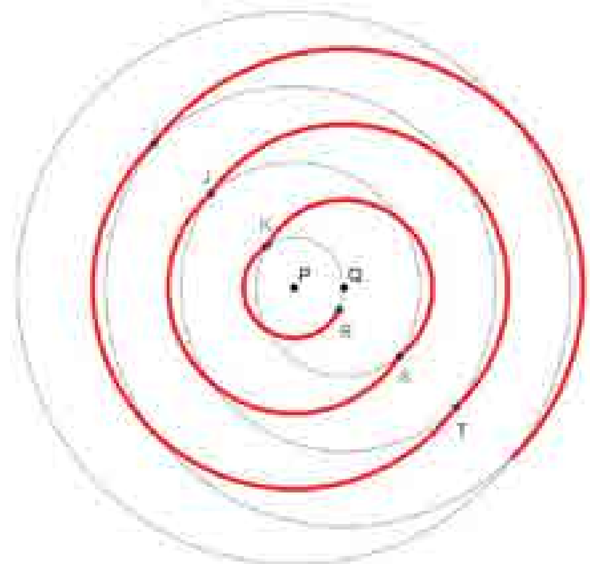


Figure 3b

Two sub-conditions emerge on altering Condition 1:

Sub-condition 1.1: The radius of the first circle is greater than the difference between the radii of the subsequent circles. That is, 'a' is greater than the common difference, 'd'. a > d

Sub-condition 1.2: The radius of the first circle is less than the difference between the radii of the subsequent circles. That is, 'a' is less than the common difference, 'd'. a < d.

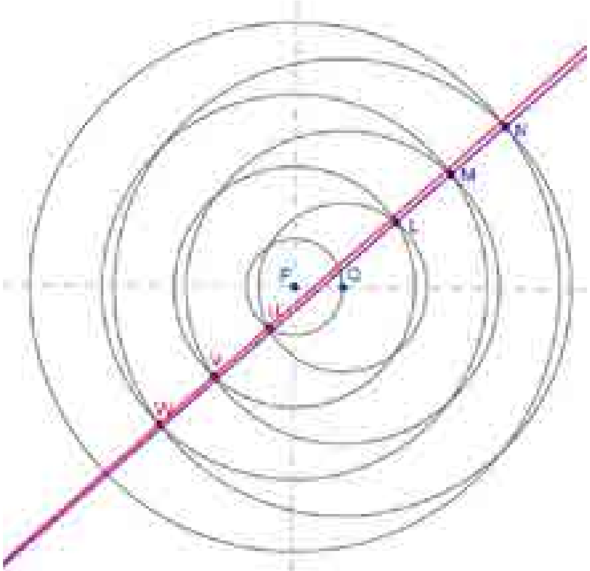
Consider Sub-condition 1.1: To illustrate the construction geometrically, we took the radius of the initial circle as 4 units and the difference between the radii of each subsequent circle as 3 units. Figure 2a emerges.

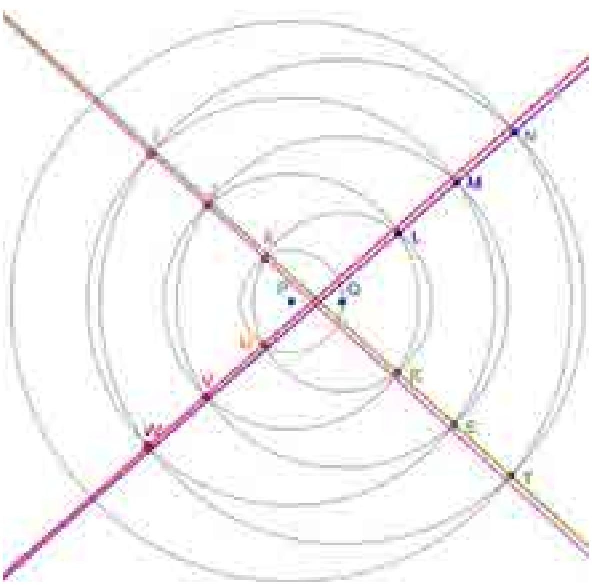
At a glance, one could see that the corresponding circles intersect each other at two points. The first two circles intersect each other at points K and U, the second and third circles at points L and R, and so on (Figure 2b). As done earlier, we took the points of intersection for making the spirals. Each point of the intersection of circles was taken as the emerging point for the next leg of the spiral. We got two pseudo-spirals, one clockwise and other anticlockwise (Figure 3a and Figure 3b).

In addition to getting the spirals, we also observed a pattern. It seemed that the points of intersection of the circles (i.e., points N, M, L, U, V, W and points I, J, K, R, S, T) are respectively collinear. Thus, we made a hypothesis, "Lines passing through the points of intersection of the circles intersect each other." In other words, our visually-based hypothesis was, "Points N, M, L, U, V, W and points I, J, K, R, S, T are collinear and that the lines passing through these points would intersect at the origin."

The hypothesis was made only on the basis of our visual perception and it soon got refuted when we joined the points on the dynamic tool. We got four lines which neither intersected nor were they mutually parallel. However - the points lying in one quadrant were seen to be collinear- the conjecture is yet to be proved. (Please note, the four quadrants were made taking the x-axis as the line on which the centres of the circles P, Q lie and the y-axis was the perpendicular line passing through the centre of the first circle, i.e., through point P.)

Alas, we couldn't go any further, even though we could sense the presence of some hidden



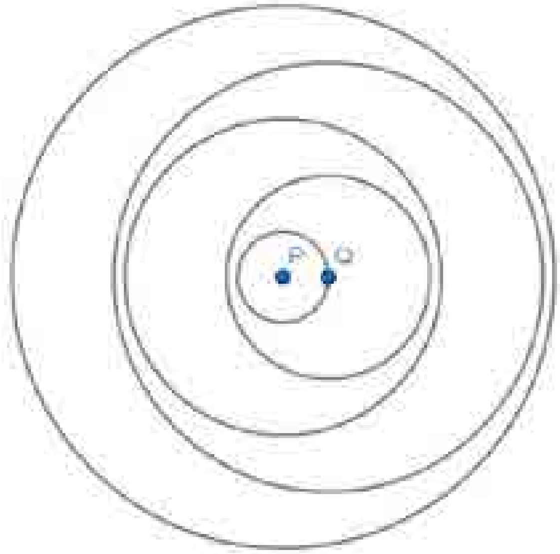


mathematical gems. Perhaps, somebody will be able to draw out more sophisticated conclusions.

Sub-Condition 1.2: Radius of the first circle is less than the difference between the radii of the subsequent circles.

To construct circles based on the above sub-condition, we took the radius of the first circle as 4 units and the difference between the radii of the consecutive circles as 5 units.

Observation: The circles do not intersect each other so it was not possible to make spirals.



P and Q are the respective centres of the circles having radii in AP: 4, 9, 14, 19, 24,...

Next, we tried altering Condition 2.

Altering Condition 2: What if a spiral is made not of successive semi-circles.

We altered the semi-circles and replaced them with semi-polygons. To make things simpler, we started with a regular quadrilateral, i.e., the square. We were now interested in making uniformly growing squares whose centres would alternate and semi-squarish spirals could be made.

In a square, the centre is fixed but to make the growing squares we had two options:

- to consider the distance from the centre to vertices in A.P. i.e., increase the lengths of half-diagonals in A.P. or
- to consider the perpendicular distances from centre to the midpoints of the edges in A.P. i.e., increase the apothems in A.P.

Both sub-cases led to two ways of making the semi-square spirals:

Sub-Condition 2.1: Half-diagonals increase in AP. Consider the distance between the centre of the first square and its vertex as the initial distance, and subsequently increase the length of every half-diagonal by the same

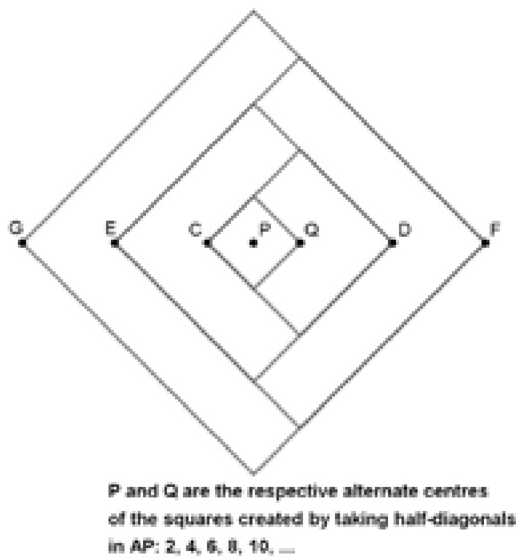


Figure 4a

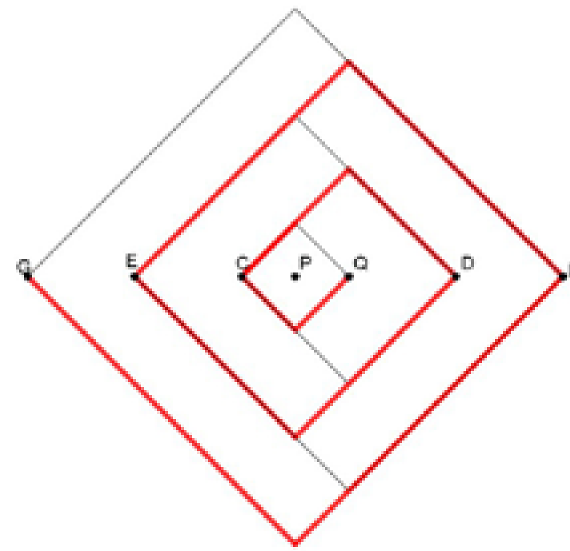


Figure 4b

magnitude. While altering Condition 2, the original Conditions 1 and 3 were kept intact. Thus, the lengths of the half-diagonals of the subsequent squares were in Arithmetic Progression with a = d, and the centres of the squares alternated. The semi-squarish spiral depicted in the Figures 4a and 4b emerged.

Sub-Condition 2.2: Apothems increase in AP.

Drop a perpendicular from the centre of the first square to the midpoint of a side and consider this as the initial distance. For each subsequent square, the length of the apothems will increase by same magnitude. The following square-spiral emerged (Figures 5a and 5b).

Similarly, one can explore more semi-polygonal spirals such as semi-pentagonal-spirals, semi-hexagonal-spirals, using GeoGebra.

And, finally altering Condition 3: What if the centres of the circles do not alternate?

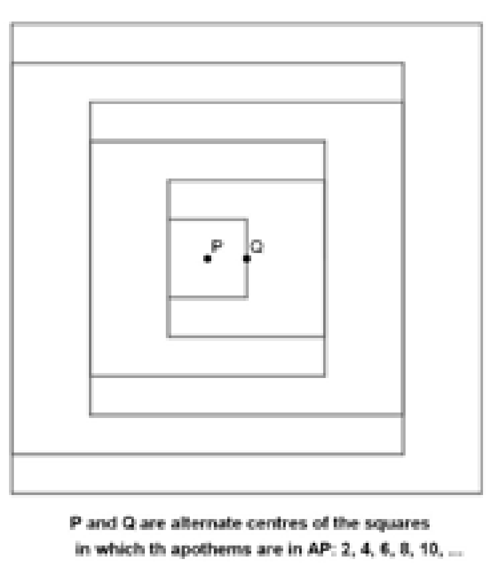


Figure 5a

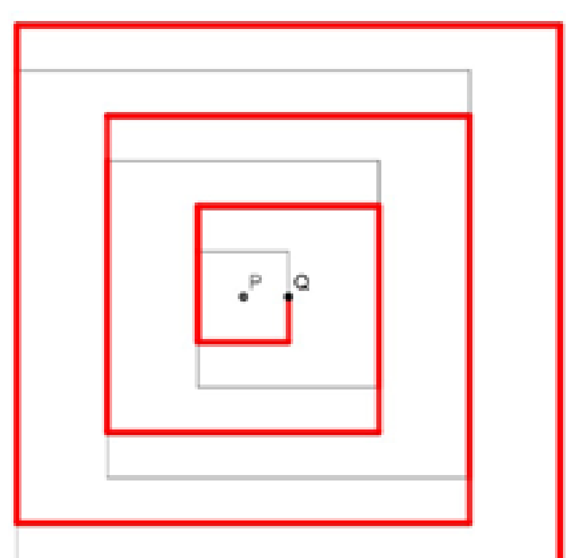


Figure 5b

If the centres do not alternate, and the other conditions remain unaltered, it will produce consecutive circles and no spirals.

Ending Remarks

While doing this exercise we wondered, why have we not opened the gates to problem-generation? There could be many reasons, one among them is that the curriculum makers or teachers frequently do not appreciate problem generation. What we mean is that teachers only value 'neat' problems which proceed on formal procedures, culminating in neater answers. Problem generation should not be pursued with a mindset of promoting a neatly framed problem which would always pave formal structures to the solution. What is needed is an appreciation to value intuition that is built on logical and justifiable observations. We need to build an acumen to make hypotheses and conjectures in a structured way without being bothered, at least at that moment, of generating proofs.

Nowhere in this activity are we claiming that we were led to solutions or any formal theorising. Lest we lose sight of the larger picture, the work shared by us only provides a glimpse on how problems can be generated or expanded from routine textbook problems. Each expanded problem may not have an answer . Often engagement with the tasks may be very different from that expected. Asking relevant questions, making conjectures based on perceptions and generating problems invokes a spirit of inquiry, a desire to explore. We propose to open avenues for discussions, explorations, observations, visualisations, patterns and generalisations to the extent possible. While trying to work on the alternative conditions, we were guided by our intuitions and visual connections. The excitement came from our thrill of observing what emerged by changing the conditions. We admit we are not experts in mathematics, but we dare to say that you needn't be an expert to let your curiosity pull you in. We were doodling mathematically. We were seeing newer problems which may not be 'neat' in a true sense but they do hold the potential of becoming sophisticated ones.

Acknowledgement. This work was part of the IMPRESS-ICSSR funded project, "Development of Mathematical Temperament in Large-sized Classrooms: A Feed-Forward for Policy Makers" (Project No. P-2706, Year 2019-2021).

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Magic Squares Using Semi-Primes

ANAND PRAKASH

Wikipedia [1] defines a semi-prime as *a natural number that is the product of two prime numbers.* The definition allows the two primes in the product to be equal to each other, so the semi-primes include the squares of prime numbers. Displayed below are the first forty semi-primes.

4	6	9	10	14	15	21	22	25	26
33	34	35	38	39	46	49	51	55	57
58	62	65	69	74	77	82	85	86	87
91	93	94	95	106	111	115	118	119	121

Many number theoretic questions of interest can be asked about the semi-primes, for example: what is the longest sequence of consecutive numbers, all of which are semiprimes? It is not possible to have a sequence of four consecutive numbers, all of which are semi-primes, for the simple reason that any sequence of four consecutive numbers must contain a multiple of 4, and the only multiple of 4 which is a semi-prime is 4 itself (and it does not contain any semi-prime adjacent to it).

However, instances of three consecutive numbers, all of which are semi-primes, are easy to find. Table 1 reveals that the first such instance is 33, 34, 35, and the next one is 85, 86, 87. Do there exist infinitely many such instances? It is difficult to say.

More instances of consecutive semi-primes

Here are all instances of three consecutive numbers under 1000, all of which are semi-primes, listed in the form of a matrix (each row gives the three numbers).

Keywords: Magic square, semi-prime

Everyone knows what a magic square is. Sometimes, it is fun to try to make a magic square in which all the numbers belong to some specified subset of the natural numbers. For example, we may want all the numbers to be primes; or we may want all the numbers to be squares; and so on. In this short note, we explore the possibility of constructing a magic square entirely using semi-primes. This seems quite difficult! However, if we relax the conditions slightly, we are able to make progress.

$$\begin{pmatrix} 33 & 34 & 35 \\ 85 & 86 & 87 \\ 93 & 94 & 95 \\ 121 & 122 & 123 \\ 141 & 142 & 143 \\ 201 & 202 & 203 \\ 213 & 214 & 215 \\ 217 & 218 & 219 \\ 301 & 302 & 303 \\ 393 & 394 & 395 \\ 445 & 446 & 447 \\ 633 & 634 & 635 \\ 697 & 698 & 699 \\ 841 & 842 & 843 \\ 921 & 922 & 923 \end{pmatrix}$$

We display below the results of two such attempts. It is rather curious that many of the triples listed above can be seen in these arrays.

A partial magic square of order 6

Here is a partial magic square of order 6, all of whose entries are semi-primes; all six of the columns have the same total (1732), but only the first four rows have that total (the other two row sums are 1996 and 1468, respectively).

634	218	217	219	301	143
445	142	303	302	94	446
201	447	87	86	697	214
202	141	95	698	393	203
215	85	635	394	34	633
35	699	395	33	213	93

Another attempt at a magic square of order 6

Here is the result of another such attempt. This time we have permitted ourselves the use of a few numbers which are not semi-primes (specifically, the triple 117,118,119), but we do obtain a complete magic square of order six, with magic constant 1442. Here it is:

634	85	301	87	118	217
202	201	447	203	86	303
215	33	214	446	393	141
213	395	219	218	95	302
143	635	119	394	117	34
35	93	142	94	633	445

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1. Wikipedia, "semi-prime", https://en.wikipedia.org/wiki/semi-prime

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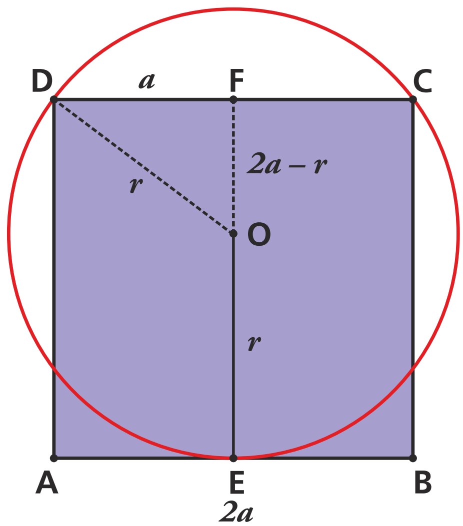
SOLUTION TO THE PROBLEM
POSED IN THE NOVEMBER 2019 ISSUE OF *At Right Angles*

CoMaC

Square *ABCD* has side *2a*. A circle of radius *r* touches side *AB* and passes through vertices *C* and *D*. The problem is to find *r* in terms of *a*. (In the actual problem, we had $2a=40$.)



Let *E,F* be the midpoints of sides *AB,CD,* and let *O* be the centre of the circle. Join *OD*. Then *O* lies on *EF*. Moreover, $OE=r$, $OF=2a-r$, $DF=a$, $OD=r$. From the right-angled triangle *OFD*, we obtain:

$$r^2 = a^2 + (2a - r)^2.$$

Hence $4ar=5a^2$, giving

$$r = \frac{5a}{4}$$

This is the required relationship. (In the given problem, we have $a=20$, so $r=25$.)

Comment: We also received a solution from Shri Tejash Patel. It uses coordinate geometry methods.

Construction of the figure

The answer suggests how we can construct such a figure, which otherwise is not at all obvious. For we obtain,

$$FO:OE=3:5.$$

So we must draw the midline of the square (i.e., the line segment *EF* joining the midpoints of a pair of opposite sides), and then locate a point *O* on *EF* such that $FO:OE=3:5$. The circle with centre *O*, passing through *E*, is then the required circle.

The Generalised Pythagoras Theorem – Another Proof

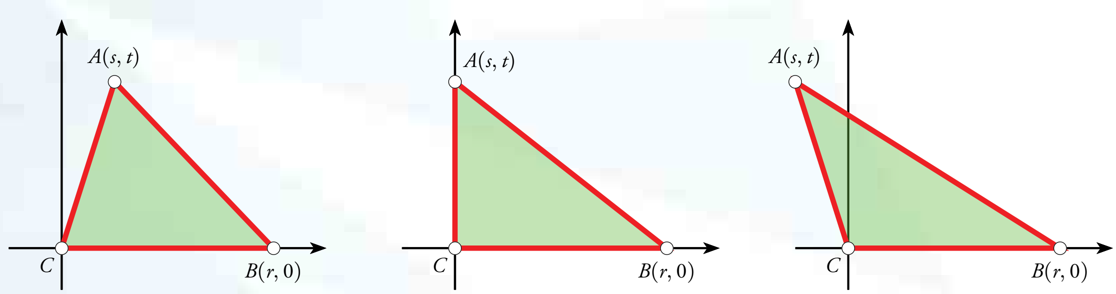
RANJIT DESAI

In this short note, we present a proof of the generalised Pythagoras theorem. We use the 'ordinary' Pythagoras theorem for the proof.

Theorem. *In any triangle ABC, we have:*

$$AC^2 + BC^2 > AB^2 \iff \measuredangle C < 90^\circ,$$
$$AC^2 + BC^2 < AB^2 \iff \measuredangle C > 90^\circ.$$

Proof. On the coordinate plane, place the triangle ABC so that vertex C lies at the origin, side CB lies along the positive x-axis, and vertex A lies in the upper half plane (i.e., in the first or the second quadrants); see Figure 1. Let the coordinates of the three vertices be as follows: $C = (0, 0)$; $B = (r, 0)$, where $r > 0$; and $A = (s, t)$, where $t > 0$.



$\measuredangle C < 90^\circ,\ s > 0$ $\quad$ $\measuredangle C = 90^\circ,\ s = 0$ $\quad$ $\measuredangle C > 90^\circ,\ s < 0$

Figure 1.

Then we have: $BC^2 = r^2$, and $AC^2 = s^2 + t^2$, so

$$BC^2 + AC^2 = r^2 + s^2 + t^2.$$

Keywords: Generalised Pythagoras theorem, triangle, sides, relations, proof

Also:

$$AB^2 = (r-s)^2 + t^2 = r^2 + s^2 + t^2 - 2rs,$$

so

$$BC^2 + AC^2 - AB^2 = 2rs.$$

Now it is clear from Figure 1 that:

$$\measuredangle C < 90^\circ \iff s > 0,$$
$$\measuredangle C > 90^\circ \iff s < 0.$$

Note that $2rs$ has the same sign as s (since $r > 0$). It follows that

$$BC^2 + AC^2 > AB^2 \iff \measuredangle C < 90^\circ,$$
$$BC^2 + AC^2 < AB^2 \iff \measuredangle C > 90^\circ.$$

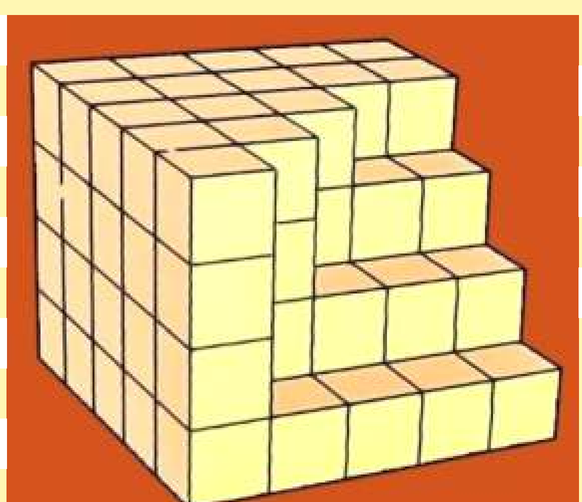
We may thus state the "generalised Pythagoras theorem" as follows.

Theorem (Generalised Pythagoras theorem)**.** *In any triangle ABC, we have:*

$$AC^2 + BC^2 > AB^2 \iff \measuredangle C < 90^\circ,$$
$$AC^2 + BC^2 = AB^2 \iff \measuredangle C = 90^\circ,$$
$$AC^2 + BC^2 < AB^2 \iff \measuredangle C > 90^\circ.$$

RANJIT DESAI started his career as a mathematics teacher in Bai Avabai High School, Valsad in 1960, and retired in 1998 as Incharge of P. G. Centre (Mathematics) from B. K. M. Science College, Valsad. In 1968, he started the Ganit Milan to connect students and teachers of schools and colleges with one another. He is actively associated with the Gujarat Ganit Mandal. He may be contacted at ranjitraimdesai@gmail.com.

CUBES: What is the minimum number of cubes required to build a full cube?



A Note on Geometric Construction

A. K. MALLIK

The whole of plane geometry is based on two figures, the straight line and the circle. Both these figures are defined by two points, say A and B. For drawing these figures, two instruments are available: (i) an unmarked straight edge for drawing a straight line joining A and B and, if necessary, extending the straight line beyond the segment AB on both sides; (ii) a compass for drawing a circle with one of the points A (or B) as centre and passing through the other point B (or A). Attention is drawn to the fact that in Euclid's original text, the compass is regarded as "collapsible." This implies that both ends of the compass—the needle and the pencil—must always be in contact with the drawing plane. The compass 'collapses' as soon as one of the ends is lifted. This, in turn, means that we cannot transfer distances by using the compass or divider in the manner routinely used in schools. It is necessary to emphasize that neither the straight edge nor the compass can be used for measurement. As Borovik and Gardiner [1] say: "Measuring is an *approximate* physical action, rather than an *exact* "mental construction," and so is not really part of mathematics." The emphasis is necessary especially in view of our school geometric box instrument set which consists of a *marked ruler* and a *divider*. The job of the divider, as said earlier, is routinely carried out in school by using the "real compass" which does not collapse like Euclid's compass. Such a compass is generally referred to as a "rusty compass." Note that the use of these instruments for measurement is perfectly acceptable for engineering drawings.

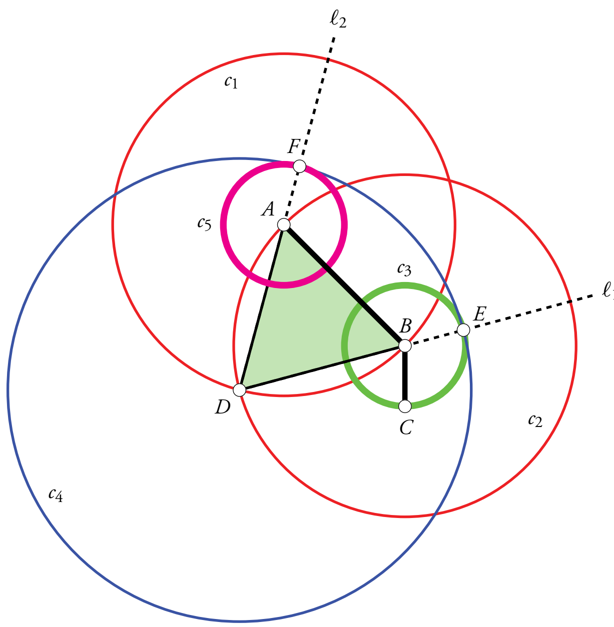
It should be emphasised that the basic figures (straight line and circle) can be drawn only after the two points defining them have been identified. A point is located at the intersection of two straight lines, the intersection of two circles, or the intersection of a straight line with a circle. In the last two cases, two points of intersection are normally generated, unless the two figures touch each other at a single point.

Keywords: Euclidean geometry, constructions, collapsible compass, rusty compass

It is known that any construction achievable by a rusty compass can also be carried out by a collapsible compass. Following [1], we reproduce below one of the construction problems discussed by Euclid to demonstrate how the use of rusty compass is avoided.

Problem. Given three points A, B, C, locate *geometrically* a point F so that line segments AF and BC are equal in length. (Equivalently: draw a circle with centre A and radius equal to the distance between B and C.)

Solution. A common mistake is to draw an arbitrary straight line through A, and then use a rusty compass to locate point F on that line. Two mistakes are thus committed. First, a line is drawn (through A) without locating two points, and then a rusty compass is used for measurement. The correct solution, from [3], is explained in Figure 1.



Steps of the construction

- $c_1 = \text{Circle}\,(A, B)$
- $c_2 = \text{Circle}\,(B, A)$
- $D = c_1 \cap c_2$
- $c_3 = \text{Circle}\,(B, C)$
- $\ell_1 = \text{Ray}\,(D, B)$
- $E = \ell_1 \cap c_3$
- $c_4 = \text{Circle}\,(D, E)$
- $\ell_2 = \text{Ray}\,(D, A)$
- $F = \ell_2 \cap c_4$
- $c_5 = \text{Circle}\,(A, F)$

Figure 1. Notation: Circle (A, B) means: ``Draw a circle with centre $A$, passing through $B$.'' Ray $(A, B)$ means: ``Draw a ray with initial point A, passing through B.'' The steps of the construction are given on the right side. (In the third step, D represents either point of intersection of c_1, c_2.)

(Note: We only consider the case when $AB > BC$. If $AB < BC$, then only small changes are needed in the procedure described in Figure 1. When $AB = BC$, the solution is trivial.)

Discussion of an interesting geometric construction problem

In [2], Jaleel Radhu, a class 11 student, discusses the following construction problem: *Given any $\triangle ABC$, show that there exist points P on side AB and Q on side AC such that $BP = PQ = QC$, and find a way of locating such a pair of points.* The steps in Jaleel's construction are shown in Figure 2.

(1) Locate any point D on side AB such that $BD < AB$ and $BD < AC$.

(2) Through D, draw a line ℓ parallel to side BC.

(3) Locate a point Q_1 on side AC such that $CQ_1 = BD$.

(4) Locate a point P_1 on ℓ such that $Q_1P_1 = BD$.

(5) Through P_1, draw a line parallel to AB; let it intersect side BC at E.

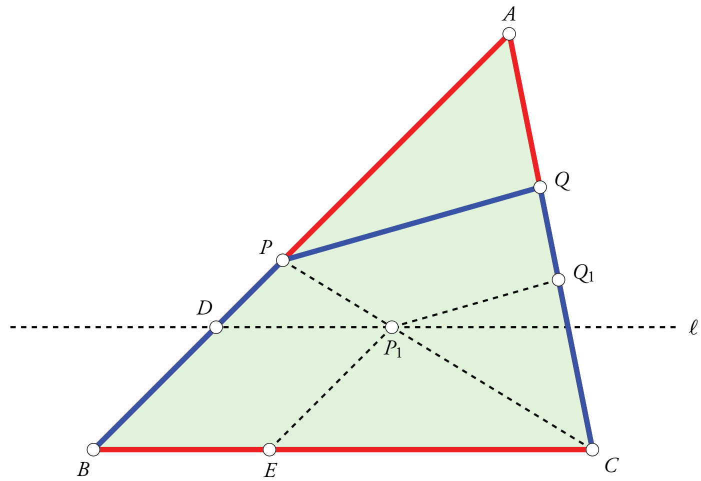


Figure 2.

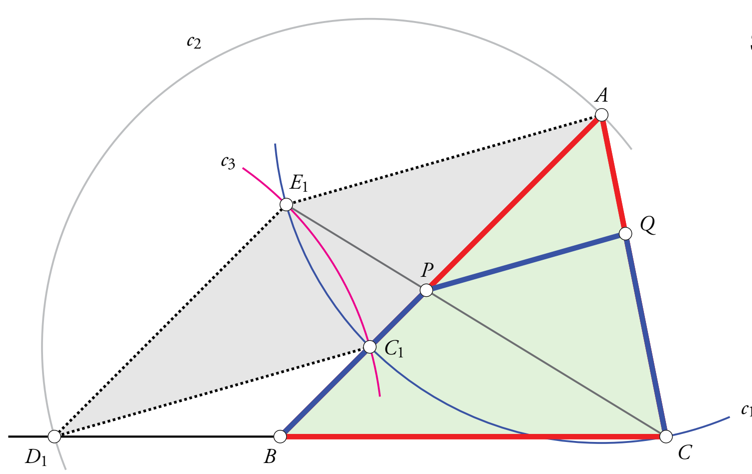
(6) Quadrilateral P_1DBE is a parallelogram; so $P_1E = DB$. Hence $EP_1 = P_1Q_1 = Q_1C$.

(7) Extend CP_1 beyond P_1; let it intersect AB at P. Through P, draw a line parallel to P_1Q_1; let it intersect AC at Q.

(8) Then P and Q are the required pair of points.

Comment on the solution. Note that the point D is located arbitrarily, neither defined in the problem, nor obtained by intersection of lines or circles. As will be shown below, the problem can be solved without the use of an arbitrary point.

Another solution to the problem. Here are the steps of the construction [4]. The diagram has been drawn assuming that $AC < AB$ (Figure 3).



Steps of the construction

- $c_1 = \text{Circle}(A, C)$
- $C_1 = c_1 \cap AB$
- $c_2 = \text{Circle}(C_1, A)$
- $D_1 = c_2 \cap \text{Ray}(C, B)$
- $c_3 = \text{Circle}(D_1, C_1)$
- $E_1 = c_1 \cap c_3$
- $P = CE_1 \cap AB$
- $PQ \parallel E_1A$

Figure 3. Quadrilateral $AC_1D_1E_1$ is a rhombus (each side equal to AC). The required points are P and Q.

Proof of construction. Consideration of the similar triangles $\triangle CPB$ and $\triangle CE_1D_1$ yields

$$\frac{CP}{CE_1} = \frac{BP}{D_1E_1}. \tag{1}$$

Consideration of the similar triangles $\triangle CPQ$ and $\triangle CE_1A$ yields
$$\frac{CP}{CE_1} = \frac{PQ}{E_1A} = \frac{CQ}{CA}. \tag{2}$$
From (1) and (2), we get
$$\frac{BP}{D_1E_1} = \frac{PQ}{E_1A} = \frac{CQ}{CA}.$$
Since by construction $D_1E_1 = AE_1 = AC$, we get $BP = PQ = QC$, as desired.

The rhombus $AC_1D_1E_1$ here lies outside triangle ABC. This is so because $AC > AB/2$.

If $AC < AB/2$, then rhombus $AC_1D_1E_1$ lies inside triangle ABC, and point Q lies on the extension of the side CA, as shown in Figure 4. (Note that if $AC < AB/2$, then, in order to satisfy the triangle inequality $AC + BC > AB$, we must have $BC > AB/2$. Now, following the same procedure, we may obtain points R on side BC and S on side AB such that $CR = RS = AS$.) Finally, if $AC = AB/2$, then P is the midpoint of AB, and Q coincides with A, resulting in $BP = PA = CA = AB/2$.

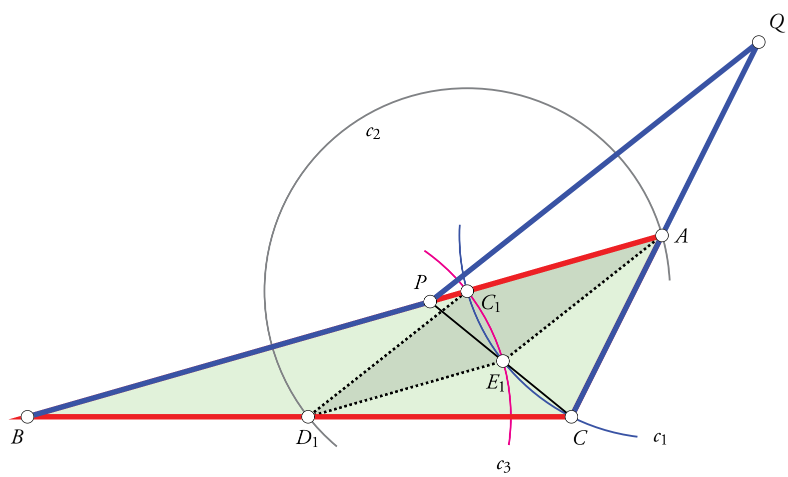


Figure 4. The construction steps are the same as earlier. Quadrilateral $AC_1D_1E_1$ is a rhombus (each side equal to AC). The required points are P and Q.

Acknowledgement. The author wishes to acknowledge the contribution of the editorial board towards improvement of this article and for adding the reference [3].

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How To Prove It

SHAILESH SHIRALI

The topic of 'proof by induction' is now a standard part of the syllabus of mathematics at the 11-12 level. Most students consider it a 'scoring topic' – they generally master the mechanics of proof by induction quickly, as the proofs follow a standard trajectory and are easy to mimic.

My experience as a mathematics teacher, however, suggests that the vast majority of students do not grasp what proof by induction is all about. While they are able to mimic all the required steps, most of them do not grasp the essential logic of such proofs. Indeed, to a good many students, these proofs give the impression of circular reasoning! For someone steeped in the culture of mathematics, it is not easy to understand why students find it so difficult to grasp the essence of such proofs. Is it because the topic is taught in haste, with insufficient time spent on the subtleties involved (and there surely are many subtleties involved)? Or is it because proof itself is inherently a difficult topic? I suppose that a great deal more research is needed to understand the core of the difficulty. It would be well worth it for teachers themselves to undertake such research, rather than wait for experts to take up the task.

In this and the following episode of *How to Prove It*, we shall dwell on some critical aspects of induction proofs (aspects which possibly are not emphasised strongly enough) and study a few examples that show how valuable and versatile it is as a proof technique.

Structure of a proof by mathematical induction

We start by listing the essential components of a proof by induction:

(1) Framing the hypothesis or conjecture.

(2) Anchoring the induction, i.e., verifying the initial step.

(3) The bridge step, i.e., establishing the link between successive propositions of the induction hypothesis.

We now elaborate on each of these steps.

Keywords: Proof by induction

Framing the hypothesis or conjecture

This step is generally not encountered at the school level; the student is simply *given* the hypothesis or conjecture to be proved and asked to prove it using the principle of induction (i.e., asked only to execute the second and third steps). So the student is simply being asked to provide the algebraic details of the proof. *But this step is crucial*. Indeed, it is this step that constitutes what is 'inductive' about proof by induction. Without it, including the word 'induction' in 'proof by induction' is meaningless; the exercise cannot be called a proof by induction. For, if we skip over this initial step, a proof by induction is actually a proof by deduction! So it is most unfortunate that this step is completely absent in the way the topic is taught in schools at present.

'Framing the hypothesis' can be expressed more simply as (and most often reduces to): *Guess the formula!* Accordingly, we shall study a few examples to illustrate this.

Example 1: Partial sums of the consecutive integers. Let's start with a simple and familiar example: guessing a formula for the sequence of partial sums of the consecutive positive integers. That is, we seek a formula for the n-th term of the sequence

$$1, \quad 1+2, \quad 1+2+3, \quad 1+2+3+4, \quad 1+2+3+4+5, \quad \ldots. \tag{1}$$

Performing the additions, we obtain the following sequence:

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ \ldots \tag{2}$$

How do we proceed now?

There is no standard way to guess a formula for a given sequence. Rather, we have to play with the sequence and hope for the best! That is, we may double all the terms; or multiply all the terms by some other number; or add some constant to all the terms; or add and subtract some constant to the terms in alternation; or divide the terms by their serial number; or express the terms in factorized form; and so on. In short, we have to perform all kinds of arithmetical operations to the given sequence, in the hope of uncovering some visible pattern, one which we can 'catch hold of' and which will enable us to formulate a suitable hypothesis or conjecture. *It is this which is the inductive step*—i.e., guessing a formula or formulating a conjecture.

In this instance, doubling all the terms yields:

$$2,\ 6,\ 12,\ 20,\ 30,\ 42,\ 56,\ 72,\ 90,\ 110,\ \ldots \tag{3}$$

Now, if we divide each term by its position number (i.e., the position where it occurs in the sequence), that is, we do $2 \div 1$, $6 \div 2$, $12 \div 3$, $20 \div 4$, …, then the pattern is instantly revealed. We get:

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ \ldots \tag{4}$$

The n-th term of this sequence is 'clearly' $n+1$, which means that the n-th term of the sequence $2, 6, 12, 20, 30, \ldots$ must be $n(n+1)$, and therefore the n-th term of the original sequence $1, 3, 6, 10, 15, \ldots$ must be $\frac{1}{2}n(n+1)$. This is what inductive thinking has led us to believe; it represents our educated guess for a formula giving the sum of the first n positive integers. At this stage, it is, of course, only a guess; we have not proved anything as yet.

Example 2: Partial sums of the squares of the consecutive integers. We consider now a more complex example: guessing a formula for the sequence of partial sums of the squares of the consecutive positive integers. That is, we seek a formula for the n-th term of the sequence

$$1^2, \quad 1^2+2^2, \quad 1^2+2^2+3^2, \quad 1^2+2^2+3^2+4^2, \quad 1^2+2^2+3^2+4^2+5^2, \quad \ldots \tag{5}$$

Performing the additions, we obtain the following sequence:

$$1,\ 5,\ 14,\ 30,\ 55,\ 91,\ 140,\ 204,\ 285,\ 385,\ \ldots \tag{6}$$

As the sequence grows more rapidly than the earlier one, we should be prepared to do more experimentation before we can guess the formula. Let's try 'cutting the numbers down to size,' by dividing each term by its position number. That is, if s_n denotes the sum $1^2 + 2^2 + \cdots + n^2$, then let's compute the values of $s_n \div n$. We get:

$$1,\ \frac{5}{2},\ \frac{14}{3},\ \frac{15}{2},\ 11,\ \frac{91}{6},\ 20,\ \frac{51}{2},\ \frac{95}{3},\ \frac{77}{2},\ \ldots \tag{7}$$

We observe quickly the repeated occurrence of the denominators 2, 3, 6. Continuing the computations, we find that the behaviour does not change; no other denominator turns up. (In passing, we note that the sequence of denominators has a nice pattern: 1, 2, 3, 2, 1, 6, 1, 2, 3, 2, 1, 6, and so on, with the string $1, 2, 3, 2, 1, 6$ repeating indefinitely.) This prompts us to multiply the latest sequence by the LCM of $2, 3, 6$, i.e., by 6. (This kind of ad hoc logic is typical of the inductive stage. We have to be prepared to do all kinds of computations. Most of the time, these efforts do not lead to anything at all.) Here's what we get:

$$6,\ 15,\ 28,\ 45,\ 66,\ 91,\ 120,\ 153,\ 190,\ 231,\ \ldots \tag{8}$$

These are the values of $6s_n \div n$. A pattern is quickly noticed when we express the numbers in factorized form:

$$2\times 3,\ 3\times 5,\ 4\times 7,\ 5\times 9,\ 6\times 11,\ 7\times 13,\ 8\times 15,\ 9\times 17,\ 10\times 19,\ 11\times 21,\ \ldots \tag{9}$$

It does not take much effort to guess that this is the sequence $(n+1)(2n+1)$. This means that $6s_n \div n = (n+1)(2n+1)$, i.e.,

$$s_n = \frac{n(n+1)(2n+1)}{6}. \tag{10}$$

This is what inductive thinking has led us to believe; it represents our educated guess for a formula giving the sum of the squares of the first n positive integers. As earlier, it is only a guess; we have not proved anything as yet.

Example 3: Spearman's rank correlation coefficient: an associated calculation. In the study of rank correlation, we wish to define a measure indicating how close two sets of ranks are to each other. Mathematically, it amounts to finding a measure of closeness of two different permutations of the string $(1, 2, \ldots, n-1, n)$, where n is any positive integer. (To start with, let us not consider the possibility of tied ranks, which naturally introduces complications.) We would like the measure to be such that it assumes a value of $+1$ when the permutations are identical to each other, and a value of -1 when the permutations are reverses of each other.

Let the two permutations of $(1, 2, \ldots, n-1, n)$ be

$$(a_1, a_2, \ldots, a_n)\,, \qquad (b_1, b_2, \ldots, b_n) \tag{11}$$

respectively. A natural measure of how different they are from each other is the following score S:

$$S = (a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2\,. \tag{12}$$

Note that we have opted to square the differences between respective ranks to avoid the situation where negative values cancel out positive values. (Instead of taking the squares, we could also have taken the absolute values. This would be perfectly acceptable.)

How do we convert the score S into a coefficient that lies between -1 and 1? Here's a simple and elegant way of doing this. The minimum possible value of S is clearly 0; this is attained when the two permutations are identical to each other. What is the maximum possible value of S? Denote this maximum value by M. Now compute the quantity

$$\rho = 1 - \frac{2S}{M}. \tag{13}$$

Observe that if the two permutations are identical to each other, then $S = 0$, leading to $\rho = 1$, which is as it should be; and if the two permutations are as far apart as possible from each other, then $S = M$, leading to $\rho = -1$, which once again is as it should be. So this formula makes sense. It remains now to find a convenient formula for M.

Let us compute the value of M for various values of n. This will be the value of the score S when the two permutations are the following:

$$(1, 2, \ldots, n-1, n), \qquad (n, n-1, \ldots, 2, 1). \tag{14}$$

That is,

$$M = (1-n)^2 + (2-n+1)^2 + \cdots + (n-1-2)^2 + (n-1)^2. \tag{15}$$

Let us now prepare a table of values of M (we could use a computer to do this, if we wish). Here is what we get.

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
M	0	2	8	20	40	70	112	168	240	330	$\cdots$

(16)

Let us now try to find a formula for the sequence

$$0,\ 2,\ 8,\ 20,\ 40,\ 70,\ 112,\ 168,\ 240,\ 330,\ \ldots \tag{17}$$

Denote the n-th term of the sequence by s_n. Observe that the sequence is growing quite rapidly. So let us produce a new sequence defined by $s_n \div n$. We obtain the following:

$$0,\ 1,\ \frac{8}{3},\ 5,\ 8,\ \frac{35}{3},\ 16,\ 21,\ \frac{80}{3},\ 33,\ \ldots \tag{18}$$

We see that a denominator of 3 occurs at regular intervals in this sequence. So let us multiply the above sequence by 3, to clear the denominators. We now obtain the following:

$$0,\ 3,\ 8,\ 15,\ 24,\ 35,\ 48,\ 63,\ 80,\ 99,\ \ldots \tag{19}$$

We could now experiment further with this sequence, but recognition is faster! We notice easily enough that each number in this sequence is 1 less than a perfect square. Indeed, if we add 1 to each number in the sequence, we simply obtain the sequence of perfect squares. It follows that the n-th term of (19) is $n^2 - 1$, hence the n-th term of (18) is $\left(n^2 - 1\right) \div 3$, hence the n-th term of (17) is $\left(n^3 - n\right) \div 3$. Hence:

$$s_n = \frac{n^3 - n}{3}. \tag{20}$$

So we have obtained a formula for the coefficient of rank correlation (defined as above):

$$\rho = 1 - \frac{6S}{n^3 - n}. \tag{21}$$

By following this path, we have hit upon the formula for an extremely famous measure of rank correlation: ***Spearman's coefficient of rank correlation***. See [2].

Important comment. It is important to note exactly what we have accomplished till now. We have been able to find formulas for the general term of three important sequences, using a mixture of play and experimentation and guesswork. So *we have completed the inductive part of the exercise*: i.e., we have been able to frame conjectures that fit the data.

It now remains to prove each of the conjectures. It is at this stage that what is generally called 'proof by induction' enters the picture. So we have to: (i) anchor the induction (i.e., verify the initial part of the conjecture); (ii) prove the bridge step (i.e., verify the link between successive propositions of the induction hypothesis).

We will elaborate on the latter two steps in a follow-up article.

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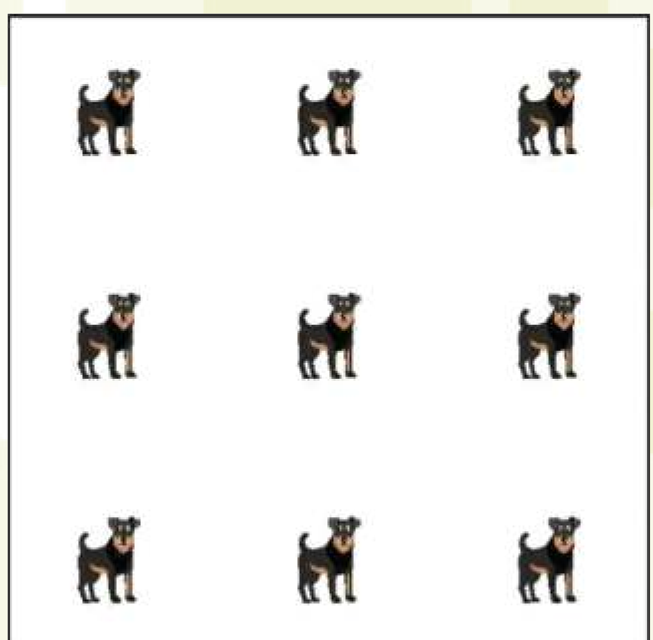
SOCIAL DISTANCING IN THE CANINE WORLD.

There are 9 dogs inside a square fence.

Construct two more square fences so that each dog is in isolation.



Secondary ClassRoom

Relations Among Lengths and Angles in General Parallelograms

A.RAMACHANDRAN

This article aims to derive some relations between lengths and angles in general parallelograms (GPs), i.e., parallelograms that are neither rectangles nor rhombi. We shall use Figure 1 for reference. In parallelogram $ABCD, AB > AD$, and $\measuredangle DAB$ is acute. Certain angles have been named p, q, r, s and t.

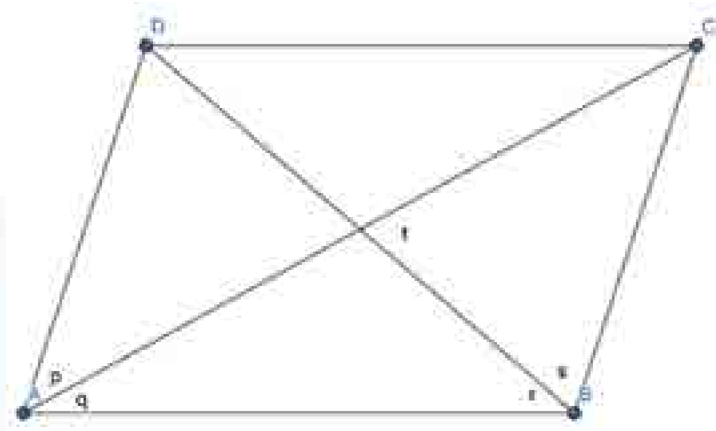


Figure 1.

Once the lengths AB, AD and $\measuredangle A$ are given, a unique parallelogram is defined. The lengths of the diagonals and angles p, q, r, s and t get fixed. (Do you see why?) We shall obtain expressions relating these to the given sides AB, AD and $\measuredangle A$.

Keywords: parallelograms, angles, sides, cosine rule, sine rule

The cosine rule of triangles can be used to obtain the following relations between the sides and the diagonals.

$$AC^2 = AB^2 + AD^2 + 2AB.AD\cos A$$
$$BD^2 = AB^2 + AD^2 - 2AB.AD\cos A.$$

These relations can be reworked to give

$$2(AB^2 + AD^2) = AC^2 + BD^2.$$

This relation is equivalent to Apollonius's theorem. It enables one to obtain one of the lengths given the other three.

The above pair of equations also lead to

$$\cos\measuredangle DAB = (AC^2 - BD^2)/4AB.AD,$$

enabling one to obtain $\measuredangle DAB$ from the four lengths.

Using the sine rule in triangles ACD, ABC, ABD and BCD in turn, we obtain the following relations between angles p, q, r, s and $\measuredangle DAB$. Note that $\sin D = \sin(180 - A) = \sin A$.

$$\sin p/\sin\measuredangle DAB = \sin p/\sin\measuredangle ADC = \sin\measuredangle CAD/\sin\measuredangle ADC = CD/AC = AB/AC,$$
$$\sin q/\sin\measuredangle DAB = \sin q/sin\measuredangle CBA = \sin\measuredangle CAB/\sin\measuredangle ABC = BC/AC = AD/AC,$$
$$\sin r/\sin\measuredangle DAB = \sin\measuredangle ABD/\sin\measuredangle BAD = AD/BD,$$
$$\sin s/\sin\measuredangle DAB = \sin s/\sin C = \sin\measuredangle CBD/\sin\measuredangle BCD = CD/BD = AB/BD.$$

To obtain a relation between angle t and $\measuredangle A$, we note the following.

Area of parallelogram $ABCD = AB.AD\sin A = \frac{1}{2}AC.BD\sin t$, giving

$$\sin t/\sin A = 2AB.AD/AC.BD$$

To summarise

$$\sin p/\sin A = AB/AC$$
$$\sin q/\sin A = AD/AC$$
$$\sin r/\sin A = AD/BD$$
$$\sin s/\sin A = AB/BD$$
$$\sin t/\sin A = 2AB.AD/AC.BD$$

Observe the neat pattern in the above set of equations. Several inferences can be made from the above.

(i) $\sin p\cdot\sin r = \sin q\cdot\sin s$

(ii) Angle inequalities: $q < p < s$ since $\frac{AD}{AC} < \frac{AB}{AC} < \frac{AB}{BD}$, and $q < r < s$ since $\frac{AD}{AC} < \frac{AD}{BD} < \frac{AB}{BD}$.

(iii) In the general case, angles p, q, r, s are distinct. Can any two of these be equal in a General Parallelogram (GP)?

If $p = q$, then $\measuredangle DAC = \measuredangle ACD$. Consequently $AD = DC$ and the figure becomes a rhombus. A similar situation arises if $r = s$. If $q = r$ or $p = s$, then the diagonals are of the same length, making the figure a rectangle. If $q = s$, then by the above inequalities, $p = q = r = s = 45^\circ$ and the figure turns out to be a square.

We now examine the pair p, r. If $p = r$, then we have $AB/AC = AD/BD$ or $AB/AD = AC/BD$. That is, the ratio of the (adjacent) sides equals the ratio of the diagonals. This is true of a square. We ask if this is possible in a GP.

Taking $AB = 1$, $AD = x < 1$, $AC = y$ and $BD = xy$, we have

$2(1 + x^2) = y^2 + x^2y^2 = y^2(1 + x^2)$, leading to

$y = \sqrt{2}$, i.e., the longer diagonal must be $\sqrt{2}$ times the longer side for angles p and r to be equal. The ratio of AD to AB can be any value between $(\sqrt{2} - 1)$ and 1. To see why, consider triangle ABD and apply the triangle inequality to its sides, which are $1, x, x\sqrt{2}$.

If $p = r$ then $t = \measuredangle A$. (This follows from the relations $\measuredangle A = p + q$ and $t = q + r$.) That is, the angle between the sides equals the angle between the diagonals. This is again true of a square but is also true of the kind of GP described above.

DUAL FIGURE CORRESPONDING TO A PARALLELOGRAM

Every parallelogram has its dual figure, whose sides are parallel to the diagonals of the former and vice versa. (Note that the concern here is only the *shape* of the figure, and not the scale. We can scale up the dual figure by any desired scale factor.) An easy way to obtain the dual of any parallelogram is to connect the mid-points of the sides sequentially, as indicated in Figure 2.

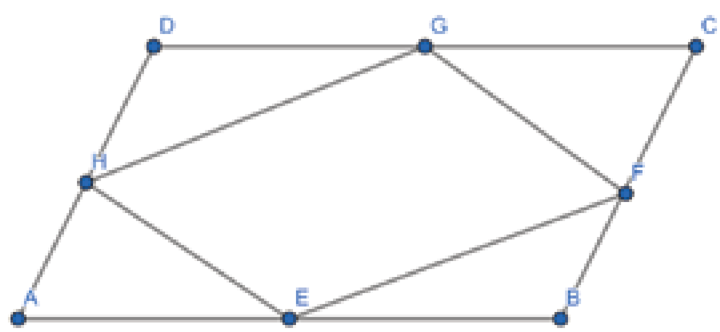


Figure 2

A rectangle and a rhombus where the angle between the sides in one is equal to the angle between the diagonals in the other are duals of each other. The dual of a square is another square. A GP of the type described in the previous paragraph, i.e., where the longer diagonal is $\sqrt{2}$ times the longer side, is its own dual.

To summarise, a member of this class of GPs shares the following properties with a square:

Ratio of sides equals ratio of diagonals. The longer diagonal is $\sqrt{2}$ times the longer side, while the shorter diagonal is $\sqrt{2}$ times the shorter side. The angle between the diagonals equals angle between sides It is its own dual.

We shall call a GP of this type a 'PSEUDOSQUARE'.

Editor's note. In the March 2017 issue of AtRiA, author Michael de Villiers describes a 'Golden rectangle' (pp. 64-69). In this figure the (adjacent) sides as well as the diagonals are in the ratio $\varphi : 1$, φ being the Golden ratio. Interestingly, the acute angle of the figure turns out to be 60°. Such a golden rectangle is a special case of a pseudosquare, as defined above, where the longer diagonal is $\sqrt{2}$ times the longer side. It can be shown that if the acute angle of a pseudosquare is 60°, then the sides and diagonals are in the golden ratio, and vice versa.

A COORDINATE GEOMETRY BASED APPROACH TO PSEUDO-SQUARES

With reference to Figure 3, $ABCD$ is a parallelogram with $AB = 1$ unit; A is placed at the origin and B on the positive X-axis. Angle q is an arbitrary angle less than 45°. $AC = \sqrt{2}$ units. Point D completes the parallelogram. The coordinates of A, B, C, D are as indicated in the figure.

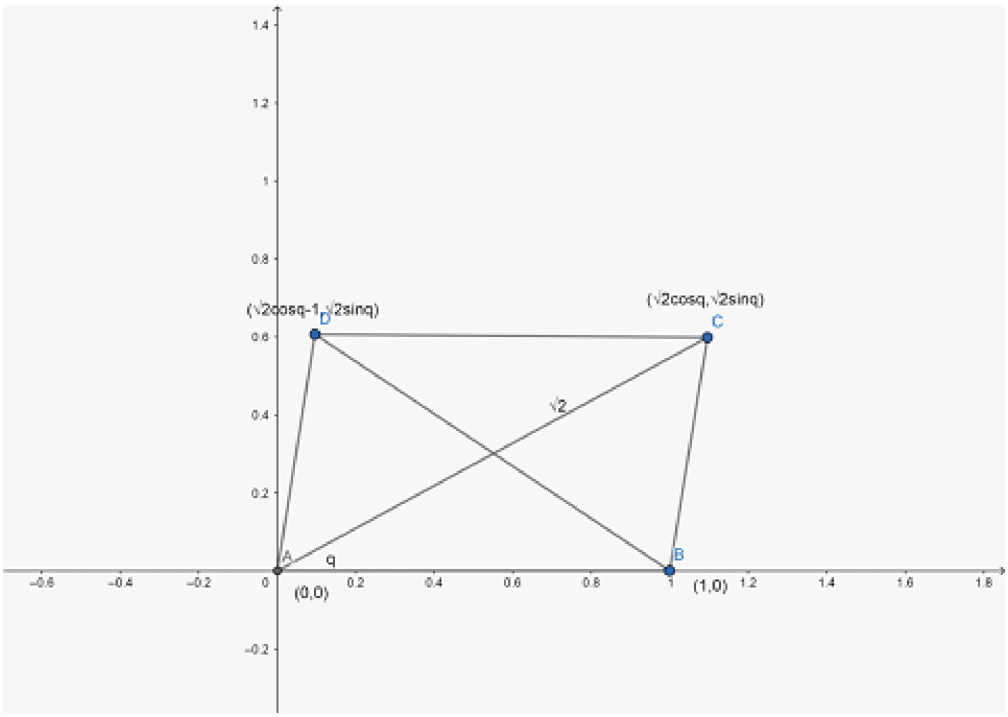


Figure 3

We first show that the ratio of sides equals the ratio of diagonals.

$AB = 1$. $AC = \sqrt{2}$.

$AD^2 = (\sqrt{2}\cos q - 1)^2 + (\sqrt{2}\sin q)^2$, which simplifies to $3 - 2\sqrt{2}\cos q$, hence

$AD = \sqrt{[3 - 2\sqrt{2}\cos q]}$.

$BD^2 = (\sqrt{2}\cos q - 2)^2 + (\sqrt{2}\sin q)^2$, which simplifies to $2(3 - 2\sqrt{2}\cos q)$, hence

$BD = \sqrt{[2(3 - 2\sqrt{2}\cos q)]}$.

Hence $BD/AD = \sqrt{2} = AC/AB$.

Next, we show that the angle between the sides equals the angle between the diagonals.

Slope of $AD = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 1}$, while

Slope of $AB = 0$.

tan value of the angle between the lines is $\frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 1} = \frac{2\sin q}{2\cos q - \sqrt{2}}$

Slope of $BD = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q - 2}$, while

slope of $AC = \frac{\sqrt{2}\sin q}{\sqrt{2}\cos q}$.

tan of the angle between the lines is again $\frac{2\sin q}{2\cos q - \sqrt{2}}$

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WHERE DO WE DRAW THE LINE?

In my opinion, students find mathematics difficult because they freely use the concepts and reasoning of mathematics in any context, without giving much thought to the deeper structural properties of the mathematical problem in each context. The lapses in understanding the mathematical concepts mainly arise due to failure to notice the salient features of a situation.

Here is one example:

According to Euclidean geometry "The sum of the three angles of a triangle is two right angles". Yet, if a sufficiently large triangle is drawn on the curved surface of the Earth and more precise instruments are used for measurement, then the sum of the three angles will be found to be exceeding two right angles. This is because the geodesic (the extremum distance between two point) in a Euclidean space is a straight line while it is an arc of the great circle on the surface of a sphere.

The geometry on the surface of a sphere is called the non-Euclidean geometry which was worked upon by Riemann. Riemann obtained his geometry in 1851 by replacing the 5th postulate (parallel postulate) of Euclid.

Euclid's fifth postulate states that *Through a point not on a given line, only one line can be drawn parallel to the given line.* Riemann replaced this with *Given a 'line' in a plane and a point outside the 'line' you cannot draw a 'line' parallel to the given 'line'*. It means there are no parallel lines on the surface of the sphere as they intersect the north pole and the south pole. [See Figure 3.]

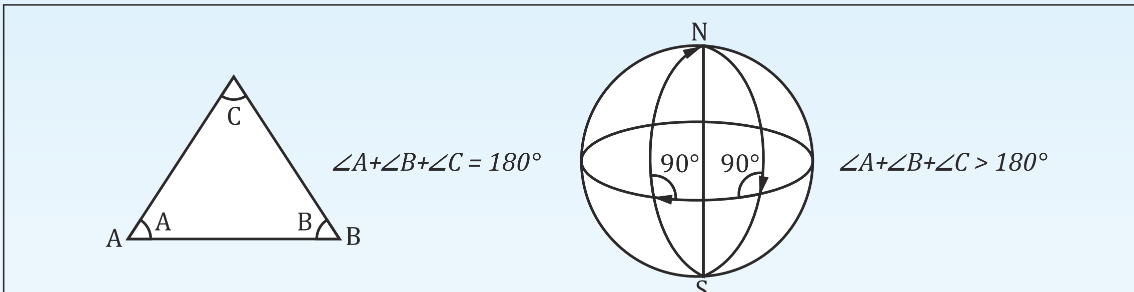


Figure 3

If a similar triangle is drawn on a surface with negative curvature then the sum of the three angles will found to be less than two right angles.

This geometry was developed by Lobachevsky and Bolyai independently in 1830. The non-Euclidean geometries due to Riemann and Lobachevsky and Bolyai are as consistent and as true as the Euclidean geometry. [See Figure 4.]

Figure 4

Triangles to Tetrahedrons and beyond…

The seed-idea of this article came from an activity from an upper primary math textbook and the modification in a subsequent edition. Students were asked to find the midpoints of the sides of an acute isosceles triangle and join them to form four smaller triangles, and then fold the triangles up to a tetrahedron. An equilateral triangle replaced the isosceles one in the subsequent edition. What caused this change? Wouldn't any triangle generate a tetrahedron? This initial exploration revealed something unexpected and the findings had an eerie resemblance to a known result. Further discussions with more math-friendly minds watered and added subsequent layers to this exploration and took it to a newer dimension – figuratively and literally! If a perpendicular is dropped from the apex (which is the top vertex of the tetrahedron where all three vertices of the triangle meet) to the base, where will the foot of this perpendicular be? For an equilateral triangle, it is the centre of the base but would it ever be coincident with any of the triangle centres, i.e., centroid, circumcentre, incentre or orthocentre of the base for other triangles? We will investigate these.

This Low Floor High Ceiling (LFHC) investigation begins by considering a neglected question on types of triangles. Then it explores a particular property that helps us classify triangles. After that, we zoom into one class of triangles and transition into 3-dimensions using nets. The nets, and the solids, in turn, generated more questions as well as helped in tackling them.

The first task considers types of triangles. How many are there?

Task 1: If we consider the sides, a triangle can be isosceles or scalene and some isosceles triangles can be equilateral too. Angle-wise, a triangle is acute, right or obtuse.

Keywords: Triangle, tetrahedron, constraint

a. How many types of triangles are there when both side-wise and angle-wise criteria are considered?

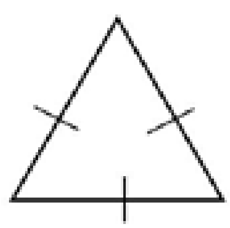
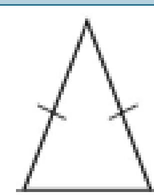
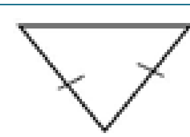
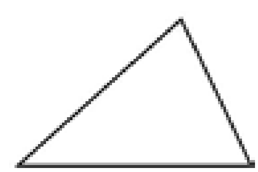
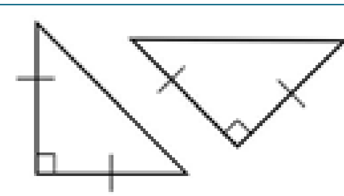
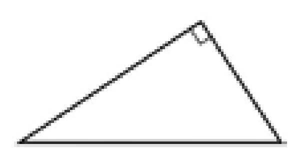
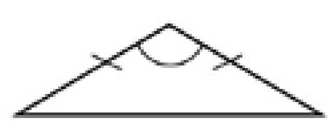
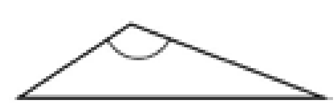
b. For each of the above categories, find at least two possible angle combinations, e.g. 30°–60°–90° and 40°–50°–90° for right scalene. Are there any types where only one angle combination is possible? If so, which one(s)?

c. Consider two acute isosceles with angle combinations (a) 20°–80°–80° and (b) 80°–50°–50°. Compare the unequal side with the equal sides. Find one more triangle with angle combination like (a) and another one like (b). How are the triangles like (a) different from those like (b)?

d. Consider the two groups of acute isosceles triangles (a) and (b) in the previous problem. Which type of triangle separates these two groups? Which type of triangle separates the obtuse isosceles from the acute ones?

Teacher Note: There are eight types of triangles as follows: three types of scalene – acute, right and obtuse, and four types of isosceles – equilateral, acute (with a different 3rd side), right and obtuse. Out of these, equilateral and right isosceles form similar class of triangles with 60°–60°–60° and 90°–45°–45° angle combinations respectively. Acute isosceles can be of two types depending on the equal sides being (a) longer or (b) shorter than the unequal one. This comes out very well if one tries to make triangles with just 10 matchsticks. The possibilities are 2–4–4 and 3–3–4 illustrating the two cases (a) and (b). The equilateral separates these two cases (a) and (b). So, there are five kinds of isosceles and they can be characterized by the angle θ between the equal sides: (i) $0° < \theta < 60°$ or the type (a) acute, (ii) $\theta = 60°$ i.e. equilateral, (iii) $60° < \theta < 90°$ or the type (b) acute, (iv) $\theta = 90°$ i.e. right, and (v) $90° < \theta < 180°$ or obtuse. Table 1 includes all eight types of triangles.

The second task was seeded by the textbook that changed the triangle from acute isosceles to equilateral in subsequent editions. So, let us explore what happens for all eight types of triangles.

Table 1	Equilateral	Isosceles	Scalene
Acute		Type a  Type b 	
Right			
Obtuse			

Task 2: Consider any ΔABC. Find the midpoints D, E and F of the sides BC, CA and AB respectively. Join DE, EF and FD.

a. Will ΔAEF, ΔBFD, ΔCDE and ΔDEF be the four faces of a tetrahedron (see Figure 1)?

b. If not, find the criteria for not getting a tetrahedron.

c. Is there any border-line case? Explain.

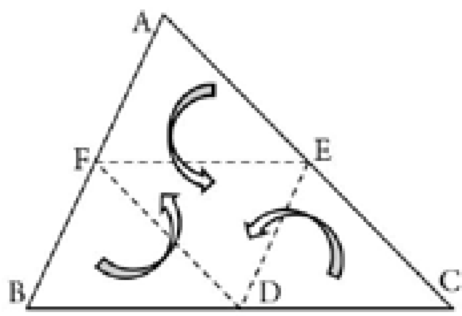


Figure 1

Teacher Note: Interestingly, not all triangles can be folded to a tetrahedron. The acute angled triangles (both isosceles and scalene) fold up and meet at a point to form a tetrahedron. But the obtuse angled triangles do not because two of the folded edges stay apart. The right triangles may create some confusion since the folded edges do match up (unlike obtuse) but no solid is formed (unlike acute). The point at which the folded sides meet is on the plane of the triangle so no solid is formed.

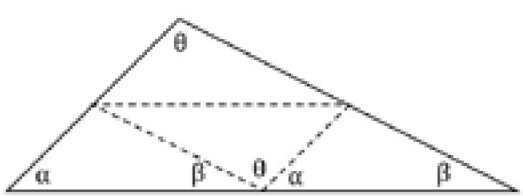


Figure 2

A closer inspection of the angles formed at the midpoint of the longest side reveals the cause (see Figure 2). In an obtuse triangle, $\alpha + \beta < 90°$ and $\theta > 90°$. Naturally α and β cannot cover all of θ (see Figure 3). Therefore, there is a gap between the folded edges. In the case of right triangles, $\alpha + \beta = 90° = \theta$ i.e. α and β cover θ exactly. So, the folded triangle flattens out with the edges meeting perfectly. Only in the case of acute triangles, $\alpha + \beta > 90° > \theta$. Therefore, α and β not only cover all of θ but actually overlap a bit. When the edges are put together to avoid the overlap, we get a solid, a tetrahedron, with a definite height.

So, a tetrahedron can form if and only if (iff) $\alpha + \beta > \theta$. It is worth noting that this angle inequality for a tetrahedron is very similar to the inequality involving the sides of any triangle ($a + b > c$, etc.).

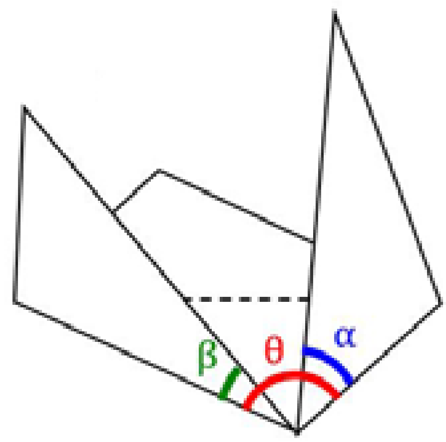


Figure 3

The following task is a scaffold towards further investigations of the tetrahedrons thus formed. It encourages the student to create a net of a solid – a task that demands imagination, spatial understanding and reasoning.

Task 3: Consider any of the three types of acute isosceles ΔABC with AB = AC. (Note that this includes equilateral as a special case.) Let D, E and F be the midpoints of the sides BC, CA and AB respectively. We know that ΔAEF, ΔBFD, ΔCDE and ΔDEF will be the four faces of a tetrahedron (see Figure 4). Now to visualize the height of this solid, it is good to split it in two halves. Note that the plane of symmetry of this tetrahedron passes through the line of symmetry of ΔABC. So, divide ΔABC along its line of symmetry AD (which intersects EF at P) and cut along AD. Fold ΔADC along the previous fold-lines to get the halved tetrahedron. Observe that this is a hallow tetrahedron with three faces viz. ΔCDE, ΔAEP (= ½ of ΔAEF) and ΔDEP (= ½ of ΔDEF). What are the sides of the missing face? Construct this missing face (on a separate piece of paper) and attach it to the net ΔADC along PD. Let Q be the third vertex of this missing triangular face. Make sure this triangle is oriented correctly. The side adjacent to AP should coincide with AP when folded. Similarly, the one next to DC should match DC. Fold this (pentagonal) net QPACD to form the halved tetrahedron and check that the fourth face fits properly.

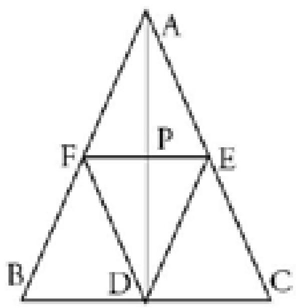


Figure 4

Teacher Note: The sides of the fourth triangle must match those of the remaining three faces. Since AE folds up with CE to form one edge of the new tetrahedron, the edges of the fourth face will be equal to AP, PD and DC. So, the fourth face is a triangle ΔDPQ on DP such that DQ = DC and PQ = AP i.e. A, C and Q will coincide in the new tetrahedron (Figure 5).

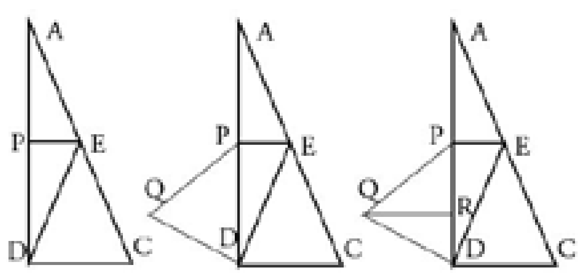


Figure 5

Note that with the fourth face, we can also draw the height of the tetrahedron which is the same as the height QR from Q to PD in ΔDPQ.

It is advisable that the net shown in Figure 5 be made for various measures of ∠ACB while the side length BC remain constant (say 12cm). Different nets can be made for the following values of ∠ABC – 50°, 55°, 60° and 70° (more can be made,

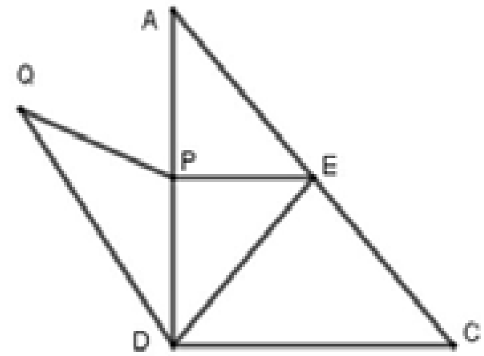


Figure 6

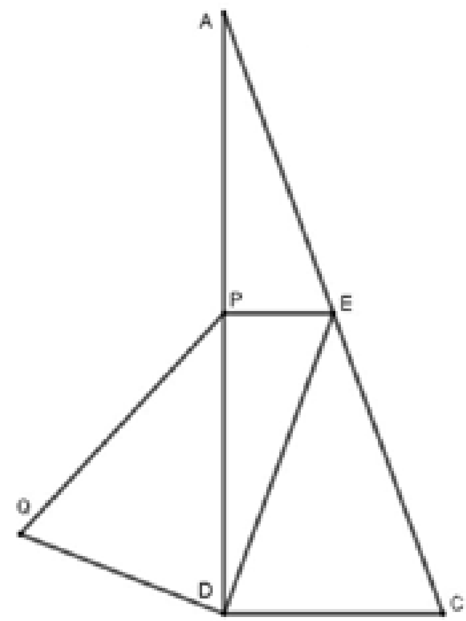


Figure 7

but these are crucial; Figures 6 and 7 include the nets corresponding to 50° and 70° respectively).

Task 4: ΔDPQ is an isosceles triangle since PD = PQ.

a. How does ∠DPQ vary with ∠ACB? Do we get all possible isosceles types described in Task 1?

b. What is the side ratio AC : BC for ΔDPQ to be a right triangle?

c. What is the ratio for ΔDPQ to be equilateral?

Teacher Note: Different groups of students can be given different values of ∠ACB and asked to create the net shown in Figure 5. Their nets can be then compared to gain further insights.

∠ACB can vary from 45° to 90° (both excluded). As ∠ACB increases, the height AD of ΔABC increases. So, for ΔDPQ, DQ remains fixed, but PD = PQ = ½ AD increases as ∠ACB increases. Therefore, ∠DPQ decreases as PD increases i.e. ∠ACB increases. ∠DPQ is obtuse for ∠ACB = 50° and acute for ∠ACB = 70°. A GeoGebra exploration with a slider for ∠ACB nicely demonstrates how ∠DPQ varies with the former.

GeoGebra steps

Chose B = (–6,0) and C= (6,0), and a slider for θ from 45° to 90°

D: midpoint of BC

Rotate B clockwise about C by θ to get B'

A: intersection of the line B'C and the y-axis

E: midpoint of AD, P: midpoint of AD

Draw circles (i) centred at P through A, and (ii) centred at D through C

Q: intersection of these two circles

Join line segments AD, AC, DC, PE, DQ and PQ

If ΔDPQ is right angled, then PD : DQ = 1 : √2. Let us take DQ = DC = ½BC = 2*a*. So, PD = √2*a*. Also, PE = ½ × DC = *a*. So, CE = DE = $\sqrt{(PD^2 + PE^2)}$ = √3*a*. Therefore AC = 2CE = 2√3*a* and BC = 2DC = 4*a* i.e. AC : BC = √3 : 2.

If ΔDPQ is equilateral, then PD = DQ = 2*a*. So, DE = √5*a* and therefore AC : BC = √5 : 2

Now that we know the different possibilities for ΔDPQ, it is a good idea to make the nets of the halved tetrahedron (as shown in Figure 5) for the following ratios of AC : BC – (i) 3 : 4, (ii) √3 : 2, (iii) 1 : 1, (iv) √5 : 2 and (v) 3 : 2. One option is to keep BC = 12cm and use AC = 9cm, 6√3cm, 12cm, 6√5cm and 18cm respectively. These should generate (i) obtuse, (ii) right, (iii) acute type (b), (ii) equilateral and (v) acute type (a) ΔDPQ respectively.

Different groups of students can tackle the different nets corresponding to the five types of ΔDPQ from here on.

From here onwards, R is the foot of the perpendicular from Q (where A, B and C coincide) to ΔDEF. By symmetry, this perpendicular or 'height' lies on the fourth face of the halved tetrahedron i.e. ΔDPQ, and R lies on PD where P is the midpoint of EF as indicated in Figure 4 and Figure 5.

The remaining tasks deal with the position of R on the mid-line PD of the base. The aim is to find R's position w.r.t. the sides of the tetrahedron and explore when R coincides with the special points viz. the centroid, the circumcentre, the incentre and the orthocentre– of ΔDEF.

Task 5: Since ΔDPQ can be any of the five possible isosceles triangles, how does the foot of the perpendicular R vary?

a. Is it always inside PD?

b. If not, when is it outside? What does that mean for the tetrahedron?

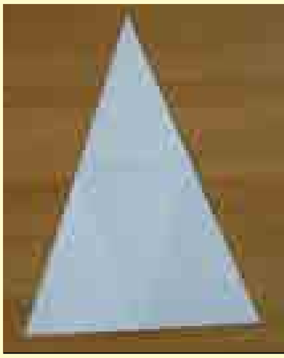
c. What is the border line case? What does that mean for the tetrahedron?

Teacher Note: This may come as a surprise but is actually a natural consequence of the types of ΔDPQ. The foot of the perpendicular R is inside PD (and therefore inside the base ΔDEF) iff ∠DPQ is acute (Figure 8).

When ∠DPQ is obtuse, R is outside PD. So, the foot of the perpendicular is outside the base ΔDEF. In this case, R would be on the ray DP, such that DR > DP. The edge AP (= QP) would lean outward from the base ΔDEF (Figure 9).

The border line case is when ∠DPQ is a right angle. Then R and P coincide, and QP is perpendicular to the base (Figure 10). Table 2 includes the nets of all five new tetrahedrons with R and QR ⊥ PD marked in each.

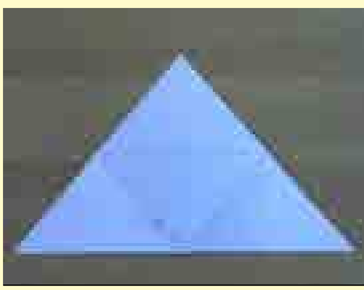
Acute isosceles type (a) triangle → folded to tetrahedron



Note that the foot of the perpendicular seems to be inside the base i.e. all 3 faces of the tetrahedron are leaning 'in'.

Figure 8

Acute isosceles type (b) triangle → folded to tetrahedron



Note that the foot of the perpendicular seems to be outside the base i.e. one face is leaning 'out'.

Figure 9

Nets of halved tetrahedron → folded to tetrahedrons (together)

Note that the foot of the perpendicular seems to be on an edge of the tetrahedron i.e. one face (the one that is halved) is perpendicular to the base.

One of the halved tetrahedrons

Another view – this is the 4^{th} face ΔDPQ

Figure 10

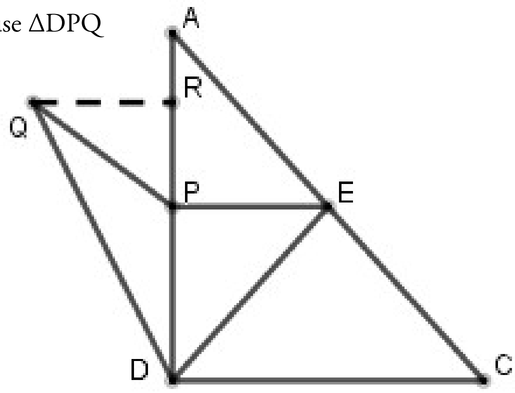
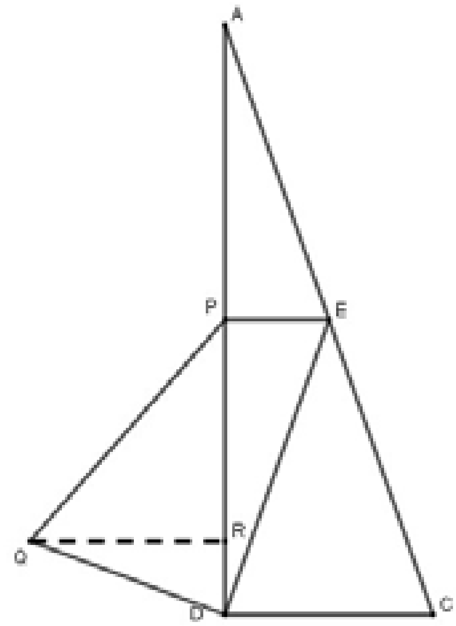
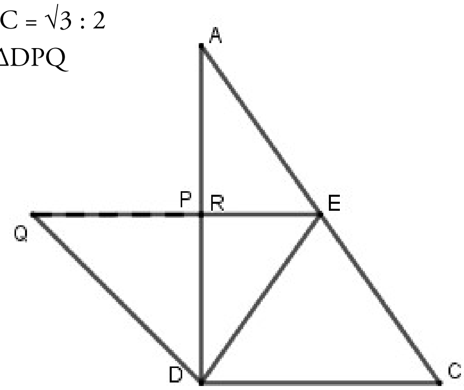
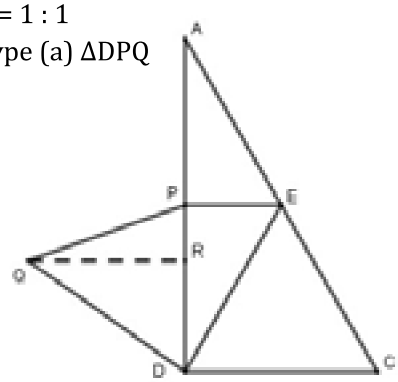
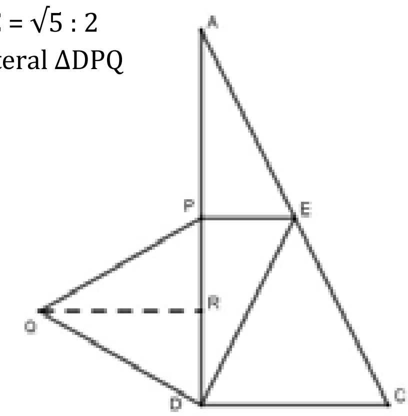
AC : BC = 3 : 4 Obtuse ΔDPQ  	AC : BC = 3 : 2 Acute type (b) ΔDPQ 
AC : BC = √3 : 2 Right ΔDPQ  	
AC : BC = 1 : 1 Acute type (a) ΔDPQ  	AC : BC = √5 : 2 Equilateral ΔDPQ 

Table 2

Having established the variance of R along the ray DP, the next questions are related to the special points of ΔDEF along PD, viz the centroid, the circumcentre, the incentre and the orthocentre of the base. Since ΔABC and therefore ΔDEF are acute triangles, all four of these points lie inside the base. The next task is about locating these special points on the net.

Task 6: Consider the net of the halved tetrahedron as shown in Figure 5. PD is the line of symmetry for the base ΔDEF. So, all four of the special points lie on PD. Locate each of these points on PD viz.

a. The centroid G of ΔDEF

b. The incentre I of ΔDEF

c. The circumcentre S of ΔDEF

d. The orthocentre O of ΔDEF

Teacher Note: The challenge is to find these points on the net that has only half of ΔDEF. So, properties of these points and the symmetry of isosceles triangle are to be utilized. One needs to draw the net on larger paper and not cut it out, so that the necessary constructions can be done.

The centroid G is a point on the median PD such that PG = 1/3 × PD. So, PD has to be trisected to find G. There is an alternative way: complete ΔDEF (such that P is the midpoint of EF) and construct one more median.

The incentre I lies on the angle bisectors. So, construct the bisector of ∠DEP and let it intersect PD at I.

Similarly, the circumcentre lies on the perpendicular bisector of each side. PD is the perpendicular bisector of EF. So, construct the perpendicular bisector of DE and let it intersect PD at S.

The orthocentre is a bit tricky. Extend EP to F so that FP = PE. Drop perpendicular from F to DE. Let it intersect PD at O.

All of these can be done on GeoGebra as well.

GeoGebra steps

(continued from before)

F: midpoint of AB

H: midpoint of DE, join FH

Centroid: G: intersection of FH and AD

b: angle bisector of ∠PED

In-centre: I: intersection of AD and b

c: perpendicular bisector of DE

Circumcentre: S: intersection of AD and c

d: perpendicular from F to DE

Orthocentre: O: intersection of AD and d

An interesting question at this point would be to explore if R coincides with any of these special points. In particular, are there different AC : BC ratios for each of these points? It can be worked out by computing various lengths and doing some tedious algebraic crunching for each of G, S, I and O. However, an alternative approach with the nets provides deeper understanding of the situation.

Task 7: Mark G, S, I, O and R on PD on all the nets corresponding to the five types of ΔDPQ. What do you observe?

Teacher Note: Considering all five nets and the five points marked in each of them, the following emerge:

1. All the five points coincide for the regular tetrahedron (as expected)
2. The points are always in the same sequence R-S-G-I-O
3. Including P and D, the order is D-R-S-G-I-O-P, for ∠ACB > 60° (Figure 11) and reverses to P-R-S-G-I-O-D for ∠ACB < 60° (Figure 12) – the pink cross indicates the position of A for regular tetrahedron i.e. ∠ACB = 60°

While 1 is quite obvious, it can be rigorously proved. The math hungry can be engaged with the task. The reverse question can also be posed to them i.e. finding AC : BC when R coincides with (i) G, (ii) S, (iii) I and (iv) O. Circumradius and inradius can be computed in terms of AB and BC.

Observation 2 is mostly known except for R. Those who are further interested can consider the various ratios along the line segment SO. Students may be introduced to the nine-point circle after this.

But in one step, this makes it clear that R coincides with all these points only for the regular tetrahedron. For any other tetrahedron or in other words if ∠ACB ≠ 60°, R remains outside the line segment SO.

These explorations start with something as basic as types of triangles which are 2D but soon leap into 3D. There it demands imagining a particular solid and unfolding the same to generate its net, and thus bringing it back to 2D. But the fun starts after that when multiple nets are created by varying an angle (or a side).

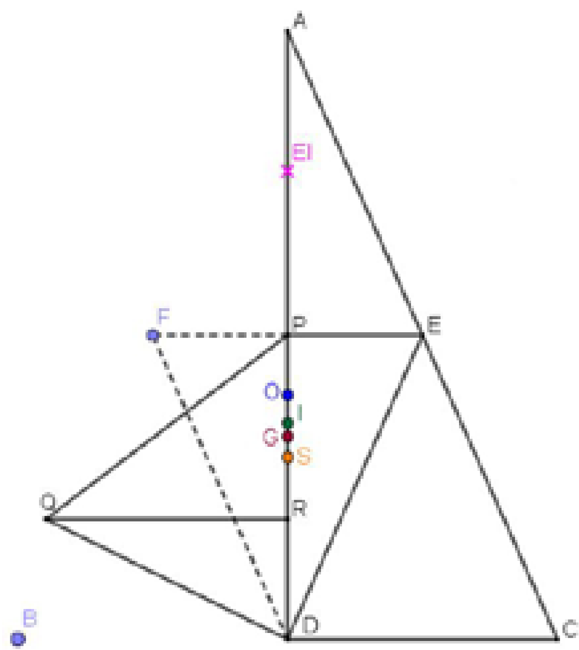


Figure 11

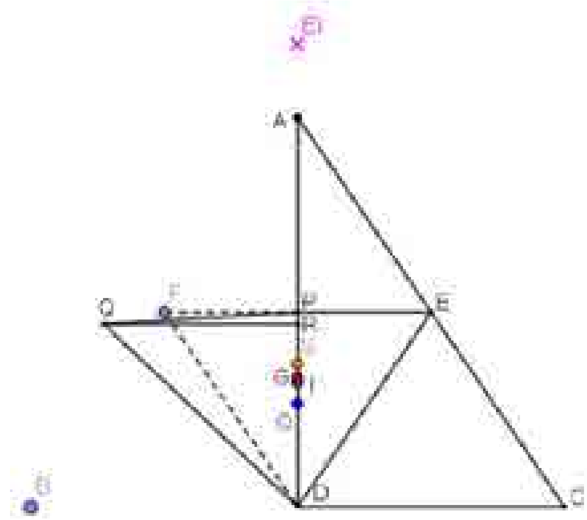


Figure 12

This angle (or side or the ratio of the sides) can be considered as an independent variable. Students get a glimpse of how other angles and position of some points (or some lengths) vary with these independent variables. This raises new questions especially about critical points when one parameter (e.g. angle or length) changes while some remain fixed. It also provides insight into the range of variation and the various possibilities.

We would like to leave the reader with a last question: Try Task 6 for the special points of ΔABC. You will be pleasantly surprised! We hope to dive into that in a subsequent article.

We would like to thank Dr. Prabuddha Chakraborty, Indian Statistical Institute for triggering the later parts of these explorations with the questions related to the foot of the perpendicular and the centres of the base triangle.

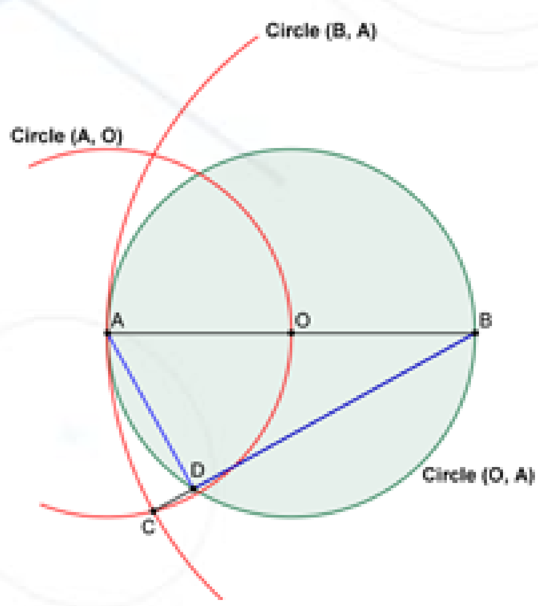
MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

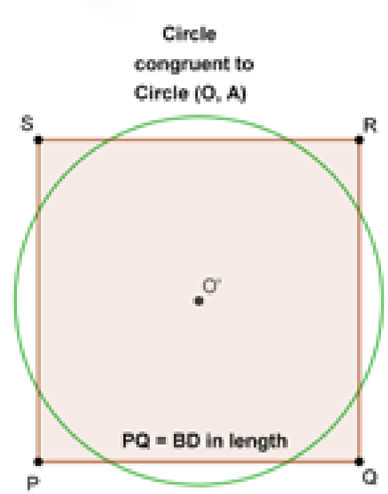
Squaring the Circle

GAURAV CHAURASIA

Construction steps

1. Draw Circle (O,A), with diameter AB and centre O.
2. Draw Circle (B,A), followed by Circle (A,O).
3. Mark C, one of the points of intersection of Circle (B,A) and Circle (A,O).
4. Join BC, and let it intersect Circle (O,A) at D.
5. Claim: BD^2 is almost equal to the area of Circle (O,A). [See the square at the right.]





Keywords: Circle, congruent, square, construction, error

Let AO = 1 unit, AB = 2 units; then BC = 2 units, and AC = 1 unit. Hence, triangle BAC is isosceles, with equal sides 2 units and base 1 unit. (In the figure, AC has not been joined.)

By construction, $\angle ADB$ is a right angle ("angle in a semicircle").

The length of the perpendicular from B to AC is $\sqrt{2^2 - \frac{1}{2^2}} = \frac{\sqrt{15}}{2}$. Therefore, the area of triangle BAC is $\frac{\sqrt{15}}{4}$ square units.

Hence, $\frac{1}{2} \cdot AD \cdot BC = \frac{\sqrt{15}}{4}$, implying that $AD = \frac{\sqrt{15}}{4}$.

Using Pythagoras's theorem, $BD^2 + AD^2 = AB^2 = 4$, so $BD^2 = 4 - \frac{15}{16} = 3\frac{1}{16}$.

Hence BD = $\frac{7}{4}$, i.e., BD is $\frac{7}{4}$ times the radius of the circle.

So the area of the square on side BD is $3\frac{1}{16} = 3.0625$, and the area of the circle on AB as diameter is $\pi \approx 3.1416$.

So the error percentage is approximately –2.51% . (The error is on the negative side.)

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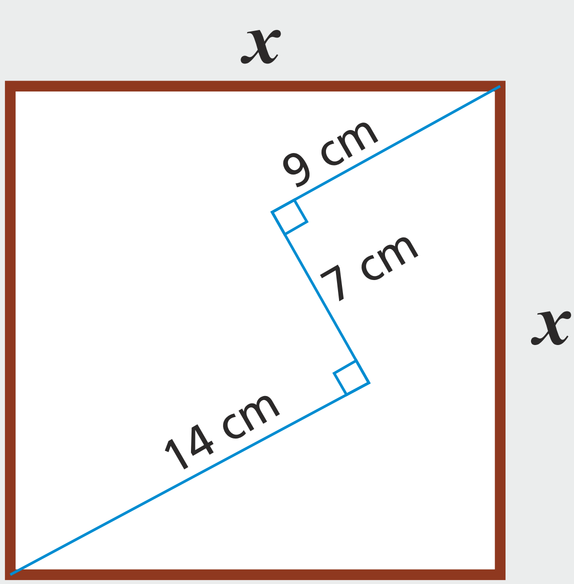
Think out of the box!



Find side x of the Square

Squaring the Circle

Editor's Note

Elsewhere in this issue, there are two articles dealing with geometric constructions, i.e., constructions using compass and ruler. One of the articles deals with the general theory behind such constructions, while the other deals with a particular construction problem from an earlier issue for which a reader had offered a solution. We would therefore like to give some background to these articles.

The ancient Greeks had invented a kind of game for themselves, in which they explored questions of the following sort: using only a compass and an unmarked straightedge (which we shall call a 'ruler' for convenience; note that it is *unmarked*), what geometric constructions are possible? Starting with two given points, with the distance between them forming the unit for measurement, what lengths can we construct? What angles can we construct? For example, can we construct a segment whose length is the square root of 2? Of course, we can; we only need to make use of Pythagoras's theorem. Can we construct an angle whose measure is 30°? Of course, we can. Can we construct an angle whose measure is 72°? We can, but this is far less obvious. Can we construct a segment whose length is the cube root of 2? It turns out that this is not possible (remember that we are constrained to use only a ruler and compass) – but this is very far from obvious! Can we construct an angle whose measure is 20°? Yet again, it turns out that this is not possible, and once again, it is very far from obvious why this should be the case. (You may wonder how we can ever know 'for sure' that certain constructions are just not possible. We will go into some of these questions in later articles.)

The ancient Greeks asked many such questions. In particular, they asked the following: Given a circle, is it possible to draw a square (using only a ruler and compass) whose area is equal to that of the circle? This question, along with a few others of its kind, troubled geometers for many centuries, until it was finally resolved in the 19th century. To the disappointment of many, it turned out that the stated task was not possible! This being so, we can at most ask for *approximate* constructions. That is, we can ask for a way to construct a square whose area is *close* to that of the circle (with not too large an error). The construction offered in the article by Gaurav is of this type.

A Property of the Modulo Operation in Number Theory

SOHAM PURKAIT

In this short note, I describe a result in number theory which I have discovered. Here is the statement.

Theorem. *Let x, a, k be integers, $k > 0$, such that the following is true:*

$$x \equiv a \pmod{k}.$$

Then the following is true for any positive integer b:

$$x^{k^b} \equiv a^{k^b} \pmod{k^{b+1}}.$$

Remark. Before proceeding, please note that x^{k^b} means $x^{(k^b)}$ and **not** $(x^k)^b$ (which is simply x^{kb}). Here, we follow the convention that any expression enclosed within brackets is evaluated first.

Example. To illustrate the theorem, let us start with the statement $5 \equiv 2 \pmod{3}$. Here $x = 5$, $a = 2$, and $k = 3$. We now observe the following relations which may be checked by actual calculation.

- Take $b = 1$. We have:

$$5^3 \equiv 2^3 \pmod{3^2}.$$

- Take $b = 2$. We have:

$$5^{3^2} \equiv 2^{3^2} \pmod{3^3}.$$

Proof. Suppose that $x \equiv a \pmod{k}$. We will build the proof of the statement in stages. Observe that:

$$x^k - a^k = (x - a)\left(x^{k-1} + x^{k-2}a + x^{k-3}a^2 + \cdots + a^{k-1}\right).$$

Keywords: Modulo, congruence, divisible, divisibility, number theory

Next, note the following relations which all follow from $x \equiv a \pmod{k}$ (note that the last relation is trivially true):

$$\begin{aligned} x^{k-1} &\equiv a^{k-1} \pmod{k}, \\ x^{k-2}a &\equiv a^{k-1} \pmod{k}, \\ x^{k-3}a^2 &\equiv a^{k-1} \pmod{k}, \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ x^1a^{k-2} &\equiv a^{k-1} \pmod{k}, \\ a^{k-1} &\equiv a^{k-1} \pmod{k}. \end{aligned}$$

Therefore, by addition of the corresponding sides, we obtain, since $ka^{k-1} \equiv 0 \pmod{k}$:

$$x^{k-1} + x^{k-2}a + x^{k-3}a^2 + \cdots + a^{k-1} \equiv 0 \pmod{k}.$$

This is the same thing as saying that k divides $x^{k-1} + x^{k-2}a + x^{k-3}a^2 + \cdots + a^{k-1}$. Since k also divides $x - a$, it follows that k^2 divides $x^k - a^k$, i.e.,

$$x^k - a^k \equiv 0 \pmod{k^2}.$$

We have thus proved the case $b = 1$ of the theorem.

To prove the case $b = 2$, we build on what we have just proved. Thus, we consider:

$$x^{k^2} - a^{k^2} = \left(x^k - a^k\right)\left(x^{k(k-1)} + x^{k(k-2)}a + x^{k(k-3)}a^2 + \cdots + a^{k(k-1)}\right).$$

Next, note the following relations which all follow from $x \equiv a \pmod{k}$ (as earlier, note that the last relation is trivially true):

$$\begin{aligned} x^{k(k-1)} &\equiv a^{k(k-1)} \pmod{k}, \\ x^{k(k-2)}a^k &\equiv a^{k(k-1)} \pmod{k}, \\ x^{k(k-3)}a^{2k} &\equiv a^{k(k-1)} \pmod{k}, \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ x^k a^{k(k-2)} &\equiv a^{k(k-1)} \pmod{k}, \\ a^{k(k-1)} &\equiv a^{k(k-1)} \pmod{k}. \end{aligned}$$

Therefore, by addition of the corresponding sides, we obtain, since $ka^{k(k-1)} \equiv 0 \pmod{k}$:

$$x^{k(k-1)} + x^{k(k-2)}a^k + x^{k(k-3)}a^{2k} + \cdots + a^{k(k-1)} \equiv 0 \pmod{k}.$$

That is, k divides $x^{k(k-1)} + x^{k(k-2)}a^k + x^{k(k-3)}a^{2k} + \cdots + a^{k(k-1)}$. Since k^2 also divides $x^k - a^k$, it follows that k^3 divides $x^{k^2} - a^{k^2}$, i.e.,

$$x^{k^2} - a^{k^2} \equiv 0 \pmod{k^3}.$$

We have thus proved the case $b = 2$ of the theorem.

The general case can now be proved inductively, using exactly the same line of reasoning. As each step is exactly of the same kind, we omit the details here.

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Pell's Equation

RAHIL MIRAJ

Introduction. In number theory, a Diophantine equation is one for which integer solutions are sought (or, sometimes, solutions in rational numbers). Typically, the number of variables is greater than the number of equations, allowing for the possibility of infinitely many solutions. The task of finding these solutions can sometimes be very challenging.

A very famous quadratic Diophantine equation is Pell's equation, which has the form

$$x^2 - dy^2 = 1. \tag{1}$$

Here, d is a given natural number which is not a perfect square, and x, y are required to be non-negative integers. We can see that the solution $x = 1, y = 0$ works for every value of d, and hence is not of much interest; we call it a 'trivial solution.' So we shall look only for nontrivial solutions of the equation.

This work is an extension of a previous work [2], where I had suggested some ways for finding the initial solution of Pell's equation.

If one nontrivial integral solution (x_1, y_1) is found, then one can find infinitely many nontrivial solutions from the following formula [1]:

$$\begin{aligned} x_n &= \frac{\left(x_1 + y_1\sqrt{d}\right)^n + \left(x_1 - y_1\sqrt{d}\right)^n}{2}, \\ y_n &= \frac{\left(x_1 + y_1\sqrt{d}\right)^n - \left(x_1 - y_1\sqrt{d}\right)}{2\sqrt{d}}. \end{aligned} \tag{2}$$

This formula is obtained by equating (respectively) the rational and the irrational parts on the two sides of the following (after expanding the expression on the right side):

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n. \tag{3}$$

Keywords: Diophantine equation, trivial solution, nontrivial solution, Python programming language, infinite loop

Illustrating the use of formula (3)**.**

d = 3: Here the equation is $x^2 - 3y^2 = 1$. An obvious initial solution is $x = 2, y = 1$. To generate more solutions, we consider the integral powers of $2 + \sqrt{3}$:

$$\left(2 + \sqrt{3}\right)^2 = 7 + 4\sqrt{3}, \text{ therefore } x = 7,\ y = 4 \text{ is a solution;}$$
$$\left(2 + \sqrt{3}\right)^3 = 26 + 15\sqrt{3}, \text{ therefore } x = 26,\ y = 15 \text{ is a solution; } \ldots.$$

d = 5: Here the equation is $x^2 - 5y^2 = 1$. An initial solution is $x = 9, y = 4$. To generate more solutions, we consider the integral powers of $9 + 4\sqrt{5}$:

$$\left(9 + 4\sqrt{5}\right)^2 = 161 + 72\sqrt{5}, \text{ therefore } x = 161,\ y = 72 \text{ is a solution;}$$
$$\left(9 + 4\sqrt{5}\right)^3 = 2889 + 1292\sqrt{5}, \text{ therefore } x = 2889,\ y = 1292 \text{ is a solution; } \ldots.$$

But it can be quite difficult to find an initial solution, particularly if d is large. Some kind of heuristic is needed.

A straightforward restatement of Pell's equation is the following. We need to look for integers $t > 1$ such that the quantity y given by

$$y = \sqrt{\frac{t^2 - 1}{d}} \tag{4}$$

is an integer. That is, we must have $d \mid t^2 - 1$, and the quotient $\frac{t^2-1}{d}$ must be a perfect square. Then, obviously, (x, y) is a solution to Pell's equation. The challenge is to find a feasible value of t in some easy manner. My explorations have helped me find some general heuristics.

- If d is of the form $k^2 + 1$, choose $t = 2k^2 + 1$; then $y = 2k$.
- If d is of the form $k^2 - 1$, choose $t = k$; then $y = 1$.
- If d is of the form $k^2 + 2$, choose $t = k^2 + 2$; then $y = k$.
- If d is of the form $k^2 - 2$, choose $t = k^2 - 1$; then $y = k$.

Here k is any positive integer. This heuristic works provided $k^2 - 2 \le d \le k^2 + 2$ for some integer $k \ge 2$. But it is difficult to choose the values of t for other forms of d.

To overcome this, I wrote a computer program, using the Python programming language. To see the code, please refer to the appendix.

The program starts with $t = 2$ and checks whether it works; if it does not, it increases the value of t by 1 and checks whether the new value of t works; and it continues in this manner indefinitely. The program thus runs in an infinite loop, with no upper limit; it keeps running till it finds a solution. So the program takes time according to the choice of d as well as the configuration of the computer.

Examples. We find these solutions of Pell equations using the computer program.

- For $d = 2$, we choose $t = 4$ and get $(x, y) = (3, 2)$.
- For $d = 3$, we choose $t = 3$ and get $(x, y) = (2, 1)$.
- For $d = 5$, we choose $t = 10$ and get $(x, y) = (9, 4)$.
- For $d = 33$, we choose $t = 24$ and get $(x, y) = (23, 4)$.
- For $d = 2019$, we choose $t = 675$ and get $(x, y) = (674, 15)$.

Comment. For some values of d, even the smallest solution has extremely large values of x, y, so it is not practical to look for a solution in this manner. For example:

- For $d = 85$, we get $(x, y) = (285769, 30996)$.
- For $d = 1000$, we get $(x, y) = (39480499, 1248483)$.
- For $d = 1729$, we get $(x, y) = (544796401, 13101974)$. (Here, 1729 is the famous 'Ramanujan number.')

Acknowledgments. I thank my father Prof. Dr. Farook Rahaman for illuminating discussions, and my computer teacher, Mr. Abhinandan Sarkar for helping me to develop the program.

Appendix: Python code

```
import math
d=int(input("d="))
t=2
while True:
    m=math.floor(math.sqrt(d))
    p=t-1
    q=(p*(t+1))/d
    x=t
    y=(math.sqrt(q))
    h=(math.floor(y))
    if m*m==d:
        print (d, "is the square of an integer.")
        break
    elif h*h==q:
        print("Here we have to choose t=",t)
        print("x=",x)
        print("y=",y)
        break
    else:
        t+=1
```

Comment. The 't' in the above code and the above analysis differs by 1 from the 't' in reference [2]. So 't = 2' in the above presentation corresponds to 't = 3' in the original reference [2]. Similarly, the expression $(t^2 - 1)/d$ above corresponds to the earlier $(t^2 - 2t)/d$, and so on. This has been done to simplify the presentation.

References

1. Titu Andreescu, Dorin Andrica, Ion Cucurezeanu, An Introduction to Diophantine Equation, Birkhäuser,(2010).

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Secondary

Student Corner

Study of an Inequality Problem from an Olympiad – Part 1

DRUHAN R SHAH & RAKSHITHA

During one of the meetings of a problem solving group in our school, we were given a problem from the British Mathematics Olympiad Round 2, 2004. We present three solutions to the problem. The first one was obtained during the session itself, and the other two were developed through discussions. In a follow-up article, we shall explore another BMO problem which we solved using a similar idea.

Problem 1. Given real numbers a, b, c with $a+b+c=0$, prove that

$$a^3+b^3+c^3>0 \quad \text{if and only if} \quad a^5+b^5+c^5>0.$$

Solution 1. This is an 'ad hoc' solution. Let $a+b+c=0$; then $c=-(b+a)$. We assume (without any loss in generality) that $a \geq b \geq c$. It follows that $a \geq 0$. If $a=0$, then $b=0=c$ as well, in which case we have $a^3+b^3+c^3=0=a^5+b^5+c^5$; so this case need not be considered. Hence we may as well suppose that $a>0$. We now have,

$$\begin{aligned} a^3+b^3+c^3 &= a^3+b^3-(a+b)^3 \\ &= -3ab(a+b). \end{aligned} \qquad (1)$$

Next,

$$\begin{aligned} a^5+b^5+c^5 &= a^5+b^5-(a+b)^5 \\ &= -5a^4b-10a^3b^2-10a^2b^3-5ab^4 \\ &= -5ab(a+b)\left(a^2+b^2+ab\right). \end{aligned} \qquad (2)$$

We claim that $a^2+ab+b^2>0$. For:

- If $b=0$, then $a^2+ab+b^2=a^2>0$, since $a>0$.
- If $b>0$, then $ab>0$, so $a^2+b^2+ab>0$.
- If $b<0$, then $c<0$ as well, since $c \leq b$ (by supposition). Since $b=-a-c>-a$, it follows that $|b|<|a|$. This implies $|ab|<|a^2|$, therefore, $a^2+ab>0$.

Keywords: Inequality, Olympiad, logarithms, symmetric function

Thus $a^2 + ab + b^2 > 0$ as claimed. Now, from (1) and (2), we observe that

$$a^5 + b^5 + c^5 > 0 \iff 5ab(a+b) < 0 \iff ab(a+b) < 0$$
$$\iff -3ab(a+b) > 0 \iff a^3 + b^3 + c^3 > 0.$$

Solution 2. Here we follow the method given in [2] (*Higher Algebra*, Hall and Knight). Consider the following identity

$$(1+ax)(1+bx)(1+cx) = 1 + qx^2 + rx^3,$$

where $q = ab + bc + ca$, $r = abc$ (there is no x-term since $a + b + c = 0$). Taking logarithms of both sides, we get:

$$\log(1+ax) + \log(1+bx) + \log(1+cx) = \log\left(1 + qx^2 + rx^3\right).$$

We now use the logarithmic series expansion (valid for all $|x| < 1$),

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

for each term. (Note. For the series expansion to work, we also need to have $|ax| < 1$, $|bx| < 1$, ..., $|rx^3| < 1$. This does not present a problem, because a, b, c, q, r are fixed quantities, and we can always take x to be small enough so that these inequalities are satisfied.) Equating the coefficients of x^n on both sides, we find that

$$(-1)^{n-1}\left(a^n + b^n + c^n\right)$$

is equal to n times the coefficient of x^n in

$$\left(qx^2 + rx^3\right) - \frac{1}{2}\left(qx^2 + rx^3\right)^2 + \frac{1}{3}\left(qx^2 + rx^3\right)^3 - \cdots.$$

Now put $n = 2, 3, 4, 5$ in the above; we get the following:

$$a^2 + b^2 + c^2 = -2q,$$
$$a^3 + b^3 + c^3 = 3r,$$
$$a^4 + b^4 + c^4 = 2q^2,$$
$$a^5 + b^5 + c^5 = -5qr.$$

We observe that

$$3\left(a^2 + b^2 + c^2\right)\left(a^5 + b^5 + c^5\right) = 3(-2q)(-5qr) = 30q^2r,$$

and also

$$5\left(a^3 + b^3 + c^3\right)\left(a^4 + b^4 + c^4\right) = 30q^2r.$$

Thus we have established the following beautiful relation: if $a + b + c = 0$, then

$$5\left(a^3 + b^3 + c^3\right)\left(a^4 + b^4 + c^4\right) = 3\left(a^2 + b^2 + c^2\right)\left(a^5 + b^5 + c^5\right).$$

Since $a^4 + b^4 + c^4$ and $a^2 + b^2 + c^2$ are never negative, and are 0 precisely when a, b, c are all 0, the assertion in the problem follows.

Solution 3. The strategy is to use the symmetric functions of the roots of a polynomial. First we denote for all $k \geq 0$,

$$S_k = a^k + b^k + c^k.$$

Thus we have $S_0 = 3$, $S_1 = 0$, $S_2 = a^2 + b^2 + c^2$, etc., and a, b, c are the roots of the cubic

$$f(x) = (x-a)(x-b)(x-c) = x^3 - S_1x^2 + (ab + bc + ca)x - abc,$$

i.e., the cubic

$$f(x) = x^3 + (ab + bc + ca)x - abc.$$

Therefore, we have

$$a^3 + (ab + bc + ca)a - abc = 0.$$

Multiplying by a^k, we get,

$$a^{k+3} + (ab + bc + ca)a^{k+1} - (abc)a^k = 0, \tag{3}$$

for $k \geq 0$. Similarly, we have

$$b^{k+3} + (ab + bc + ca)b^{k+1} - (abc)b^k = 0. \tag{4}$$

and

$$c^{k+3} + (ab + bc + ca)c^{k+1} - (abc)c^k = 0. \tag{5}$$

By adding these three relations, we get, for $k \geq 0$:

$$S_{k+3} = -(ab + bc + ca)S_{k+1} + (abc)S_k. \tag{6}$$

Put $k = 2$ in (6):

$$S_5 = -(ab + bc + ca)S_3 + (abc)S_2. \tag{7}$$

Now observe that

$$2(ab + bc + ca) = (a + b + c)^2 - \left(a^2 + b^2 + c^2\right) = 0 - S_2 = -S_2,$$

which simplifies to

$$ab + bc + ca = -\frac{1}{2}S_2.$$

and from $a^3 + b^3 + c^3 = 3abc$, we get

$$abc = \frac{1}{3}S_3.$$

Now, using these in equation (7), we have

$$S_5 = \frac{1}{2}S_2S_3 + \frac{1}{3}S_2S_3 = \frac{5}{6}S_2S_3 \tag{8}$$

Now note that $S_2 = 0$ implies $a = b = c = 0$. In that case, we get $S_3 = S_5 = 0$. If a, b, c are not all 0, then S_2 is a positive number, so S_5 is a positive number times S_3. It follows that

$$S_5 > 0 \iff S_3 > 0,$$

as required.

Comment from the editor. Observe that the strategy followed in the third solution is the same as that followed in the "extreme algebra" problem explored elsewhere in this issue.

Exercises for the reader

(1) Given real numbers a, b, c with $a + b + c = 0$, prove that

$$a^7 + b^7 + c^7 > 0 \iff a^5 + b^5 + c^5 > 0.$$

(2) Given real numbers a, b, c, d with $a + b + c + d = 0$, prove that

$$a^3 + b^3 + c^3 + d^3 > 0 \iff a^5 + b^5 + c^5 + d^5 > 0.$$

(3) Given real numbers a, b, c with $a + b + c = 0$, find all $k \in N$, such that

$$a^k + b^k + c^k > 0 \iff a^3 + b^3 + c^3 > 0.$$

(4) Given real numbers a, b, c with $a + b + c = 0$, find all $k, m \in N$, such that

$$a^k + b^k + c^k > 0 \iff a^m + b^m + c^m > 0.$$

References

1. UK Mathematics Trust, https://www.ukmt.org.uk/
2. H. S. Hall and S. R. Knight, *Higher Algebra*, Macmillan, 1964.

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A Friendly Pair of Triangles

AVIPSHA NANDI

Terminology. Given a triangle ABC, we say that a triangle PQR is 'inscribed' in ABC if the vertices of PQR lie on the sides of ABC, one vertex per side (e.g., if P lies on side BC, Q lies on side CA, and R lies on side AB). See Figure 1 for an example.

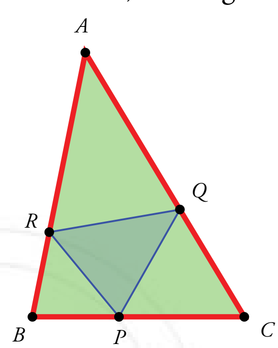


Figure 1.

Two nice inscribed triangles

In this article, which is a continuation of my earlier article [1], I show that for any given triangle ABC, there exist two triangles that are inscribed in it and have the following properties: the two triangles have equal area, and their centroids are equidistant from the centroid of $\triangle ABC$. As the two triangles are related in such a nice way, I call them a "friendly pair" of triangles.

The two triangles are the following.

- Let the incircle of $\triangle ABC$ touch the sides BC, CA, AB at points P, Q, R respectively. Join these three points together to form $\triangle PQR$.
- Let the ex-circle of $\triangle ABC$ opposite vertex A touch side BC at U. Similarly, let the ex-circle of $\triangle ABC$ opposite vertex B touch side CA at V. Finally, let the ex-circle of $\triangle ABC$ opposite vertex C touch side AB at W. Join these three points together to form $\triangle UVW$.

Keywords: Incircle, ex-circle, inscribed triangle, centroid, area formula, position vector

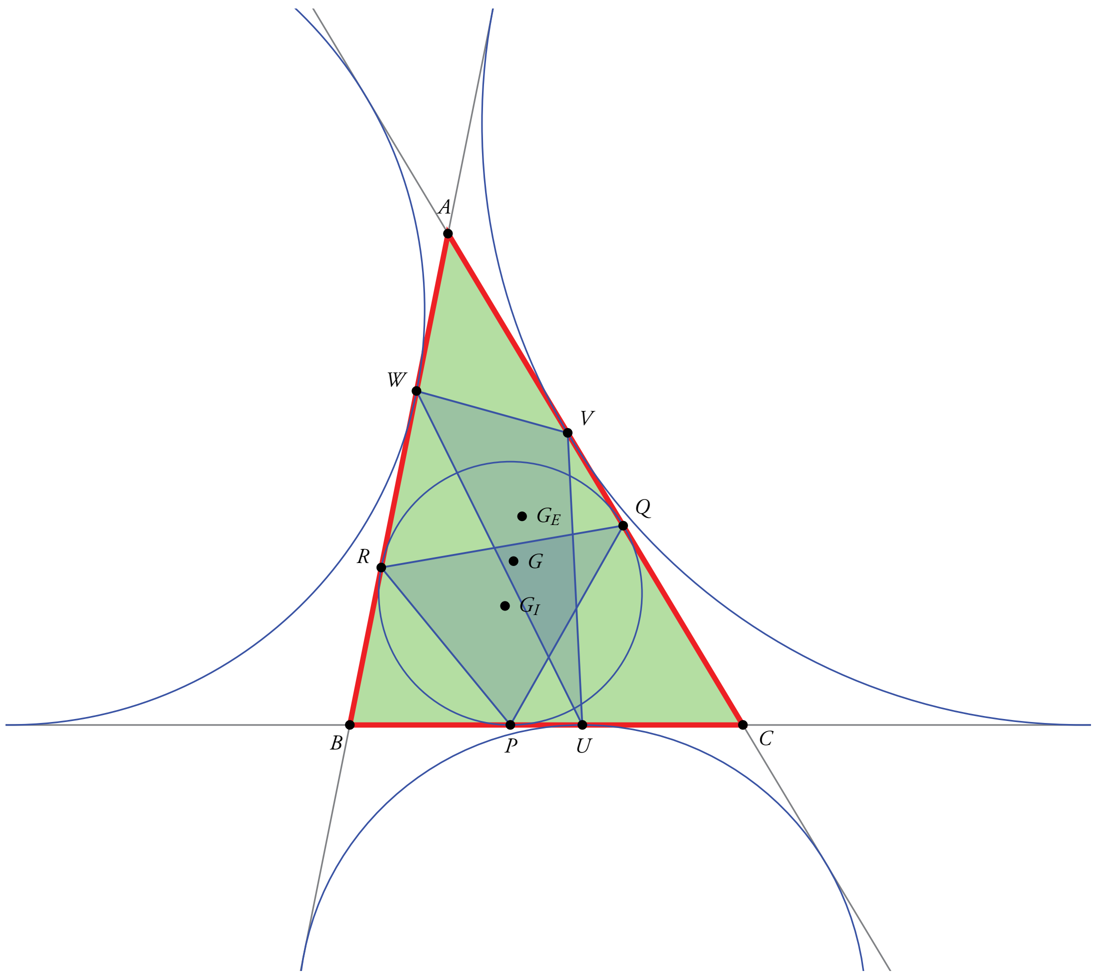


Figure 2.

Then PQR and UVW are a friendly pair of triangles. (See Figure 2.) That is, they have equal area, and their centroids are equidistant from G, the centroid of $\triangle ABC$.

Actually, more can be said. If G_I is the centroid of $\triangle PQR$, and G_E is the centroid of $\triangle UVW$, then G is the midpoint of the segment G_IG_E.

Proofs of the claims

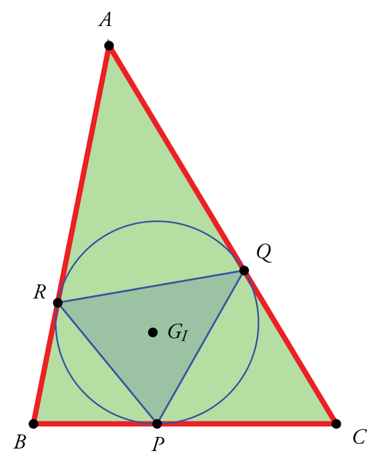
We use standard notation: a, b, c for the sides of the triangle; s for the semi-perimeter; Δ for the area; r for the radius of the incircle; and r_A, r_B, r_C for the radii of the three ex-circles.

Consider first the points where the incircle touches the sides of $\triangle ABC$ (see Figure 3). The lengths of BP, PC, CQ and so on are given at the side.

To see why $AQ = s - a$, denote AQ by x. Then $AR = x$, $BR = c - x$, $BP = c - x$, $CQ = b - x$, $CP = b - x$. Since $BP + CP = a$, we get $(b - x) + (c - x) = a$, so $2x = b + c - a = 2s - 2a$, and $x = s - a$. Similarly for the others.

It follows that:

$$\begin{cases} BP : PC = s - b : s - c, \\ CQ : QA = s - c : s - a, \\ AR : RB = s - a : s - b. \end{cases} \tag{1}$$



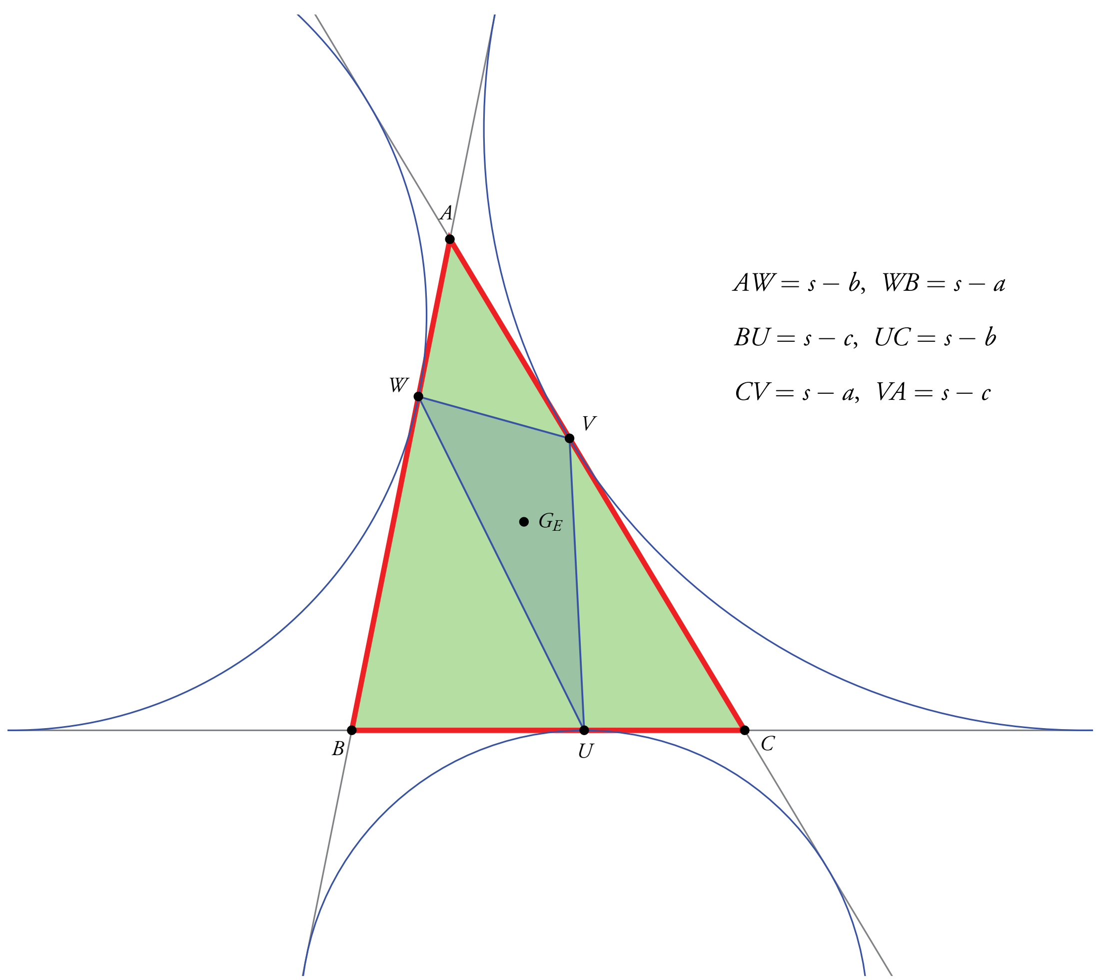
$AQ = AR = s - a$

$BP = BR = s - b$

$CP = CQ = s - c$

Figure 3.

Next, consider the points where the ex-circles touch the sides of $\triangle ABC$ (Figure 4). The lengths of BU, UC, CV and so on are given at the side.



$AW = s - b,\ \ WB = s - a$

$BU = s - c,\ \ UC = s - b$

$CV = s - a,\ \ VA = s - c$

Figure 4.

To see why $BU = s - c$ and $CU = s - b$, note that $AB + BU = AC + CU$, both sides being equal to the length of the tangent from A to the ex-circle opposite vertex A. Hence $c + BU = b + CU$. Also,

$BU + CU = a$. Therefore $c + BU = b + a - BU$, so $2BU = a + b - c = 2s - 2c$, and $BU = s - c$. The other relations follow in the same way.

$$\begin{cases} BU : UC = s - c : s - b, \\ CV : VA = s - a : s - c, \\ AW : WB = s - b : s - a. \end{cases} \tag{2}$$

From (1) and (2) we see that:

- P and U are symmetrically placed on segment BC (i.e., $BP = CU$), so the midpoint D of PU is also the midpoint of BC.
- Q and V are symmetrically placed on segment CA (i.e., $CQ = AV$), so the midpoint E of QV is also the midpoint of CA.
- R and W are symmetrically placed on segment AB (i.e., $AW = BR$), so the midpoint F of RW is also the midpoint of AB.

(Points D, E, F are not marked in the figures.)

The above relations may be expressed using vectors. Denoting the position vector (p.v.) of A by $\mathbf{A}$, the p.v. of B by $\mathbf{B}$, and so on, we have:

$$\begin{cases} \mathbf{P} + \mathbf{U} &= \mathbf{B} + \mathbf{C}, \\ \mathbf{Q} + \mathbf{V} &= \mathbf{C} + \mathbf{A}, \\ \mathbf{R} + \mathbf{W} &= \mathbf{A} + \mathbf{B}. \end{cases} \tag{3}$$

The centroids of $\triangle ABC$, $\triangle PQR$, and $\triangle UVW$ are G, G_I and G_E, respectively. Clearly:

$$\begin{cases} 3\,\mathbf{G} &= \mathbf{A} + \mathbf{B} + \mathbf{C}, \\ 3\,\mathbf{G_I} &= \mathbf{P} + \mathbf{Q} + \mathbf{R}, \\ 3\,\mathbf{G_E} &= \mathbf{U} + \mathbf{V} + \mathbf{W}. \end{cases} \tag{4}$$

It follows from (3) and (4), by addition, that

$$\mathbf{G_I} + \mathbf{G_E} = 2\,\mathbf{G}. \tag{5}$$

Therefore G is the midpoint of the segment connecting G_I and G_E.

Therefore the two centroids are equidistant from G.

Area computations. Now we show that the areas of triangles PQR and UVW are equal. The geometric result we use is the following. Given $\triangle ABC$, let points K and L lie on sides AB and AC respectively (see Figure 5). Then:

$$\frac{\text{Area of } \triangle AKL}{\text{Area of } \triangle ABC} = \frac{AK \cdot AL}{AB \cdot AC}. \tag{6}$$

This is true, because we have:

$$\frac{\text{Area of } \triangle AKL}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}\, AK \cdot AL \cdot \sin A}{\frac{1}{2}\, AB \cdot AC \cdot \sin A} = \frac{AK \cdot AL}{AB \cdot AC}.$$

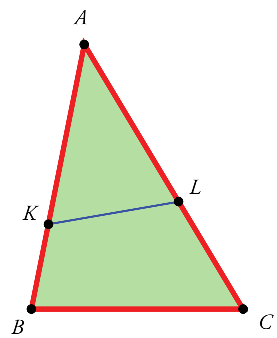


Figure 5.

Applying this result to Figure 3, we get, since the area of $\triangle PQR$ is equal to the area of $\triangle ABC$ minus the area of $\triangle BPR$ minus the area of $\triangle CQP$ minus the area of $\triangle ARQ$:

$$\begin{aligned}\frac{\text{Area of }\triangle PQR}{\text{Area of }\triangle ABC} &= 1 - \frac{(s-b)^2}{ca} - \frac{(s-c)^2}{ab} - \frac{(s-a)^2}{bc} \\ &= 1 - \frac{a(s-a)^2 + b(s-b)^2 + c(s-c)^2}{abc}. \qquad (7)\end{aligned}$$

Similarly applying the result to Figure 4, we get:

$$\begin{aligned}\frac{\text{Area of }\triangle UVW}{\text{Area of }\triangle ABC} &= 1 - \frac{(s-a)(s-c)}{ca} - \frac{(s-a)(s-b)}{ab} - \frac{(s-b)(s-c)}{bc} \\ &= 1 - \frac{(s-a)(s-b)c + (s-b)(s-c)a + (s-c)(s-a)b}{abc}. \qquad (8)\end{aligned}$$

Hence, proving that triangles PQR and UVW have the same area is equivalent to proving that the following two quantities

$$\begin{cases} a(s-a)^2 + b(s-b)^2 + c(s-c)^2, \\ (s-a)(s-b)c + (s-b)(s-c)a + (s-c)(s-a)b, \end{cases} \qquad (9)$$

are identically equal. Since $2s = a + b + c$, this is equivalent to showing that the following two quantities

$$a(b+c-a)^2 + b(c+a-b)^2 + c(a+b-c)^2, \qquad (10)$$

$$(b+c-a)(c+a-b)c + (c+a-b)(a+b-c)a + (a+b-c)(b+c-a)b, \qquad (11)$$

are identically equal. This can be done by directly simplifying both the expressions. But here is another way of proceeding. Look at the coefficients of each kind of term in (10) and (11). In both the expressions, every term has degree 3.

- The coefficient of a^3 is 1 in both (10) and (11). Likewise for b^3 and c^3.
- The coefficient of a^2b is $-2 + 1 = -1$ in both (10) and (11). Likewise for the terms a^2c, b^2a, b^2c, c^2a and c^2b.
- The coefficient of abc is $2 + 2 + 2 = 6$ in both (10) and (11).

As these are the only terms of degree 3 possible, it follows that expressions (10) and (11) are identically equal to each other. Therefore $\triangle PQR$ and $\triangle UVW$ have equal area.

By simplifying these expressions, we can express this area in terms of the elements of the triangle. We use the following identities. (For the proofs, please see the appendix.)

$$\begin{aligned} ab+bc+ca &= r^2+s^2+4Rr, \\ a^2+b^2+c^2 &= 2\left(s^2-r^2-4Rr\right), \\ a^3+b^3+c^3 &= 2s\left(s^2-3r^2-6Rr\right). \end{aligned}$$

Consider the expression $a(s-a)^2+b(s-b)^2+c(s-c)^2$ in (7). Let us simplify it using the above relations. We have, using the short forms $\sum a^3$ for $a^3+b^3+c^3$, $\sum a^2$ for $a^2+b^2+c^2$, and $\sum a$ for $a+b+c$:

$$\begin{aligned} & a(s-a)^2+b(s-b)^2+c(s-c)^2 \\ &= s^2\cdot\sum a-2s\cdot\sum a^2+\sum a^3 \\ &= 2s^3-2s\left(2s^2-2r^2-8Rr\right)+2s\left(s^2-3r^2-6Rr\right) \\ &= 2s\left(s^2-2s^2+2r^2+8Rr+s^2-3r^2-6Rr\right) \\ &= 2s\left(2Rr-r^2\right)=2sr(2R-r), \end{aligned}$$

i.e.,

$$a(s-a)^2+b(s-b)^2+c(s-c)^2=2\Delta(2R-r). \tag{12}$$

The expression $(s-a)(s-b)c+(s-b)(s-c)a+(s-c)(s-a)b$ simplifies to the same quantity, $2\Delta(2R-r)$. Hence:

$$\begin{aligned} \frac{\text{Area of }\triangle PQR}{\text{Area of }\triangle ABC} &= 1-\frac{a(s-a)^2+b(s-b)^2+c(s-c)^2}{abc} \\ &= 1-\frac{2\Delta(2R-r)}{abc}=1-\frac{2\Delta(2R-r)}{4R\Delta} \\ &= 1-\frac{2R-r}{2R}=\frac{r}{2R}. \end{aligned}$$

Acknowledgement. The author thanks her math mentor for his support, motivation and cooperation. She also thanks an anonymous referee for his/her kind suggestions, all of which led to a better presentation of this paper. Finally, she thanks her parents Subrata Kumar Nandi and Ranju Nandi for their continued support, motivation and cooperation.

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Appendix: proofs of some of the background results

Formulas connecting the lengths of the sides of the triangle.

$$ab+bc+ca=r^2+s^2+4Rr, \tag{13}$$

$$a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right), \tag{14}$$

$$a^3+b^3+c^3=2s\left(s^2-3r^2-6Rr\right). \tag{15}$$

Proofs. We take the following relations as well-known:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \Delta = rs, \qquad \Delta = \frac{abc}{4R}.$$

We start with the relation $rs = \sqrt{s(s-a)(s-b)(s-c)}$. Squaring both sides and dividing by s, we get:

$$\begin{aligned} r^2 s &= (s-a)(s-b)(s-c) \\ &= s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc \\ &= s^3 - 2s\cdot s^2 + (ab+bc+ca)s - 4R\cdot rs, \\ \therefore\ ab+bc+ca &= r^2 + s^2 + 4Rr. \end{aligned}$$

Next,

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4s^2 - 2\left(r^2+s^2+4Rr\right) \\ &= 2\left(s^2 - r^2 - 4Rr\right). \end{aligned}$$

To find an expression for $a^3+b^3+c^3$, we consider the cubic polynomial whose roots are a, b, c. Let the polynomial be

$$x^3 + ux^2 + vx + w. \tag{16}$$

Then we must have the following equalities for the coefficients u, v, w:

$$\begin{aligned} u &= -(a+b+c) = -2s, \\ v &= ab+bc+ca = r^2+s^2+4Rr, \\ w &= -abc = -4Rrs. \end{aligned}$$

From (16) we obtain:

$$x^3 = -ux^2 - vx - w,$$

that is,

$$x^3 = 2sx^2 - \left(r^2+s^2+4Rr\right)x + 4Rrs. \tag{17}$$

This equation must be satisfied by each of a, b, c, so we get, by substitution:

$$a^3 = 2sa^2 - \left(r^2+s^2+4Rr\right)a + 4Rrs,$$

and two other such equalities in which a is replaced by b and c in turn. By adding these three equalities, we get the following relation (where we have used the short forms $\sum a^2$ for $a^2+b^2+c^2$, and $\sum a$ for $a+b+c$):

$$\begin{aligned} a^3+b^3+c^3 &= 2s\cdot\sum a^2 - \left(r^2+s^2+4Rr\right)\cdot\sum a + 12Rrs \\ &= 4s\cdot\left(s^2-r^2-4Rr\right) - 2s\cdot\left(r^2+s^2+4Rr\right) + 12Rrs \\ &= 2s\left(s^2-3r^2-6Rr\right). \end{aligned}$$

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Secondary

TechSpace

Dynamic Geometry Software:

A Conjecture Making Tool

JONAKI GHOSH

Digital technologies have afforded many possibilities for the teaching and learning of mathematics and this has become a major area of research in Mathematics Education. A particular class of digital tools, known as Dynamic Geometry Software (DGS), allows mathematical concepts to be explored in a visual – dynamic way. In a DGS, geometrical figures or diagrams can be dragged and manipulated thus making them dynamic. The 'dynamism' of these figures provides opportunities to students to experience mathematical properties very differently from static figures which they generally experience in a traditional geometry class. In this article, we shall illustrate how middle school students experienced the angle sum property of a triangle, a fundamental geometrical idea, using GeoGebra, an open source DGS. A dynamic geometry software is also referred to as a dynamic geometry environment (DGE).

Keywords: DGS, geometry, algebra

Getting started with GeoGebra

GeoGebra enables the user to explore mathematical concepts using multiple representations. The reader may wish to download the software and explore. The steps for downloading the software are given at the end of the article. As soon as a new document is opened in GeoGebra, it offers a Graphics view, an Algebra view and an Input Bar at the bottom of the window (Figure 1).

Other views can be accessed using the View option on the toolbar (see Figure 2). For example, by selecting Spreadsheet from the View option one can bring in a spreadsheet as well (see Figure 3). The Graphics view, Algebra view and Spreadsheet view can be used simultaneously to explore properties of a figure geometrically, symbolically as well as numerically.

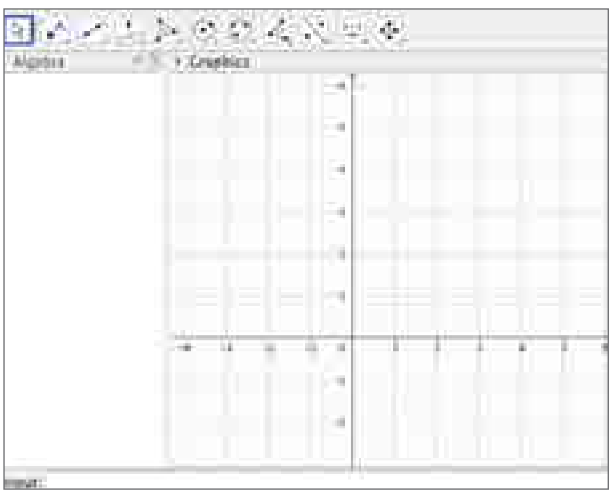

Figure 1. The Algebra view, Graphics view, and Input bar in a new GeoGebra document.

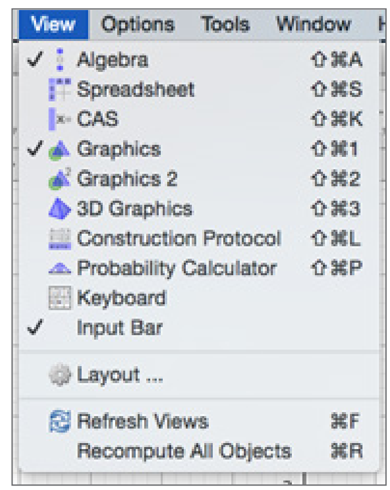

Figure 2. The View option allows the user to select different views.

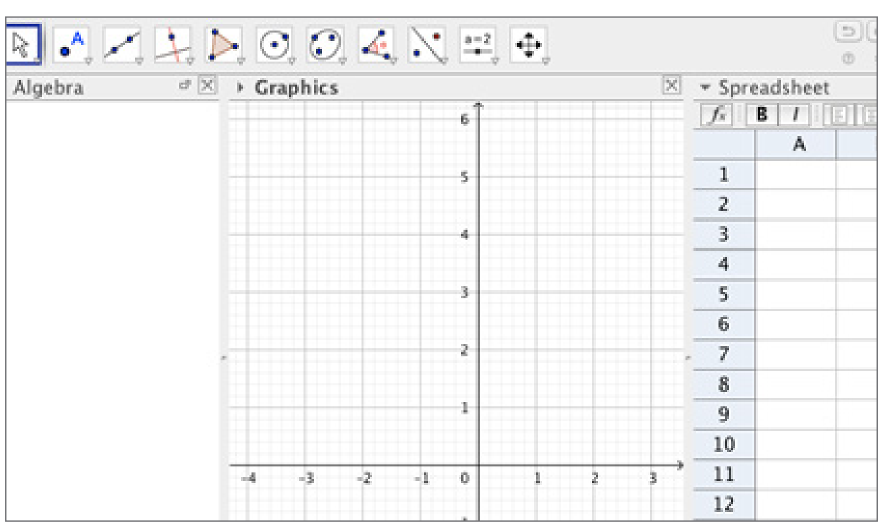

Figure 3. The Algebra view, Graphics view, and Spreadsheet view in a new GeoGebra document.

To understand this better let us draw a triangle in the Graphics view using the **Polygon tool** of the GeoGebra toolbar as shown in Figure 4. (To do this, select the Polygon icon from the tool bar and click on the Graphics view. GeoGebra will mark the first point as A. Proceed to draw a triangle by clicking on two other locations, which will be labelled as points B and C. To complete the triangle, return to point A and click on it again. This completes the triangle ABC.) GeoGebra immediately displays the coordinates of the vertices A, B, C, the lengths of the sides *a*,*b*,*c*, and the area of the triangle (displayed as t1) in the Algebra view. This reiterates the point that a triangle is a two-dimensional plane figure, a polygonal region in the plane, an aspect often neglected while drawing shapes using chalk on a blackboard!

Selecting one of the vertices, say A, and dragging it on the Graphics view will lead to varying experiences of the triangle while its measurements will change dynamically in the Algebra view. Figure 5 shows an obtuse angled triangle as well as (an almost) right-angled triangle obtained by dragging.

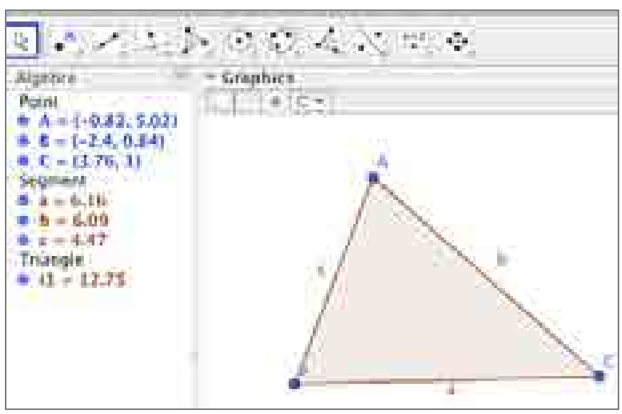

Figure 4. A triangle has been drawn using the Polygon tool whose parts are named and described in the Algebra view.

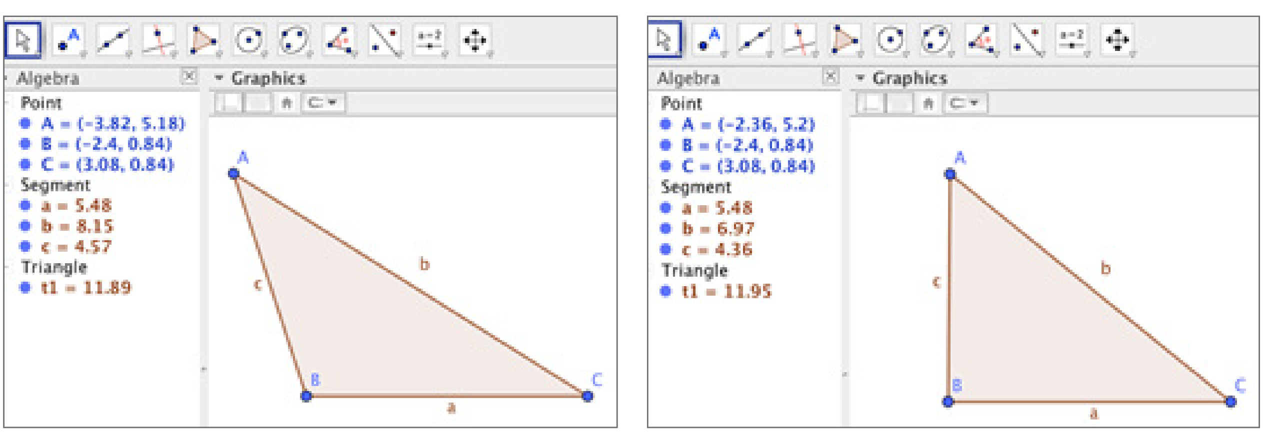

Figure 5. Varying appearances of a triangle obtained by dragging vertex A.

Dragging as a tool

The dragging feature is perhaps the most powerful aspect of a DGS. While dragging parts of a figure in GeoGebra, the Algebra view as well as the Graphics view reveal that certain attributes of the figure vary whereas others remain invariant. According to Leung (2012)

> A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities….. the variations of the moving image are perceived in contrast to what simultaneously remains invariant (p 2).

Being able to discern what varies and what remains invariant is key to experiencing a mathematical concept or property. This important feature can be exploited to enable students to make conjectures while performing geometrical explorations.

Leung (2003) explains

> …..when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of "mental animation", i.e. mentally visualizing variations of conceptual objects in hope of "seeing" patterns of variation or invariant properties (p 1).

To illustrate this aspect let us see how grade VI students abstracted the angle sum property of a triangle using the dragging feature in GeoGebra. Thirty five grade VI students in a school in New Delhi were being introduced to GeoGebra as a tool to explore geometrical concepts in their curriculum. The fact that the sum of the three interior angles of a triangle is equal to 180° was demonstrated by their teacher when they studied the topic of triangles. This was done through an activity where, given different triangles, students were required to measure the three angles (using a protractor) and obtain the sum. This task led to some confusion, as the angles did not turn out to be whole numbers (for some students) and the sum was close to 180 degrees but not exactly 180 degrees. In another activity students were given triangular cut-outs and were made to cut out the corners and bring them together to form a 'straight line angle.' Such visual representations were often treated as proofs in the classroom. However, it should be emphasized that such a model only helps to visualize the angle sum property and is not a proof in the mathematical sense.

The students were familiarized with the basic drawing tools of GeoGebra before they proceeded to explore the angle sum property. They worked in pairs and used the **Polygon Tool** to draw a triangle as shown in figure 5. Each pair of students dragged the vertices of their respective triangles and observed the changes in the Algebra view. In the following step, they were required to measure the angles of the triangle by using the **Angle tool**. Prior to this, they were instructed to go to **Options -> Rounding -> 0 decimal places** so as to avoid angle measures appearing in decimals. A click inside the triangle after selecting the **Angle tool** from the toolbar marked the angles inside the triangle as $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ along with their respective values. The teacher instructed the students to define a variable called **anglesum** and enter the following in the **Input Bar:**

anglesum = $\underline{\alpha} + \underline{\beta} + \underline{\gamma}$.

As a result of this step, anglesum = 180° appeared in the Algebra view (as shown in Figure 6).

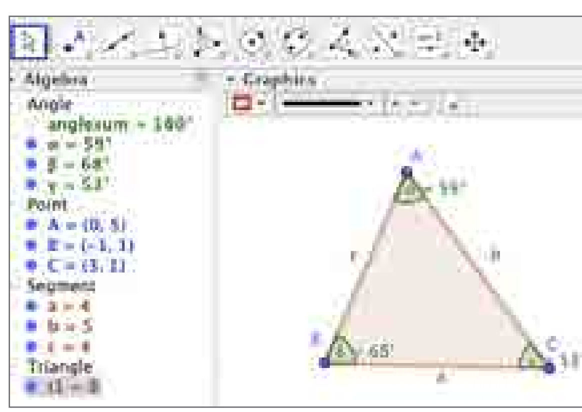

Figure 6. GeoGebra measures the sum of interior angles as 180° in the Algebra view.

Students were then asked to drag any vertex of their triangle and observe the Algebra view again. This time students noted that the anglesum remained constant (at 180°) for each type of triangle on their screen, while other aspects varied. Their observation of the invariance of the sum of the three angles amidst variation of the side lengths, the angle measures and the area of the triangle, led them to conjecture that $\underline{\alpha} + \underline{\beta} + \underline{\gamma}$ is 180°. Although students had explored the angle sum property through other activities, GeoGebra offered an exciting and different way to explore the problem, along with the added advantage of being able to examine many triangles very quickly to develop the idea.

To add another dimension to the exploration, the teacher asked the students to bring in a spreadsheet. The values of the angle measures $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ were entered in the columns A, B and C respectively after putting the headers $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ in the cells A1, B1 and C1 respectively.

Explorations using technology can often lead to surprises and a teacher using technology in her classroom must be prepared to deal with them. One of the student pairs had obtained a figure as shown in Figure 8 on their GeoGebra screen. When they clicked inside their triangle (after selecting the **Angle tool**), GeoGebra marked the reflex angles instead of the interior angles! This happens when the triangle is drawn in the anticlockwise sense. This unexpected output fascinated everyone as they eagerly tried to find out why GeoGebra produced this output. Further, they dragged the vertices of the triangle and conjectured that the angle sum (of reflex angles) remains constant at 900°. This was followed by a spreadsheet exploration, which confirmed their conjecture numerically. Figure 8 shows the output.

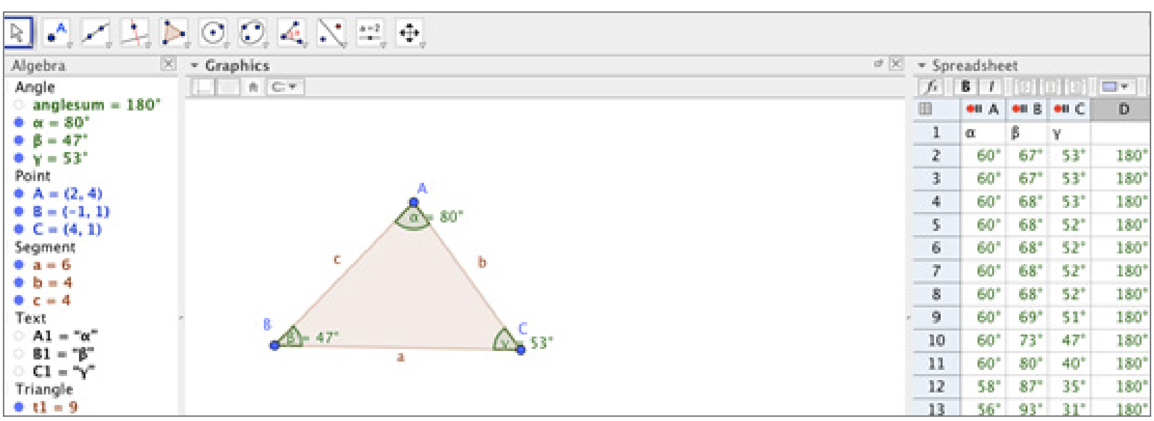

Figure 7. Algebraic, graphic and spreadsheet views illustrating the angle sum property.

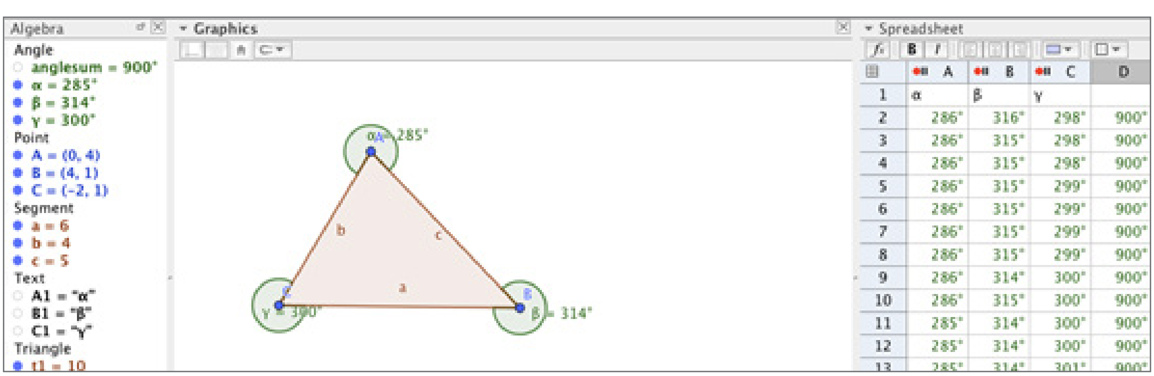

Figure 8. Algebraic, graphic and spreadsheet views, illustrating that the sum of reflex angles of a triangle is 900°.

Thus dragging the vertices of a triangle led to two invariants, namely the interior angle sum which is equal to 180° and the reflex angle sum which is 900°. The simultaneous occurrence of these invariants along with variations in various attributes of the triangle, such as the changing side lengths, coordinates of vertices and area was the key in making conjectures regarding the angle sum properties. The teacher asked the class to justify the new finding (regarding the reflex angle sum). Is the sum of reflex angles always 900°? If so, why? Can you explain? In order to scaffold students' thinking, she pointed out that at any vertex of the triangle, say at A, the sum of the interior and reflex angles is equal to 360°. Students carried forward the reasoning and arrived at the following explanation.

The total sum of interior and reflex angles of a triangle is 3×360° = 1080°. The interior angle sum is 180°, thus the reflex angle sum must be equal to 1080°- 180° = 900°. This was very satisfying for the teacher as students had used their reasoning to justify a new finding and had taken the lesson to a new level, even though she had not anticipated it or prepared for it.

One of the key features of a DGS is that students learn geometry in an environment where conjectures can be made and quickly examined. Students can use the Algebra and Spreadsheet views to test their conjecture while varying parts of a figure through dragging. A large number of examples can be explored in a relatively small amount of time and this helps to make generalizations regarding properties of geometrical figures. In the simple triangle exploration task described in this article, the students abstracted and generalized a pattern in the angles of a triangle and formed a conjecture regarding the angle sum property. Zoltan Dienes (1963) had explained this process of abstraction and generalization as the key components of mathematical thinking. According to him, mathematical learning is based on the perceptual variability principle as well as the mathematical variability principle:

> "The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved." (p.158)

Perceptual variability principle (also referred to as the multiple embodiment principle) suggests that children learn mathematical concepts best when they experience a concept through a variety of physical contexts in the form of concrete materials and manipulatives. Thus, to abstract the concept of the fraction ½, various physical embodiments such as circular cut-outs, rectangular rods, beads and counters may be used to represent the fraction in multiple ways. In the context of the triangle exploration by the grade VI children, experiencing the angle sum property by measuring angles of different triangles using a protractor and by using triangular cut-outs, led to perceptual variability of the angle sum concept.

The mathematical variability principle suggests that in order to abstract a concept, even the irrelevant attributes of a concept must be experienced to enable the learner to single out the general mathematical concept. For example, to abstract the concept of a square, the learner must be able to decipher that the equality of the sides and all angles being right angles are relevant invariant properties of a square whereas side length and orientation are not. Thus, when a square is experienced through mathematical variability, one can see squares of different side lengths and orientation, all sharing the common property of the equal sides and right angles. Returning to the context of the triangle exploration, GeoGebra allowed the students to vary the mathematical attributes of the triangle such as side lengths and angles (which are the irrelevant attributes) and observe that the angle sums (both interior and reflex) remain invariant. This helped them to quickly abstract and generalize the notion that the interior angle sum of a triangle is 180° while the reflex angle sum is 900°. The formal proof of these results are usually addressed at the secondary school stage. However, GeoGebra may be used to help students to visualise these results (and other similar results) through hands-on explorations. Using the approach suggested in this article the reader may try to verify that the sum of the interior angles of a quadrilateral is 360° and the reflex angle sum is 1080° and work towards a generalisation for polygons with n sides.

Through this article, we have tried to illustrate that the dragging feature of a dynamic geometry software such as GeoGebra, can enable students to explore properties of geometrical shapes. Varying different attributes of a shape brings to the fore its invariant properties amidst variation which, in turn, plays a pivotal role in forming conjectures. Conjecture making is an important aspect of learning geometry, as it is a precursor to argumentation and proof and can be easily accessible to students in a dynamic geometry environment.

Downloading GeoGebra

In order to download GeoGebra please proceed as follows

Step 1: Go to https://www.geogebra.org/download

Step 2: Click on **App downloads** on the left hand side of the screen

Step 3: Scroll down to **GeoGebra Classic 5** on the right hand side of the screen and click on **Download**.

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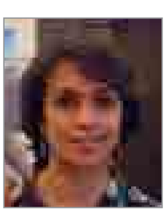

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Problems for the MIDDLE SCHOOL

A. RAMACHANDRAN

Problem-IX-2-M-1. All prime numbers except 2 are odd numbers. Two consecutive odd numbers can both be prime. Such a pair is termed 'twin primes.' Except for the set (3,5,7), three consecutive odd numbers cannot be prime, since one of them must be a multiple of 3.

Two sets of twin primes can occur separated by a composite odd number. That is, out of five consecutive odd numbers, suppose all except the middle one are prime. [That is, if we denote the middle number by x, then $x - 4$, $x - 2$, $x + 2$ and $x + 4$ are all primes.] Such a set of prime numbers is termed a prime quadruple. Show that, with the exception of the case (5,7,11,13), x must then be a multiple of 15. (The next few sets that do follow the above rule are (11,13,17,19), (101,103,107,109) and (191,193,197,199).)

Problem-IX-2-M-2. Going a step further from the situation of the earlier problem, one can find a string of 9 consecutive odd numbers, out of which 6 are prime (such a set is termed a prime sextuplet), in the pattern described below. [If x is again taken to be the middle member (which is composite), then $x - 8$, $x - 4$, $x - 2$, $x + 2$, $x + 4$ and $x + 8$ are all prime.] The first such example is (7,11,13,17,19,23). Show that in all such instances except for the above, x must be a multiple of 105. The next two such sextuplets are (97,101,103,107,109,113) and (16057,16061,16063,16067,16069,16073); they do follow this rule.

Problem-IX-2-M-3. Suppose that the GCD of two numbers a and b is 10 and their LCM is 4200 (we write this as GCD $(a,b) = 10$ and LCM $(a,b) = 4200$) , what are the possible values of a and b?

Keywords: Prime numbers, LCM, GCD

Problem-IX-2-M-4. Given the three positive integers 390, 462 and 1190, find:

i. their pairwise LCM's
ii. their pairwise GCD's
iii. the LCM of all three
iv. the GCD of all three.

Now show that the following relations hold for these numbers:

A. $\dfrac{\textit{Product of pairwise LCM's}}{\textit{Product of pairwise GCD's}} = \left\{\dfrac{\textit{LCM of all three}}{\textit{GCD of all three}}\right\}^2$

B. $\textit{Product of pairwise GCD's} \times \left\{\dfrac{\textit{LCM of all three}}{\textit{GCD of all three}}\right\} = \textit{product of given numbers}$

C. *GCD of the pairwise LCM's = LCM of the pairwise GCD's.*

Solutions

Pedagogical Note: As a certain amount of algebra is unavoidable when discussing the general case, we use it as sparingly as possible in the following explanations and highlight it in blue font. However, the teacher is advised to use as many numerical and visual examples as possible and to refer to the general case only when the student is absolutely sure of the logic being followed.

1. We are given five consecutive odd numbers, all greater than 5, of which the first two are prime numbers, the last two are prime numbers, and the number in the middle is composite. We must show that this middle number is a multiple of 15.

We will use these two facts: (i) out of three consecutive odd numbers, one and only one is a multiple of 3; (ii) out of five consecutive odd numbers, one and only one is a multiple of 5.

Call the five consecutive odd numbers A, B, C, D, E. We are told that A and B are prime numbers, and so are D and E. We must show that the middle number C is a multiple of 15.

Among the three consecutive odd numbers A, B, C, one and only one is a multiple of 3. This can only be C, as A and B are prime numbers.

Among the five consecutive odd numbers A, B, C, D, E, one and only one is a multiple of 5. This can only be C, as A, B, D and E are prime numbers.

This means that C is a multiple of 3 and also a multiple of 5. Therefore, it is a multiple of 15.

2. We have just shown that x is a multiple of 15. It is sufficient if we show that x is a multiple of 7. One and only one out of seven consecutive odd numbers must be a multiple of 7. So if we took a prime sextuplet as described and x is not a multiple of 7, then one of the other odd numbers must be a multiple of 7. You can try it as above. Suppose the centre number is not a multiple of 7 and one of the other numbers is. As one out of 7 consecutive odd numbers must be a multiple of 7, either $x - 6$ or $x + 6$ must be a multiple of 7. [If $x - 6$ is a multiple of 7, then $x + 8$ differs from $x - 6$ by 14, so $x + 8$ must be a multiple of 7. Similarly for $x - 8$ and $x + 6$ which differ by 14. As this contradicts what is given, we can say that x must be a multiple of 7, and thereby of 105.]

3. For the given numbers, there are 8 solutions:

(10,4200), (40,1050), (50,840), (30,1400), (70,600), (200,210), (120,350) and (280,150).

The general case can be argued as follows:

We can take the numbers a and b to be of the form $a = pq$ and $b = pr$, with q and r mutually prime. Then GCD$(a, b) = p$ and LCM$(a, b) = pqr$. We also have $\frac{LCM\ (a,\ b)}{GCD\ (a,\ b)} = qr$. Both a and b are multiples of p, the GCD. So we need to partition the prime factors of qr into two sets with no common factor, in as many ways as possible. Multiplying p with two complementary sets of prime factors of qr would yield a possible solution to (a, b). Students may be guided to observe that in the general case the number of solutions is given by 2^{d-1}, d being the number of *distinct* prime factors of qr.

4. The pairwise LCM's are 30030, 46410, and 39270. The pairwise GCD's are 6, 10, and 14. The LCM of all three numbers is 510510, while their GCD is 2. In relation A, both sides of the equation would equal 65,155,115,025. In relation B, both sides would equal 214,414,200. In relation C, both sides would equal 210.

Addendum: To prove that this is true in the general case, we note that three numbers could have (a) one common factor, (b) pairwise common factors and (c) factors unique to each of the given integers. So we could assume the numbers to be $P = ab_1b_2c_1$, $Q = ab_2b_3c_2$ and $R = ab_3b_1c_3$. From this we obtain the following:

$LCM_{PQ} = ab_1b_2b_3c_1c_2$
$LCM_{QR} = ab_1b_2b_3c_2c_3$
$LCM_{RP} = ab_1b_2b_3c_3c_1$

$GCD_{PQ} = ab_2$
$GCD_{QR} = ab_3$
$GCD_{RP} = ab_1$

$LCM_{PQR} = ab_1b_2b_3c_1c_2c_3$
$GCD_{PQR} = a.$

Proceeding from this it should be easy to show that the above relations hold true in the general case.

Secondary

Problem Corner

Problems for the Senior School

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem IX-2-S.1

The numbers 4 and 52 share the following features: both are sums of two squares; both exceed another square by 3. Thus:

$$4 = 0^2 + 2^2, \qquad 4 - 3 = 1^2;$$
$$52 = 4^2 + 6^2, \qquad 52 - 3 = 7^2.$$

Show that there are infinitely many numbers that have these two characteristics. [CRUX]

Problem IX-2-S.2

Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n. [CRUX]

Problem IX-2-S.3

Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with? [Kvant]

Problem IX-2-S.4

Note that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. Determine conditions for which $\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$, where a, b, c are positive integers. [CRUX]

Problem IX-2-S.5

Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs such that if the numbers in each pair are added, and the sums are then multiplied together, the result is a perfect square. [Tournament of Towns]

Keywords: Sums of two squares, divisible

Solutions of Problems in Issue-IX-1 (March 2020)

Solution to problem IX-1-S.1

The midpoints of two sides of a triangle are marked. How can the midpoint of the third side be found using only a pencil and a straightedge?

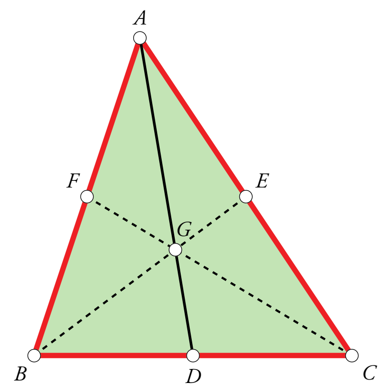

Figure 1.

Let us name the triangle ABC and let E and F be the midpoints of AC and AB, respectively (Figure 1). Join BE and CF and suppose G is their point of intersection. Since BE and CF are medians, G is the centroid of ABC. Join A and G and extend it to meet BC in D. AD is the median from A onto BC and therefore D is the midpoint of BC.

Solution to problem IX-1-S.2

Is it possible to cut several circles out of a square of side 10 cm, so that the sum of the diameters of the circles is 5 metres or more?

Yes, it is possible. Divide the square into 1 mm $\times$1 mm squares by drawing lines parallel to the adjacent edges of the given square. Inscribe circles into each of these $100 \times 100 = 10000$ squares. The sum of the diameters of all these circles is 10000 mm, i.e., 10 metres. Now cut them!!

Solution to problem IX-1-S.3

Suppose in a given collection of 2020 integers, the sum of every 100 of them is positive. Is it true that the sum of all the 2020 integers is necessarily positive?

Yes. To see this note that there cannot be more than 99 non-positive integers in the given collection. Otherwise the sum of some 100 of them will be non-positive contrary to the assumption. If $x_1, x_2, \ldots, x_k$ are the non-positive integers for some $k \leq 99$, then, by assumption,

$$(x_1 + x_2 + \cdots + x_k) + (x_{k+1} + x_{k+2} + \cdots + x_{100}) > 0.$$

The remaining integers (i.e., $x_{101}, x_{102}, \ldots$) are anyway positive, so the sum of all the 2020 integers is positive.

Solution to problem IX-1-S.4

Suppose integers a, b and c are such that $ax^2 + bx + c$ is divisible by 5 for any integer x. Prove that each of a, b and c is divisible by 5.

Setting $x = 0, 1, -1$ in succession, we find that c, $a + b + c$ and $a - b + c$ are divisible by 5. Therefore $(a + b + c) - (a - b + c) = 2b$ is divisible by 5, and since 2 and 5 have no factor in common, b is divisible by 5. Hence $a + c = (a + b + c) - b$ is divisible by 5. Hence $a = (a + c) - c$ too is divisible by 5. Hence a, b and c are all divisible by 5.

Solution to problem IX-1-S.5

The altitude dropped from A to BC in triangle ABC is not shorter than BC, and the altitude dropped from B to AC is not shorter than AC. Find the angles of triangle ABC.

Let X and Y be the feet of the altitudes dropped from A and B to BC and AC, respectively. Then $AX \geq BC$ and $BY \geq AC$. Also since AX and BY are altitudes, $AX \leq AC$ and $BY \leq BC$. Thus

$$AC + BC \leq AX + BY \leq AC + BC,$$

implying

$$AX + BY = AC + BC,$$

and therefore $AX = BC$, $BY = AC$. But then $BY = AC \geq AX$ and $AX = BC \geq BY$ implying $AX = BY$. Therefore:

$$AX = AC = BC = BY,$$

implying $X = Y = C$, i.e., X, Y, C coincide. Hence $\measuredangle C = 90^\circ$ and $\measuredangle A = \measuredangle B = 45^\circ$.

An 'Extreme Algebra' Question

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

We came across this problem in a Facebook post [2] which in turn pointed to a YouTube video [3]: *Given that a, b, c are numbers such that $a+b+c=1$, $a^2+b^2+c^2=2$ and $a^3+b^3+c^3=3$, find the value of $a^5+b^5+c^5$.*

In [3], the author describes this as an 'extreme algebra' problem, as the solution involves an 'extreme amount' of algebra. He solves the problem in a direct, straightforward manner, by multiplying the corresponding sides of the second and third equalities (thereby getting the expression $a^5+b^5+c^5$) and then somehow getting rid of the unwanted terms. (As we have already indicated, there is a substantial amount of algebra involved.)

We shall solve the problem by using an approach which draws on *the theory of equations*. This is an approach of great versatility and we highly recommend it to the reader.

A simpler example. We shall solve the following problem: *Given that a, b are numbers such that $a+b=1$ and $a^2+b^2=2$, find the value of a^5+b^5.*

Consider the quadratic equation whose roots are a and b. Let this equation be

$$x^2+ux+v=0. \tag{1}$$

The values of u and v may be found using the given data. Before doing so, let us see what we can glean from (1).

Multiplying through (1) by x^n, where n is any positive integer, we obtain:

$$x^{n+2}+ux^{n+1}+vx^n=0. \tag{2}$$

Keywords: extreme algebra, theory of equations, recurrence relations

Since a and b are solutions of (1), they must be solutions of (2) as well. Therefore we have:

$$\begin{cases} a^{n+2} + ua^{n+1} + va^n = 0, \\ b^{n+2} + ub^{n+1} + vb^n = 0. \end{cases} \tag{3}$$

Let $S_n = a^n + b^n$. (So $S_0 = 1 + 1 = 2$, $S_1 = a + b = 1$, $S_2 = a^2 + b^2 = 2$, and so on.) From (3), we obtain by addition, $S_{n+2} + uS_{n+1} + vS_n = 0$, i.e.,

$$\boxed{S_{n+2} = -uS_{n+1} - vS_n.} \tag{4}$$

The above relation, which shows how the members of the sequence $S_1, S_2, S_3, S_4, \ldots$ can be computed in terms of earlier members of the same sequence, is an example of a ***recurrence relation***. The study of such relations is of great importance in mathematics. See [1] for an introduction to this topic. A large number of such references can be found on the web.

Using the given data, $S_1 = 1$ and $S_2 = 2$, we may obtain as many terms as we wish of the sequence $S_1, S_2, S_3, S_4, \ldots$; we only need the values of u and v. To obtain these, we recall the theory of quadratic equations to deduce that

$$a + b = -u, \quad ab = v. \tag{5}$$

Since $a + b = 1$, we get $u = -1$; and since $a^2 + b^2 = 2$, we get

$$2ab = (a+b)^2 - \left(a^2 + b^2\right) = 1^2 - 2 = -1, \quad \therefore\ ab = -\frac{1}{2}, \quad \therefore\ v = -\frac{1}{2}.$$

It follows that

$$S_{n+2} = S_{n+1} + \frac{1}{2}S_n. \tag{6}$$

Using (6) repeatedly, we are able to compute successive terms of the sequence. We have displayed them in the table below.

n	0	1	2	3	4	5	6	7	8
S_n	2	1	2	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{3}{4}$	$6\frac{1}{2}$	$8\frac{7}{8}$	$12\frac{1}{8}$

We have described the method in detail, and we will now apply it to the given problem.

Back to the original problem. We return to the problem quoted at the start: *Given that a, b, c are numbers such that $a + b + c = 1$, $a^2 + b^2 + c^2 = 2$ and $a^3 + b^3 + c^3 = 3$, find the value of $a^5 + b^5 + c^5$.*

Let a, b, c be the roots of the following cubic equation:

$$x^3 + ux^2 + vx + w = 0. \tag{7}$$

By the factor theorem, the following identity must hold:

$$x^3 + ux^2 + vx + w = (x-a)(x-b)(x-c), \tag{8}$$

implying the following relations between $\{u, v, w\}$ and $\{a, b, c\}$:

$$\begin{cases} u &= -(a+b+c), \\ v &= ab + bc + ca, \\ w &= -abc. \end{cases} \tag{9}$$

Let $P_n = a^n + b^n + c^n$. Since $P_1 = 1$ (given), it follows that $u = -1$. Since $P_2 = 2$ (given), it follows from the identity

$$2(ab + bc + ca) = (a+b+c)^2 - \left(a^2 + b^2 + c^2\right) \tag{10}$$

that $2v = 1^2 - 2 = -1$, and so $v = -1/2$.

We now set up a recurrence relation for the sequence $P_1, P_2, P_3, \ldots$ just as we did earlier. Thus we have, from (7),

$$x^3 = -ux^2 - vx - w = x^2 + \frac{x}{2} - w.$$

Therefore, for all positive integers n,

$$x^{n+3} = x^{n+2} + \frac{x^{n+1}}{2} - wx^n. \tag{11}$$

Each of a, b, c must satisfy (11). Hence, summing over a, b, c, we obtain

$$P_{n+3} = P_{n+2} + \frac{P_{n+1}}{2} - wP_n. \tag{12}$$

We know that $P_3 = 3$ (given). Trivially, it is true that $P_0 = a^0 + b^0 + c^0 = 3$. Substituting in (12) with $n = 0$, we get

$$3 = 2 + \frac{1}{2} - 3w, \qquad \therefore \ w = -\frac{1}{6}.$$

It follows that:

$$\boxed{P_{n+3} = P_{n+2} + \frac{P_{n+1}}{2} + \frac{P_n}{6}.} \tag{13}$$

We are now in a position to compute as many members of the sequence as we wish. Using the recurrence relation (13) repeatedly, we obtain the table displayed below.

n	1	2	3	4	5	6	7	8	9	$\cdots$
P_n	1	2	3	$4\frac{1}{6}$	6	$8\frac{7}{12}$	$12\frac{5}{18}$	$17\frac{41}{72}$	$25\frac{5}{36}$	$\cdots$

This yields the required answer, $P_5 = 6$, but as may be observed, we have obtained much more.

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2. Valdosta maths club, https://www.facebook.com/groups/valdostamathclub/permalink/855020178186192/

3. Extreme algebra question, https://www.youtube.com/watch?v=1TBVeuOcY1w&fbclid=IwAR0ceGeNmSjm1vE_Y-iHz8hNNNtSmZJoZnpbKmCBrSIRv0wD9xfCPaW4nLE

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Secondary

Problem Corner

Partial Sum of Consecutive Integers Equal to a Square

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

In this short note, we study the following problem.

> *For which positive integers n is it true that the sum of the positive integers from 1 till n is a perfect square?*

Since the sum of the positive integers from 1 till n is $\frac{1}{2}n(n+1)$, the problem may be restated as follows: find all pairs (m, n) of positive integers satisfying the following equation:

$$\frac{n(n+1)}{2} = m^2. \tag{1}$$

To solve this equation, we start with the following simple observations:

- For any positive integer n, the integers n and $n+1$ are co-prime (i.e., they have no factors in common other than 1).
- Precisely one of the integers n and $n+1$ is even.
- If the product ab of two co-prime positive integers a and b is a perfect square, then both a and b are perfect squares.

Hence the following may be stated:

(i) If n is even, then $\frac{1}{2}n$ and $n+1$ are co-prime integers, so if $\frac{1}{2}n(n+1)$ is a perfect square, then both $\frac{1}{2}n$ and $n+1$ are perfect squares.

(ii) If n is odd, then n and $\frac{1}{2}(n+1)$ are co-prime integers, so if $\frac{1}{2}n(n+1)$ is a perfect square, then both n and $\frac{1}{2}(n+1)$ are perfect squares.

Keywords: Partial sum, consecutive integers, perfect square, coprime

It follows that there are two categories of positive integers n for which the sum of the positive integers from 1 to n is a perfect square, namely:

(i) n is even and both $\frac{1}{2}n$ and $n+1$ are perfect squares. This means that we have $n = 2x^2$ and $n+1 = y^2$ for some positive integers x and y.

(ii) n is odd and both n and $\frac{1}{2}(n+1)$ are perfect squares. This means that we have $n = x^2$ and $n+1 = 2y^2$ for some positive integers x and y.

In case (i) we have $y^2 - 2x^2 = 1$, and in case (ii) we have $x^2 - 2y^2 = -1$. Observe that both these equations are of the following kind:

$$u^2 - 2v^2 = \pm 1, \tag{2}$$

where u and v are positive integers. So we must solve equation (2) over the positive integers.

This is a familiar equation; we have met it many times in the past. One way of generating the solutions is to consider the powers of the irrational number $1+\sqrt{2}$. To be specific, let k be any positive integer, and let the quantity

$$\left(1+\sqrt{2}\right)^k$$

simplify to $u + v\sqrt{2}$, where u and v are integers. Then we may show that $u^2 - 2v^2 = \pm 1$, thus providing a solution to (2). Moreover, *every* solution to (2) may be obtained in this manner, simply by giving different values to k. For example,

- $k = 1$ yields $u = 1$ and $v = 1$. Here $u^2 - 2v^2 = -1$, so we have $n = u^2 = 1$. This corresponds to the not-particularly-interesting relation $1 = 1^2$.
- $k = 2$ yields $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, i.e., $u = 3$ and $v = 2$. Here $u^2 - 2v^2 = 1$, so we have $n = 2v^2 = 8$. This corresponds to the first interesting instance of the property we are looking for:
$$1 + 2 + 3 + \cdots + 7 + 8 = \frac{8 \times 9}{2} = 36 = 6^2.$$
- $k = 3$ yields $(1+\sqrt{2})^3 = 7 + 5\sqrt{2}$, i.e., $u = 7$ and $v = 5$. Here $u^2 - 2v^2 = -1$, so we have $n = u^2 = 49$. This corresponds to the relation
$$1 + 2 + 3 + \cdots + 48 + 49 = \frac{49 \times 50}{2} = 1225 = 35^2.$$

Proof that the procedure works. The most effective way of showing that this procedure invariably yields a solution is through induction. However, on this occasion we opt to use a non-inductive approach

As is well-known, if we have

$$(1+\sqrt{2})^k = u + v\sqrt{2}, \tag{3}$$

where u, v are integers, then we also have

$$(1-\sqrt{2})^k = u - v\sqrt{2}. \tag{4}$$

The most direct approach now is to make use of the following two facts:

$$(1+\sqrt{2}) \cdot (1-\sqrt{2}) = -1,$$

and

$$(u + v\sqrt{2}) \cdot (u - v\sqrt{2}) = u^2 - 2v^2.$$

We now get, from (3) and (4):

$$u^2 - 2v^2 = (-1)^k = \pm 1,$$

as required.

A slightly more cumbersome approach is to find explicit expressions for u and v in terms of k, and then to work with those expressions. For convenience, we write $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. From (3) and (4), we obtain, by addition and subtraction respectively:

$$u = \frac{\alpha^k + \beta^k}{2},$$

and

$$v = \frac{\alpha^k - \beta^k}{2\sqrt{2}}.$$

Hence:

$$\begin{aligned} u^2 - 2v^2 &= \frac{\alpha^{2k} + 2(\alpha\beta)^k + \beta^{2k}}{4} - \frac{\alpha^{2k} - 2(\alpha\beta)^k + \beta^{2k}}{4} \\ &= (\alpha\beta)^k = (-1)^k, \quad \text{since } \alpha\beta = -1. \end{aligned}$$

Though this approach is, as noted, more cumbersome, it has the advantage of yielding explicit expressions for u and v in terms of k.

Every solution? It remains to be shown that the above procedure generates every possible solution to the given problem. As one may expect, this part is more challenging.

One approach is to use the above idea in reverse. This enables us to generate solutions using smaller numbers from solutions using large numbers. To make this more clear, we consider the identity

$$(u + v\sqrt{2}) \cdot (1 + \sqrt{2}) = (u + 2v) + (u + v)\sqrt{2}.$$

This shows that from a solution $(x, y) = (u, v)$ to the equation $x^2 - 2y^2 = \pm 1$, where u, v are positive integers, we may generate another positive integral solution $(x, y) = (u + 2v, u + v)$, and this clearly features larger numbers than the original solution. For example, starting with the solution $(x, y) = (1, 1)$ and iterating the map $(u, v) \longmapsto (u + 2v, u + v)$, we obtain the following infinite chain of solutions:

$$(1,1), \quad (3,2), \quad (7,5), \quad (17,12), \quad (41,29), \quad (99,70), \quad \ldots.$$

Now we apply this idea *in reverse*. Write $u' = u + 2v$ and $v' = u + v$. Then clearly:

$$u = 2v' - u', \quad v = u' - v'.$$

From this we infer that given a solution $(x, y) = (u, v)$ to the equation $x^2 - 2y^2 = \pm 1$, where u, v are positive integers, we may generate another solution $(x, y) = (2v - u, u - v)$, and this features *strictly smaller numbers* than the original solution.

Now note the following (here u, v are non-negative integers):

- If $u^2 - 2v^2 = \pm 1$ and $v > 1$, then $2v > u > v > 1$.
- If $u^2 - 2v^2 = \pm 1$ and $v > 1$, then $(u, v) > (1, 1)$ and also $(2v - u, u - v) > (1, 1)$.

Reasoning in this manner, we see that starting with any solution to the given equation $x^2 - 2y^2 = \pm 1$ and iterating the map described above, $(u, v) \longmapsto (2v - u, u - v)$, we obtain a decreasing sequence of solutions. As it is not possible to have an infinitely long strictly decreasing sequence of positive integers, it must happen at some stage that we reach a solution with $v = 1$, which means that we have reached the solution $(1, 1)$, at which point the decrease necessarily comes to a halt.

Reversing the map again, we infer that every solution to the equation belongs to the chain shown above:

$$(1,1), \quad (3,2), \quad (7,5), \quad (17,12), \quad (41,29), \quad (99,70), \quad \ldots.$$

But this implies that the procedure we have described does generate every possible solution to the problem. No solution is missed out.

Recalling the connection between the integer pairs (u, v) for which $u^2 - 2v^2 = \pm 1$ and the integers n for which the sum of the positive integers from 1 till n is a perfect square (namely: if v is odd, then $n = u^2$, and if v is even, then $n = 2v^2$), we see that the integers n for which the sum of the positive integers from 1 till n is a perfect square are the following:

$$1, \quad 8, \quad 49, \quad 288, \quad 1681, \quad 9800, \ldots.$$

Moreover, this is a complete list; no solution is missed out.

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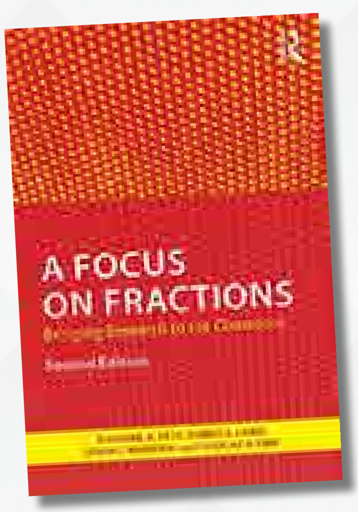
Book Review

A Focus on Fractions – Bringing Research to the Classroom

By Marjorie M. Petit, Robert E. Laird, Edwin L. Marsden
Reviewed by Rajat Sharma



The reviewer works with the Azim Premji Foundation and in the course of his work on content development, he had to review a module made on *Operations on Fractions*. This book - he had earlier read its first chapter - was recommended as support for the task. A complete read led to his understanding of the usefulness of such books and this review is intended to share his experience. All illustrations used have been reproduced from the book.

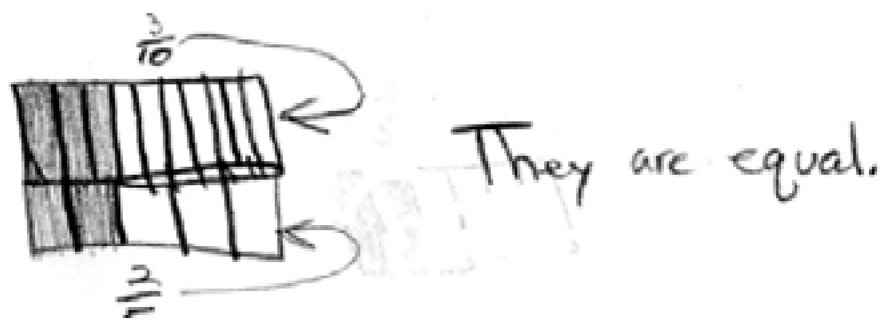
The relevance of this book to my work is due to the fact that it is a terrific blend of conceptual understanding of fractions and the challenges faced in classrooms by the teacher during transaction of this concept. It made me re-visit my experiences in the field of education. This book has dual insights- data from cited research studies as well as from classroom work and is a valuable resource both for teachers as well as for teacher educators. Threads have been tied beautifully from start to finish of each chapter.

Chapter One (Modelling and Developing Understanding of Fractions) is a must-read for everyone who has earlier struggled during their own school days or is now struggling as a teacher/ teacher educator while dealing with the concept of fraction. Mathematical models or in the authors' words "mental maps," are used as tools to understand the concept and generalize the concerned mathematical idea. For fractions, the three most commonly used models are the Set Model (collection of discrete objects), the Linear Model (or Number Line) and the Area Model. I was able to see how well all three models complemented each other for the first time – as a student, these appeared to me as different, unrelated topics. I needed such a detailed and precise understanding for my daily work with government school teachers. For example, if one is drafting a worksheet for participants with regards to the meaning of fraction, one could easily do a diagnostic test with questions based on all three models

Keywords: Fractions, models, representation, misconception, pedagogy

and according to the responses, make out how to proceed with his/her session. In many classrooms around the country, fraction is restricted to only one model (generally, the area model and specifically, area of a circle and/or rectangle). One may ask why all three models are needed - when students know all of them, they learn to select and use the model most applicable to the problem that they are solving.

In a few instances, where the fine motor skills of younger students are yet to be fully developed or when older students are dealing with fractions which are not significantly different from each other, models fail to make their expected impact and that's where manipulatives come into the picture. For example, when a student was asked to compare 3/10 to 2/5, he drew this



and wrote that they are equal. For such instances, manipulatives – such as Cuisenaire rods, paper folding, pattern blocks – and usage, have been brought into the picture.

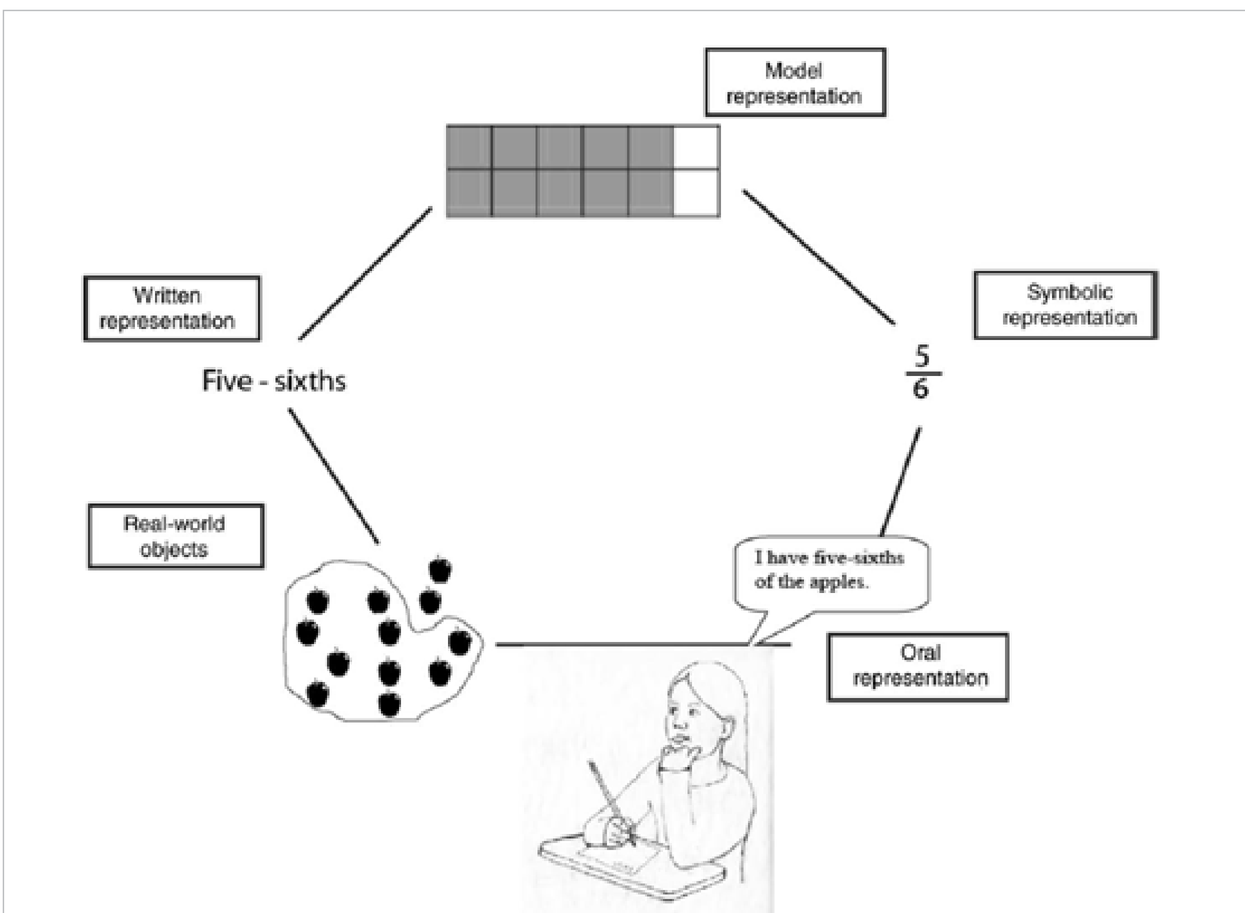
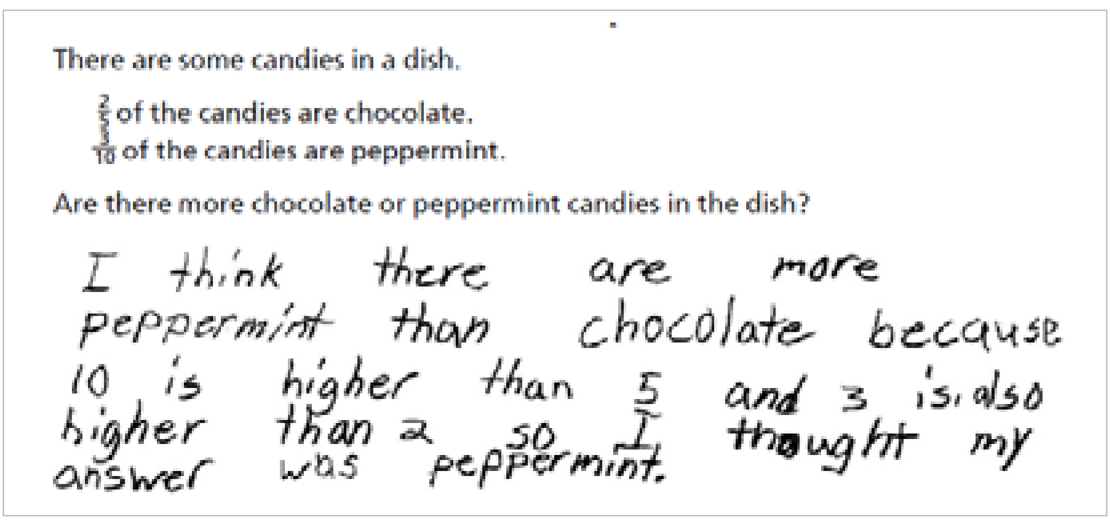


Figure 1 - Five ways to represent the concept of Fraction

Not that modelling or manipulatives are self-sufficient - a big point that came out of this chapter was that modelling is only a means to understanding the concept of fraction and is, very certainly, not the end.

> *"Learning is facilitated when students interact with multiple models that differ in perceptual features causing students to continuously rethink and ultimately generalize the concept." -* ***(Dienes, cited in Post & Reys, 1979)***

Chapter Two deals with a common misconception (or rather a preconception) that most of the students beginning the study of fractions develop. This is applying the analogy of the concept of whole number. In Chhattisgarh where I work, a student is first introduced to fractions in the third standard, right after s/he has spent close to two and a half years forming concepts around whole numbers. So as soon as a fraction is presented to such a student, and if it hasn't been handled well, they start seeing fraction as two whole numbers rather than as a single quantity. When a student is presented with the number ½, s/he considers it to be two numbers – 1 and 2. It's all downhill from there and they start lagging behind as there seems to be no cohesion between concepts already formed in their minds. In the next year, where they begin operations on fractions, students start to shy away from the subject of Mathematics altogether. The book cities an example from their research-



I really do empathise with such students. To quote the position paper on Teaching of Mathematics (2006), by NCERT -

> *"There is some evidence that the introduction of operations on fractions coincides with the beginnings of fear of mathematics."*

Which brings us to the all-important question that is also the title of Chapter Three (What is the Whole?) Could you define the whole, for me? Is it the Earth or the Milky Way galaxy or could it be just one chapatti? Behr and Post said - *"The concept of the whole underlies the concept of a fraction." (Behr & Post, 1992, p. 13)*. A fraction is nothing if not for the whole. Finding/identifying the whole when there is more than one part is often a challenge for students, as research has found. Not being able to visualize the whole becomes a major hurdle for students and when they start losing "sight" of the whole then they also possibly stop thinking about the meaning of the fraction in a given context. For example, students are taught that half is less than a whole. But half of a bag of 20 chocolates is greater than a whole bag of 8 chocolates!

Many students find similarities between fractions and the operation of division for several reasons, the symbolic representation predominant among them. Part of it is theoretically linked via Chapter Four of this book – Partitioning. Equal division of the whole is an important concept to be developed by students whatever the model used may be. This gets a notch higher when they're solving problems with several operations or solving problems through "partitioning in their heads," a catchy phrase that they have used.

Comparing and Ordering of Fractions come next. According to the authors, researchers have found that students use fives types of reasoning when they're able to correctly compare and order fractions –

- *Reasoning with unit fractions*
- *Extension of this reasoning to non-unit fractions*
- *Reasoning based on models*
- *Reasoning through the use of a common reference fraction such as ½*
- *Reasoning involving equivalence.*

The Unit Fraction is defined as a fraction having 1 as the numerator and any natural number as denominator. So it becomes easier to compare and order between 1/10, 1/5 and 1/3 as a tenth is a smaller part than a fifth which is smaller than a third. While comparing 7/8 (1- 1/8) and 4/5 (1-1/5), students use an extended version of the same unit fraction reasoning. They see how far away 7/8 is from the whole and similarly, how far away 4/5 is from the whole. Now, using unit fraction reasoning, it is comparatively easier to decipher that 1/8 (the distance of 7/8 from the whole) is smaller than 1/5 (the distance of 4/5 from the whole), hence we can conclude that 7/8 is greater than 4/5. Another way is to use benchmarks/milestones like 0, ½ and 1. Comparing 13/24 to 24/50 becomes easier if a student compares them first individually with ½ and then as they lie on either side of the half, they can say that 13/24 > 24/50.

The next two chapters deal with the number line and density of fractions (which could also be extended later to rational numbers). They describe a few advantages of using the linear model over the area and set models and how this helps students. This is followed by a description of some of the disadvantages and difficulties faced by them.

It is mentioned that students who think sequentially about the number line face more difficulties than the ones who think about it proportionally. Sequential thinking is where the numbers are just aligned in a sequential manner from 0 to 4, 5 and beyond and the distance between these points isn't given any weightage. Thinking proportionally means, that in addition, the same distance is maintained between consecutive numbers. Their argument that number lines help students understand important aspects of fractions has been observed to some extent in our classroom experiences, though there remains a sense of doubt about supplementary pedagogy. The density of fractions has been explained with a simple example of a TV remote's volume feature where, say, the sound level of 4 units is not audible in a room while that of 5 units is too loud as per the user's expectations. Obviously, there ought to be something in between which is louder than Level 4 but quieter than Level 5 but alas, there isn't any in whole number, whereas there are an infinite number of fraction values between any two fraction or whole numbers. I still feel that this remains a challenge to explain to a grade 5 student in practical scenarios. We try certain questions during our workshops with teachers, such as, name five fraction values between two fraction numbers and participants, of course, come up with various answers- often ending with a question on where there is a limit to fraction values. Could a similar line of questioning be used in a classroom with students? This is an area that hasn't been explored much in my experience. Authors also use the concept of average to elaborate on density as an average of any two numbers is always a number between them. Then, successive averages could be figured between numbers that are closer together, to find a midpoint between them. This successive use of averages generates an unending list of different fractions between two fraction numbers.

Roger Antonsen mentions in his Ted Talk (https://www.youtube.com/watch?v=ZQElzjCsl9o), *Math is the hidden secret to understanding the world,* that math could be of great use in understanding different perspectives of various people; similarly, the same analogy could be used for understanding equivalent fractions (Chapter – 8), which is nothing but (infinitely) different names (symbols) given to a fraction. Equivalence becomes a necessity when students are to add & subtract fractions. So having a procedural fluency in it is a pre-requisite although merely applying algorithm without understanding that 2/5 is the same as 4/10 (even though one has multiplied numerator and denominator by two) doesn't go too far.

Chapter 9 – Addition and Subtraction of Fractions talks about how procedural fluency and conceptual understanding work together to deepen students' understanding of fraction addition and subtraction. The following quotes from National Research Council [NRC], 2001, explain further:

> *"Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently."*

> *"Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful."*

The number line helps in dealing with the subtraction of mixed fractions. Estimation becomes an important skill while operating with fractions though some students tend to forget the meaning of fraction and do not apply them as soon as they start to solve problems that have operations in them. Being accustomed to different models and then being able to apply one when required forms the baseline of successful operations on fractions.

The last, but definitely not the least, chapter is, of course – probably the most difficult part for students, i.e., multiplication and division of fractions. The generalized concepts of whole numbers with regard to these two operations don't apply as we are now dealing with fractions. Ideas such as "multiplication makes numbers bigger while division makes numbers smaller" start conflicting with previously learnt concepts in students' minds. Visual representation of multiplication and division of fractions becomes harder and harder for many teachers and so does its abstract understanding for their students. Visually seeing that multiplying by ¼ is the same as dividing by 4 is necessary, be it using only the area model. Teachers should still make sure that students aren't beginning to generalise here as well. For example, dividing fractions will result in a bigger number or vice-versa for multiplication. This will fall short when mixed fractions are involved. Day-to-day life experience is also provided to throw some light where if one was preparing 2½ times the recipe, the calculation would be 2½ × ¾ cups of flour and the recipe would require more than ¾ cup of flour.

Each chapter of the book ends with a section, namely *Instructional Link—Your Turn*. These are prompts in table format which enable teachers to reflect on their lessons with questions such as 'Do you encourage students to use a variety of models in all aspects of understanding fractions?' or 'Are students provided with the opportunity to answer questions in which models are used?' Such prompts enable the reader to action their understanding in a very practical manner.

Just as Antonsen mentioned in his Ted Talk, the concept of fractions too, has, in my opinion, a hidden secret of understanding the world and this book comes as a good resource. For all educators and teachers out there, do try to get your hands on it.

Review: Arrow Cards

By Math Space

Arrow cards are a simple manipulative to grasp place value or more generally the base-ten number-writing system that we use. Dr. Maria Montessori invented the static cards shown in Figure 1. These cards are used along with proportional material like static beads (unit = single bead, ten = 10 beads strung together forming a line, hundred = 10 tens strung together to form a square and thousand = 10 hundreds strung to form a cube) to gain a sense of numbers – the quantities they indicate and the numerals that represent them and how they are linked. When these cards are superimposed, they form the multi-digit number 1232 as shown in Figure 2.

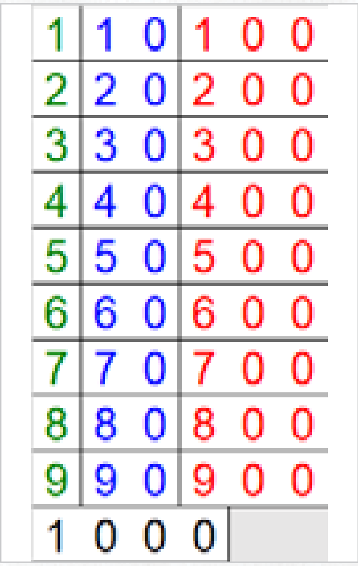


Figure 1: Static cards – Montessori

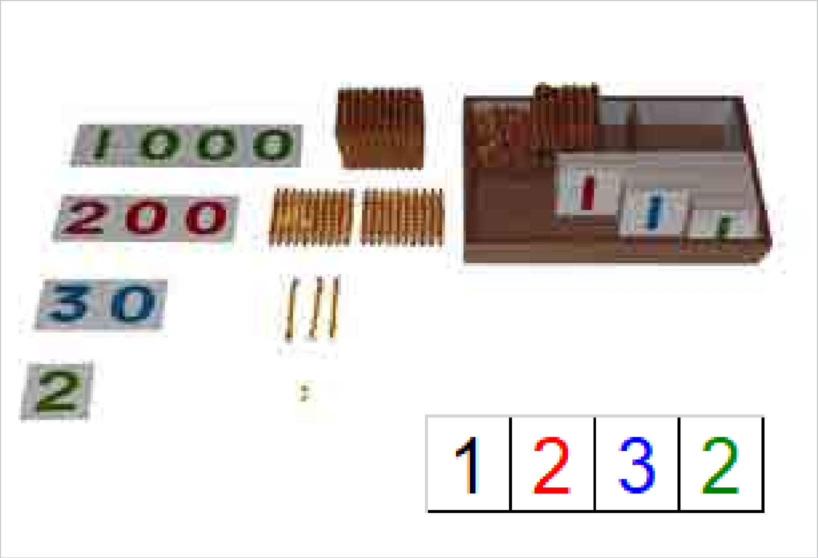


Figure 2: Combined static cards representing a multi-digit number

Keywords: manipulatives, arrow cards, numbers, place value

When these transitioned to regular schools, an arrow got added so that the cards can be held up for an entire class to see. The cards are supposed to be held only by one hand holding the arrows together. This ensures that a number like 327 can't be made with 300, 2 and 7. Even if a child tries to do that and succeeds thanks to friction, one flick of the hand would send the 2 flying out! So, the only way to make 327 would be to use the cards 300, 20 and 7 which is essentially a self-corrective feature, common to many Montessori materials. Figure 3 illustrates the arrow cards. Ideally, these should be introduced with some proportional material like (i) ganitmala, (ii) bundle-sticks, or (ii) 2D base 10-blocks, known as flats-longs-units (FLU) and ideally after introduction of zero.

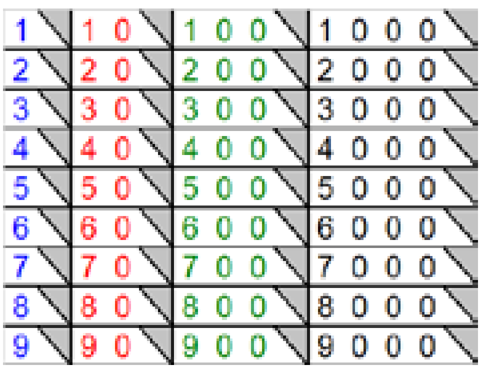


Figure 3

There are two versions commonly available: (i) the colour-coded version as shown and (ii) the colour-less one where black is the only font colour. The utility of the second version is demonstrated in the following: Scatter all the cards in front of the children and ask them to pick say, forty – they have to distinguish 40 from 4, 40, 400, 4000, etc. However, the colour-coded version has many uses.

1. Expansion of a multi-digit number e.g. 327 = 300 + 20 + 7 becomes automatic and effortless.

2. It can be linked to the absence of a particular 'bundle' in a number with 0 as an in-between digit. For example, in 307, the colour of ten (red) is missing. Observe that all 30 tens in this number are inside the 3 hundreds and there is no ten on its own outside a hundred. The absence of a ten (on its own) is associated with the absence of its corresponding colour in the arrow card representation of the number. Similarly, for 4050, all hundreds are inside the thousands, and no hundred is outside a thousand. Likewise, all units (or ones) are inside tens or thousands, and none is on its own. The absence of units and hundreds correspond to the absence of their colours in the arrow card representation. See Figure 4.

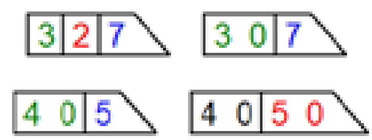


Figure 4

3. The colours provide a vital clue as to why the stick-multiplication or Napier board works (see Figure 5). We encourage the reader to check the following Power Point presentations for further details:

a. http://teachersofindia.org/en/presentation/initiating-multiplication– initiating multiplication with FLU and arrow cards

b. http://teachersofindia.org/en/presentation/deciphering-stick-multiplication– unpacking stick multiplication with colour-coded sticks and arrow cards

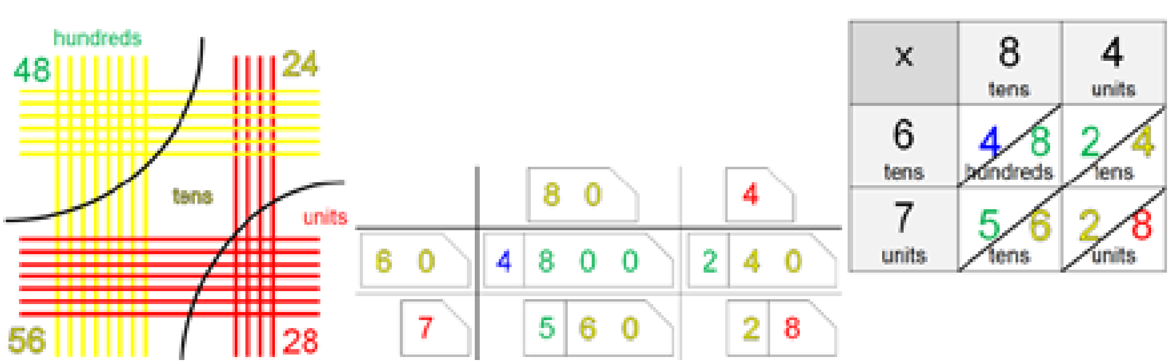


Figure 5: Stick multiplication with arrow cards

In addition, arrow cards play quite a powerful role to decipher any algorithm based on place value. It can help understand the division algorithm of finding square roots as well!

However, it should be noted that arrow cards just by themselves do not foster the understanding of 240 as 24 tens or 1500 as 15 hundreds, etc. That aspect is understood better with proportional material like FLU. Arrow cards help in understanding that a 32 is 30 + 2 and not 3 and 2.

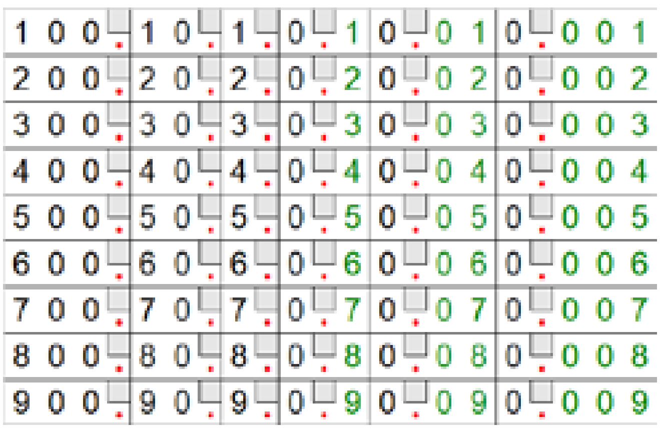


Figure 6

But more interestingly, note that the arrow cards can be expanded for larger numbers beyond 4-digits as much as one wants. Also, it can be

Figure 7

modified to include decimals (Figure 6). The arrow is replaced by a groove for the decimal point. So, the cards have to be held together at the decimal point (Figure 7). This facilitates the notion of lining up numbers so that the decimal points are aligned – a crucial step in column addition-subtraction with decimal numbers.

Introduction to decimal arrow cards should also be done with the decimal version of FLU.

Note that the decimal arrow cards can also be extended beyond hundreds on the left and beyond thousandths on the right, and thus can be used to represent any decimal number.

While both kinds of arrow cards can be bought from various resource organizations, they can be made easily with old A3 size (or larger) posters or chart papers. More details on how to make can be found at http://teachersofindia.org/en/article/making-your-own-arrow-cards with suggested size and possible layouts.

We hope to review FLU and many other manipulatives in future issues.

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] both in terms of uses and regarding possibility of low-cost versions that can be made from boxes etc. It tries to address both fear, hatred and dislike for mathematics as well as provide food for thought to those to like or love the subject. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Review

Introducing Numberphile

Reviewed by Shashidhar Jagadeeshan

Ever since we decided to reintroduce our readers to Numberphile (https://www.numberphile.com/), I have been obsessively watching videos hosted via YouTube on the site! I have also listened to some podcasts. The content is both mathematics and interviews with mathematicians about a range of topics from“Why do people hate mathematics?”to “Fame and Admiration."

The YouTube videos (*Numberphile* and *Numberphile2*) range from short videos (sometimes less than 5 minutes) to about an hour, and host a plethora of topics. Apart from the obvious and popular numbers like π, the Golden Ratio and 1729, they also introduce the viewers to different topics in mathematics. My most recent favourite is one in which Zvezdelina Stankova introduces the technique of inversion to prove Ptolemy’s Theorem.

The Podcasts (*The Numberphile Podcast*) are longer and have in-depth interviews with well-known mathematicians.

AtRiA readers will be glad to find (in the videos and the podcasts) not only many topics that have been covered in the magazine, but also the work of many of the mathematicians we have featured here. For example, there is an interview with Edward Frenkel whose book *Love and Maths* was reviewed in the March 2015 issue. There are interviews with Steven Strogatz (November 2018) and the Fields medallist Timothy Gower (November 2019).

Keywords: Numberphile, math podcast, math videos

What unites these accomplished mathematicians is not only their love of mathematics, but also the desire to communicate it to a lay audience and infect them with their passion. Of course, "*The man who knew infinity*" – Srinivasa Ramanujan – has to be featured, because if there ever was a lover of numbers, and that too numbers with hitherto unknown properties, Ramanujan was the lord of them all.

What I really like is that beautiful and deep ideas are introduced to the viewer using mostly a large brown paper on which the mathematics is derived in a written form. The background knowledge needed is mostly school mathematics. The explanations are very clear, no steps are skipped, the enthusiasm of the teacher is palpable, and one ends up learning some amazing mathematics. All this, produced with excellent quality. I could watch several such videos a day!

As a teacher, you can use Numberphile in a variety of ways. First, learn new mathematics that you can share with students. If you have access to the internet in the classroom, you can watch the videos together with students. You can even use a short video on days when class is over before time! Together with your students, you can learn about the history of math and understand why mathematicians do what they do.

Here is how Dr Shirali uses them in his Math Club at Sahyadri School:

> *Let's take a well-known geometric problem associated with a configuration of three identical squares in a row (forming a 3 x 1 rectangle). Call the unit squares ABCD, DCEF and FEGH. Join A to C, E and G. Find the sum of angles ACB, AEB and AGB.*
>
> *There are amazingly many extremely elegant solutions to this problem. One solution is presented in one of the videos (https://www.youtube.com/watch?v=m5evLoL0xwg). I had given this problem to the students, and they slogged on it for a week, then I showed them this video. Some of them had found alternate solutions.*
>
> *In the same way, I have shown other videos. In some cases, without any preparation; e.g., the video on the Collatz problem.*
>
> *Now that John Conway has passed away, I will certainly show them one of the videos featuring Conway (see tribute of Conway by Prof Ramanujam in this issue).*

Numberphile is created by former BBC video journalist Brady Haran. Haran is also well known for YouTube channels like Periodic Videos and Sixty Symbols.

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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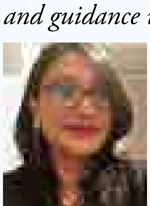
The Closing Bracket . . .

We feature here Mathematics teachers who we encounter in the course of the Foundation's work, unsung heroes and heroines who are often the gateway into and out of mathematics for many a student. Math phobia has been endlessly discussed, it is time to focus on those whose dedication to the subject creates lifelong learners.

In this issue, we feature Sandhya Sundas, primary teacher, presently posted at C S Rai Govt Sec School, Arinthang, Gangtok, Sikkim. This is one of the schools in which the Class IV and Class V textbooks created and designed by Sikkim SCERT in association with Azim Premji Foundation is being piloted. Here, in Sandhya's own words is how she reacted to the pandemic situation.

When lockdown was first declared, we all were not aware that it would last more than three months and that school would remain closed for such a long period. Like every other teacher, all my school/teaching related items were closed in the school desk. This included the class register in which the contact numbers of the children were recorded. I was left with no means to connect with them. When the situation started getting worse and there were fewer chances of opening the school, at that moment a thought came into my mind that "No you cannot sit at home like this, you must do something so that the teaching learning process goes on." I felt that I must reach the children somehow. So I posted a video on Facebook where I tried to get connected with the students through their parents. But, it did not work well from my personal Facebook page. Then I thought of an alternative - if any digital media platform would give me their support, then I could not only get connected to my students, but perhaps I could help other students too, as like me, most of the teachers were unable to help them. So I requested "Sikkim Varta" a digital medium platform to support me. They asked me to visit their office - that was during the peak lockdown period! I convinced my family who were worried about my going out due to the pandemic situation and I walked to the Sikkim Varta office. On the way, the police stopped me but when I shared my idea with them, they let me proceed. At the office, when I described my initiative, they very warmly accepted my request and provided me with their media platform to conduct a session of Class V maths online. On April 12th, I made my first attempt to reach my students and got a very good response. At least 40 to 60% of the parents and guardians in Gangtok, as well as within and outside Sikkim were able to access the program. After that, every day, I walk to the studio and make my presentation as if I am teaching in the real classroom. But an online class in front of the camera is very different. It was definitely not as easy as I thought. To start with, I had nothing with me – not even a board or chalk. I then approached an organization near my house and they provided me with their blackboard and chalk – it was in poor condition and I couldn't even buy polish for it because all the shops were closed. I managed however, and then, one day, I posted a request for a board on Facebook. Luckily, a teacher, Mr Deoraj Bagdas readily shared his white board. So things were getting better in small steps. But I missed the interactivity of my classroom, so often, I deviated from my lesson plan because of the surprising responses from the students. Here, I had to assume that they were with me all the way. Sikkim Varta, in spite of their busy schedule, supported me to reach maximum students through the media – I often had to wait in the studio for my turn, then I had to spend 3 to 4 hours with their editor to edit the video before telecast. Sometimes it would get dark by the time I returned home. Walking on the silent street, made me feel like I was in a horror show – again, the responsible Sikkim Varta staff accompanied me whenever I got late!

And it is going on. I share the links of my talks with my students and then they reach out to me with their doubts. The new textbook was designed to make the child think and reason and understand mathematics. Very often, the children have to rely on parents and older siblings who have been given a procedural understanding of mathematics. So it is a difficult task to get them to take the harder but more interesting route. I am also learning a lot - and last, but not the least, the team of SCERT and APU are always supportive with their books and guidance whenever I need them.



Here is a link to one of Sandhya's talks.

Sandhya: https://www.facebook.com/watch/live/?v=2622640997842256&ref=watch_permalink

Now you know the effort and dedication behind that clip!

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

5th National Social Enterprise Idea Challenge

Azim Premji University

Image credit : latimes.com

Inviting innovative solutions from students to problems posed by the Covid-19 pandemic

This year's Idea Challenge initiative is specifically looking for solutions that will

- Empower communities (strengthening livelihood, public health, service delivery, local governance, education etc.)
- Support NGOs/Philanthropic organizations/Citizen groups in their pandemic response
- Equip our front-line workers- ASHA, ANM, ICDS, Nurse, NGO frontline workers, etc;
- Strengthen the monitoring systems at any scale
- Enhance specific measures by Hospitals, municipalities, Gram Panchayats, Block Panchayats, etc.

Register by Oct 2, 2020

Registration Link: https://azimpremjiuniversity.edu.in/seidea

Azim Premji University

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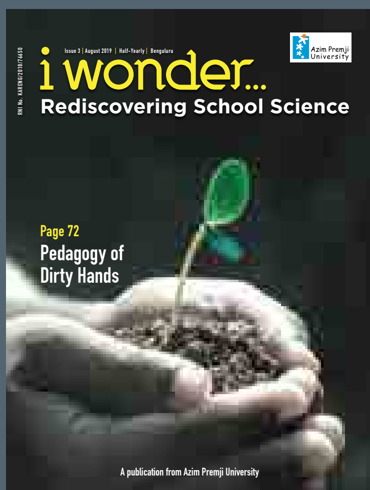
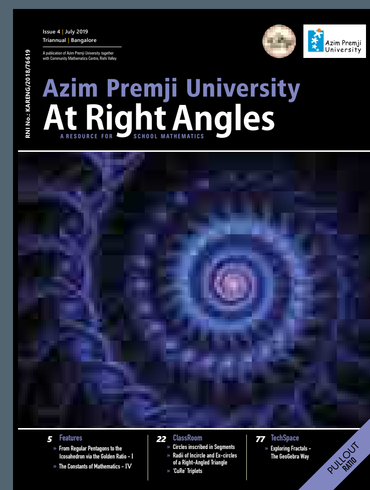
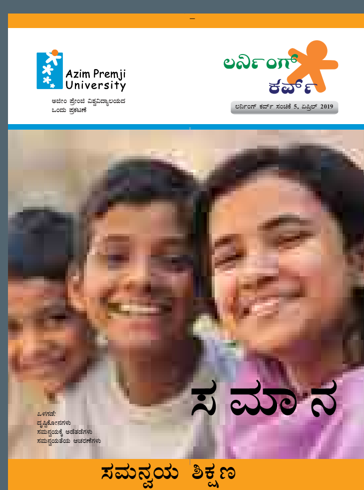
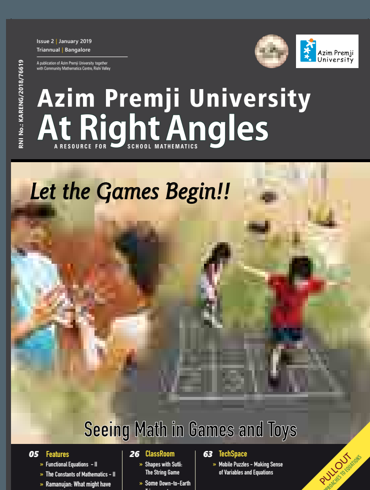
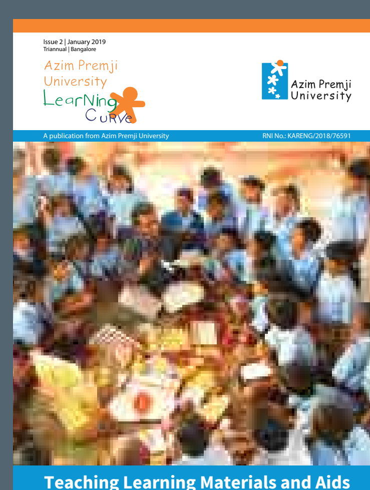








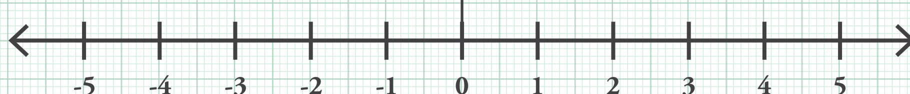


At Right Angles

A Resource for School Mathematics

PARALLEL LINES

PADMAPRIYA SHIRALI



Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

PARALLEL LINES

Many elements in buildings – beams, pillars, windows, doors, window bars, flooring tiles – incorporate parallel lines. Line dividers on roads, railway lines, power lines are all examples of parallel lines. Parallel Lines assume a lot of importance when marking out roads or pedestrian crossings, sports courts, athletic tracks and airport runways.

Visually, we are so used to parallel-ness that most of us find it uncomfortable if we see a tube light not aligned parallel to the ceiling, or a tilted picture frame.

While we see them in everyday objects, we must ask, do they have any other importance?

They lie at the centre of many properties involving geometry. Drawing a transversal across pairs of parallel lines creates angles that have special properties.

In this pullout, we explore parallel lines and transversal properties. Solving any geometric problem requires knowledge as well as developing a geometric eye. Hence, in the teaching of geometry, we need to develop certain skills in children that will help to open a geometric eye. How does this geometric eye develop?

Over a period of 30 years, I have met students who were exposed to plenty of activities involving visual posers and who had a chance to play with Jigsaw puzzles, spot the hidden figure problems, spot visual patterns and so on in their primary school years. I have also met several who learnt geometry only in a limited classroom situation. I began to notice that the manner and skill with which these two groups of students approached a problem were quite different. The students who had greater exposure to visual challenges and greater contact with activities involving shapes had a higher ability to visualise and unpack the problem. While one cannot draw absolute conclusions about the kind of experiences that help to develop visual dexterity, I feel quite confident about this: exposing students to visual challenges does have a beneficial effect. Also, an intelligent guess at what one is looking for and having a sound knowledge of the geometric concepts aids the process.

Geometric problems require students to spot a specific object or a relationship. For example, an X (vertically opposite angle); a pair of adjacent supplementary angles; a pair of lines that are perpendicular to each other; or a pair of similar triangles; and all these in a figure which has many crisscrossing lines, angles and triangles. At times, we may have to turn the figure around to spot some of these things. There is a need to focus on relevant information and to ignore the rest of the data. In some ways, one has to turn a blind eye to irrelevant data.

How do we develop this geometric eye? What skills do we need to emphasise? I list a few here.

- Spot hidden geometric shapes.
- Spot right angles and straight angles.
- Spot pairs of equal line segments, perpendicular lines, and parallel lines.
- Spot pairs of shapes that are rotated relative to one another.
- Look at the same shape through different orientations: top-down, left-right.
- Find common line segments or angles in intersecting shapes.
- Spot shapes that are reflections and rotations of each other. Spot symmetries. Spot patterns.
- Hide some features of a diagram and highlight some others, using one's fingers or actual highlighting.

ACTIVITY 1

Objective: Warm up visual exercises of a general nature

Most of the exercises here are self-explanatory. They are all aimed at increasing observational skills. These include applying logic, recognising symmetry, visualising rotations and getting geometric ideas.

Jigsaw Puzzle: Draw an outline of the square, cut the shapes and ask the children to reassemble the pieces in the square.

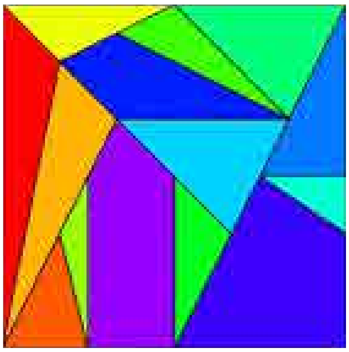


Figure 1

Complete the reflection.

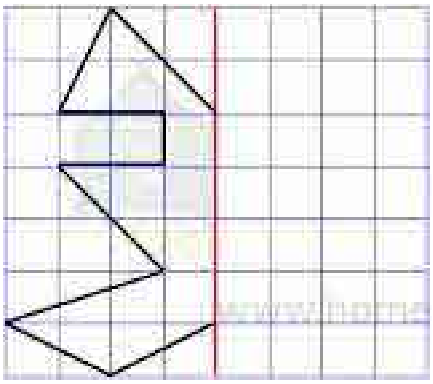


Figure 2

Here are some rotations and reflections of F. Make some rotations and reflections of P.

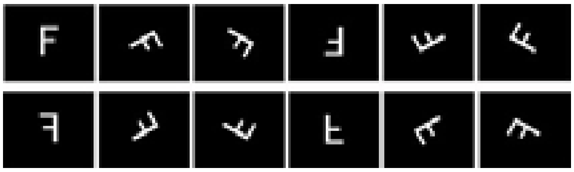


Figure 3

How many rectangles do you see in this figure?

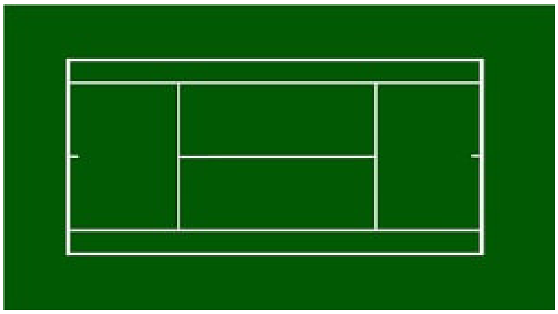


Figure 4

Describe the relative positions of squares A and B.

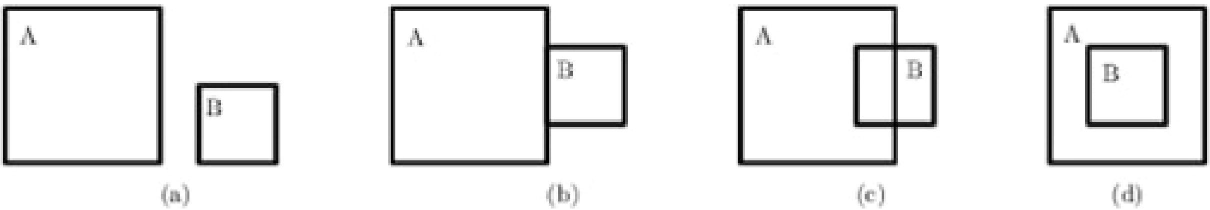


Figure 5

ACTIVITY 2

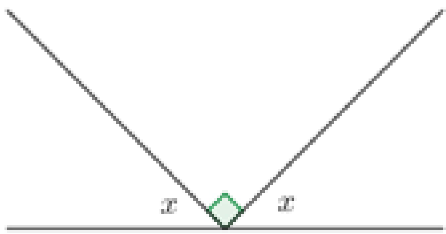
Objective: Warm-up activities to review necessary geometric facts

Complementary angles and supplementary angles property.

Angles on a straight line add up to 180 degrees.

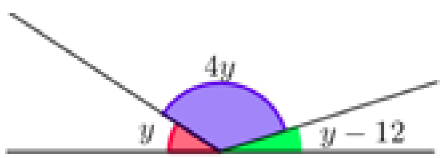
Opposite angles formed by two intersecting lines are equal.

Sum of the angles at a point is 360 degrees.



$x =$

Figure 6



$y =$

Figure 7

If X = 20 degrees and Y is its supplement, what is Y – X?

If L = 50 degrees, M and L are complements, and N is a supplement of M, what is N?

What are a,b,c and d? (See Figure 8.)

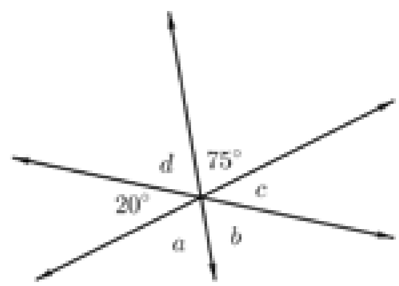


Figure 8

ACTIVITY 3

Objective: Introduction to the concept of parallel lines, definition

Figure 9

Figure 10

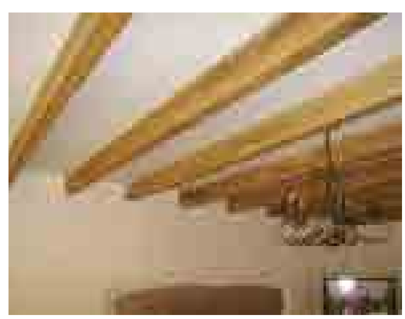


Figure 11

Show the students these pictures and ask them to comment on what they notice in them.

What is the common thing that they see in all these pictures?

Ask them whether these lines will ever meet. Help them to come up with a definition for parallel lines in their own words initially. Give them the definition only after that.

Parallel lines are a pair of lines that lie on the same plane, and do not meet however far we extend them beyond both ends.

It is important that the lines should lie on the same plane. A line drawn on a table and a line drawn on the board may never meet but that does not make them parallel.

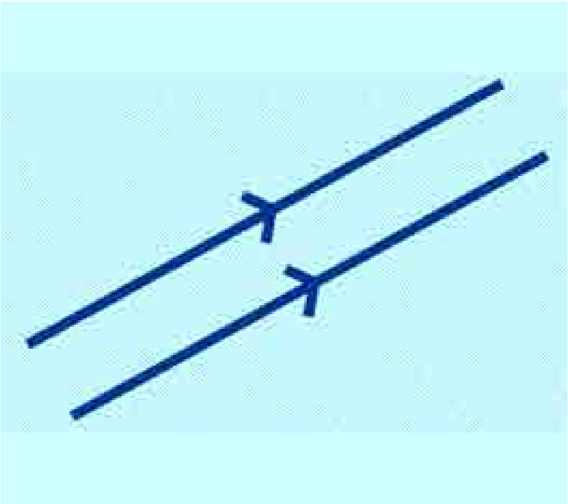


Figure 12

ACTIVITY 4

Objective: Spotting parallel lines

Note to the teacher: Drawings made on the board or on a ground are only approximations of parallel lines.

Let students look at various objects in the class to spot parallel lines in them.

Ex. Notebook edges, blackboard

Point out that when there are two sets of parallel lines, we use double arrows for the second set as shown in the picture.

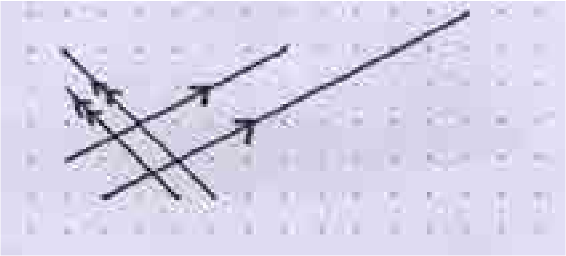


Figure 13

How many sets of parallel lines do you see in this picture?

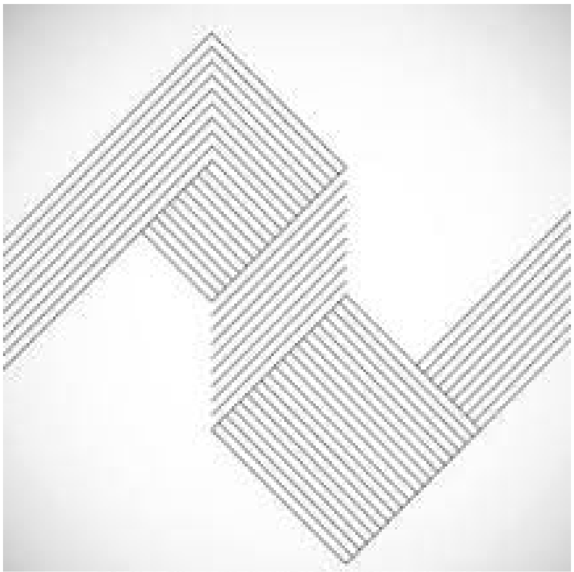


Figure 14

ACTIVITY 5

Objective: Explore parallel lines on dot paper

Materials: Square dot and isometric dot paper

Students can explore creating sets of parallel lines on square dot and isometric dot paper.

Figure 15

Can the students reason out why these pairs of lines will never meet? Let the students give explanations in their own words.

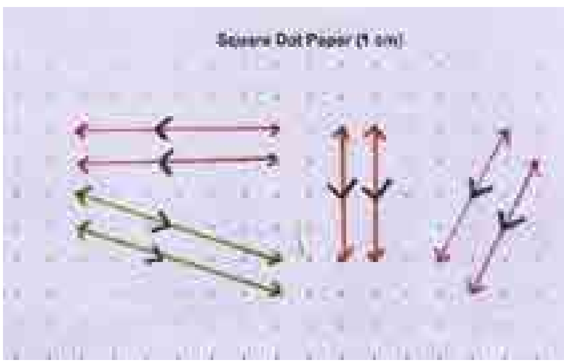


Figure 16

Ensure that the sets of lines are parallel.

It is easy to connect wrong dots and make non-parallel lines.

(Look up AtRiA Jul 2018 TearOut)

Note: It is easier to draw vertical and horizontal lines and the ones inclined at 45° (on rectangular dot sheets), but drawing a line parallel to one which has a different slope (i.e. $m \neq \pm 1$) is slightly harder.

Do students see that lines do not need to be of the same length for them to be parallel?

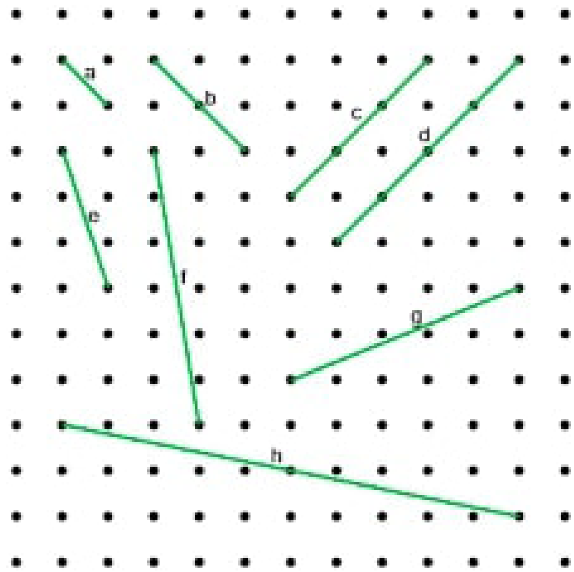


Figure 17

Which sets of lines are parallel in figure 18?

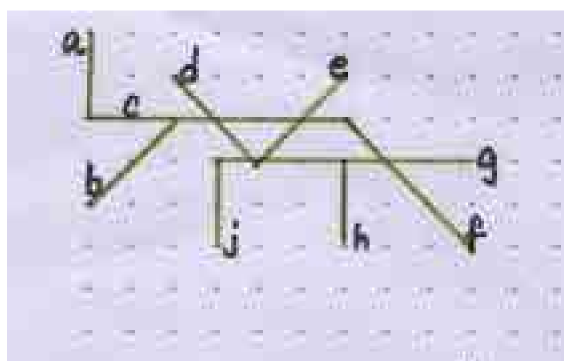


Figure 18

ACTIVITY 6

Objective: To show that when parallel lines differing only in position meet a given line (called the 'transversal'), they all make the same angle with it.

Definition: A straight line drawn across a set of given lines is called a transversal.

Let students use a scale and a set square to draw some parallel lines, as shown in figure 19.

The line x is a transversal.

Notice the angles that the lines make with the given line x. What can be said about the angles?

They make the same angle with the transversal. Only their positions are different.

Figure 19

ACTIVITY 7

Objective: Through a given point, there can be only one straight line parallel to a given straight line.

Materials: Plain paper

Ask students to draw a straight line on the paper. Let them identify and mark a point that is not on the line. Ask them to draw a line parallel to the given line using a scale and a set square as in the previous activity.

Where will they place the scale?

Discuss the previous situation of activity 5 and see if they can adapt the same idea in this case.

Now ask them if they can make a different parallel line to the first line going through the same point.

What do they notice?

The teacher can help them to state that through a given point there can be only one straight line parallel to a given straight line.

How many lines can be drawn through the same point that are not parallel to the given straight line?

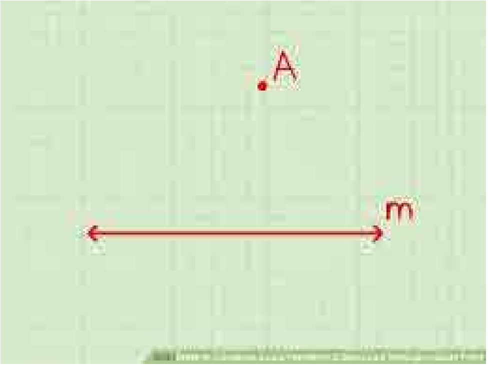


Figure 20

ACTIVITY 8

Objective: To highlight the notion of transversal and the angles that it creates.
Materials: Rigid parallel frame in one colour and one long strip in a different colour
Vocabulary: Interior angles

A rigid frame *(note the pasted thin sticks in the picture to create a rigid frame)* and a strip serve as handy material for studying parallel lines and transversal properties.

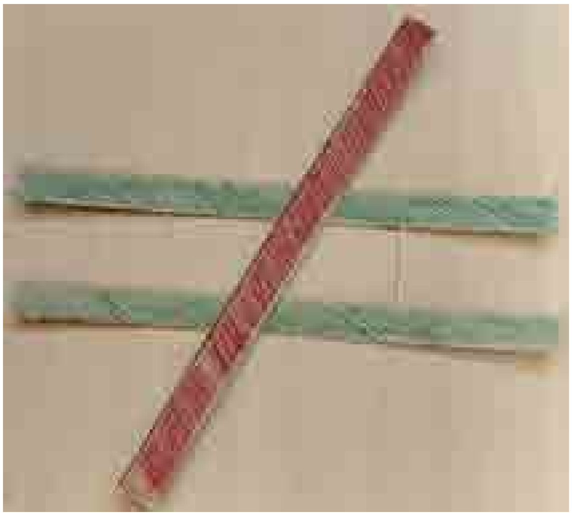


Figure 21

(If the model is not satisfactory the students can place a scale on paper and draw lines along both sides, then draw any transversal and measure the angles formed.)

Place the transversal initially in one position so that you can measure the different angles of the top intersection accurately. (Hold the transversal strip firmly by clipping or stapling to the parallel strips.)

Ask the question: "If you know the measure of one angle (of the top intersection), can you figure out the other angles?"

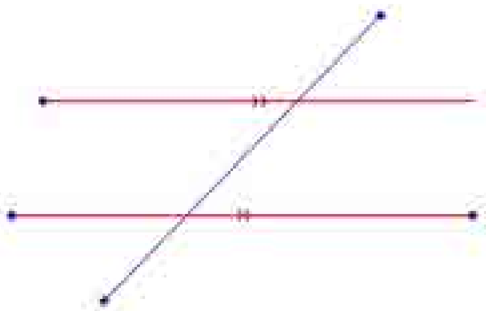


Figure 22

Are the students able to use their knowledge of vertically opposite angles and a linear pair?

Let the students now measure one angle of the bottom intersection and deduce the other angles.

They can now measure the angles using a protractor to verify.

Students should make a drawing of this and note down the information. What do they notice? Let them record the angles.

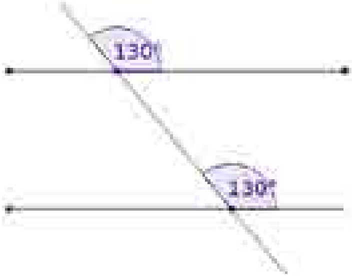


Figure 23

Ask them to color all angles of the same size in one color. How many colors did they use?

Do they notice any pattern? How many different angles are there?

Students can place the transversal in a new position and observe the angles that have formed.

Let the students make a second drawing, note down the information and color as earlier.

Do they see the same pattern again? Let them number the angles in an anti-clockwise manner as shown.

Which sets of angles are equal?

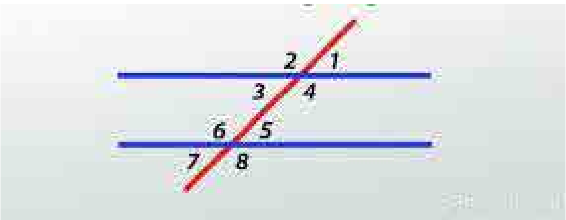


Figure 24

They can repeat the process once more to verify if the pattern repeats.

Students can describe the property (when a transversal intersects two parallel lines ….) in their own words and the teacher can give a formal definition later.

What do they notice about the interior angles? What about exterior angles?

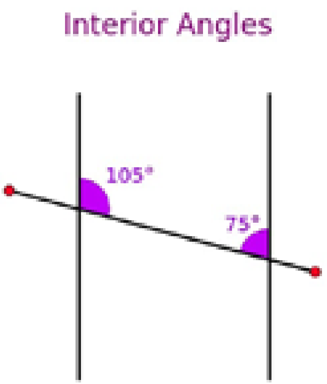


Figure 25

Raise the question: Is there a situation when all the 8 angles are equal?

A special transversal: Students should be able to see that a transversal can form a 90 degrees angle with a set of parallel lines in some situations.

Note that when two pairs of parallel lines intersect each line acts as a transversal for the other set.

Figure 26

ACTIVITY 9

Objective: To verify if two given lines are parallel to each other

Materials: Tracing paper

Students can be given sets of parallel and non-parallel lines with transversals to verify if these sets are parallel.

They use the tracing paper to copy the angle of the top intersection and match it with the lower intersection to verify if it is the same.

Note: This act of tracing and verifying is a slow activity and aids in remembering corresponding angles.

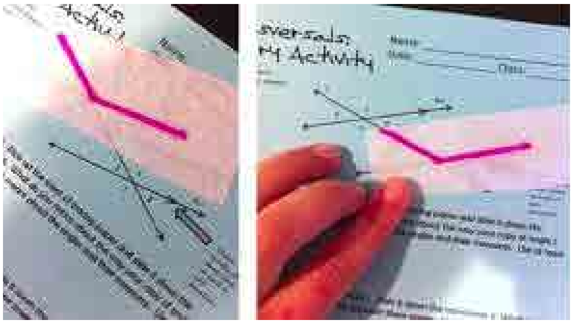


Figure 27

Source: https://www.mathgiraffe.com/blog/transversals-parallel-lines

ACTIVITY 10

Objective: To help students to recognise corresponding angles ('F angles')

Note: Students should be taught to identify various angles in a graded way and not all at one time. In this activity, the focus is on corresponding angles.

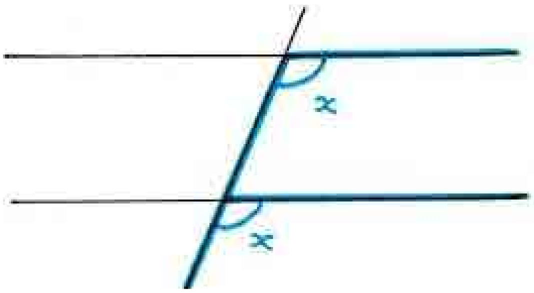


Figure 28

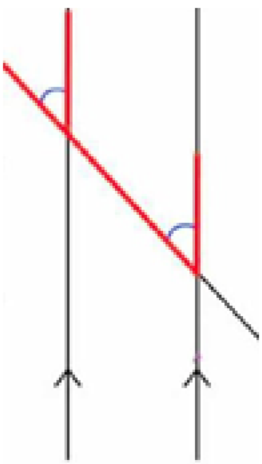


Figure 29

The blue lines and red lines in Figures 28 and 29 are parallel. Any pair of parallel lines makes an F shape with a line that crosses them.

The marked angles are called 'F angles.' Notice the 'F shape' in them.

The F angles are equal. Reflect F and rotate F to recognise F in different orientations.

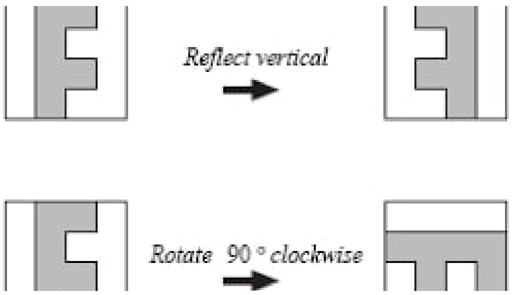


Figure 31

Here we show different orientations of F to be able to recognise them in various situations.

In the diagram given below, the students should find different F's to list all the corresponding angles.

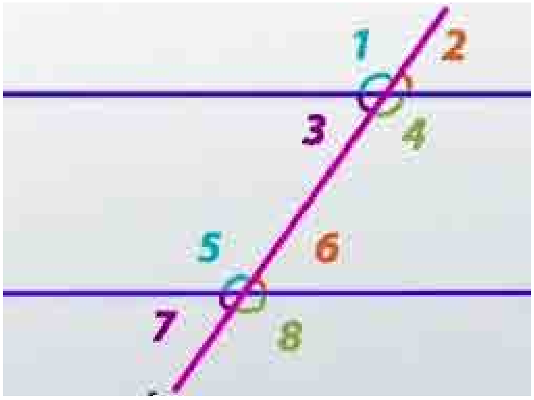


Figure 32

Angle 1 = Angle 5, Angle 2 = Angle 6, Angle 3 = Angle 7, Angle 4 = Angle 8

Let students practise many problems that require them to identify F (corresponding) angles.

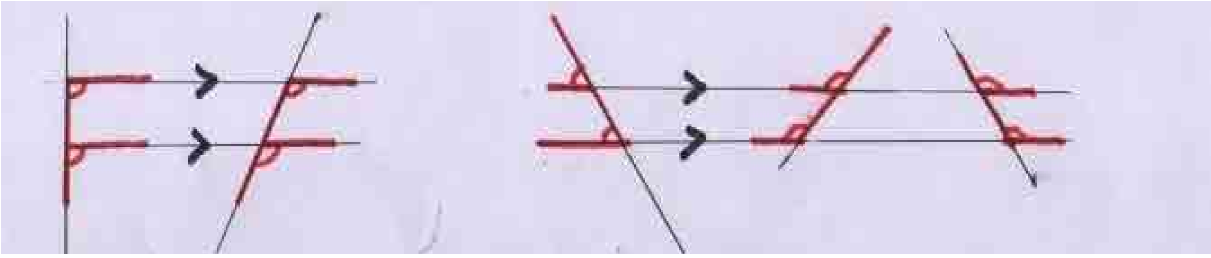


Figure 30

ACTIVITY 11

Objective: To help students to recognise alternate angles ('Z angles')

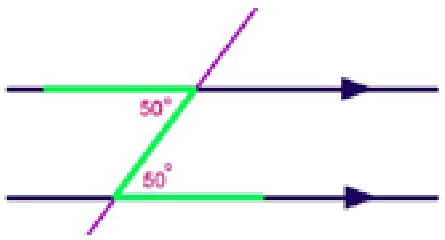


Figure 33

Let them draw different orientations of the parallel lines to spot reflection and rotation of Z angle.

Figure 34

Note: In this activity, the focus is on alternate angles. Since students are already familiar with vertically opposite angles, we may not need more reinforcement for X angles.

The Z shape is made by two parallel lines and a transversal, as shown in the pictures.

Here again, students should be able to spot rotations of the Z shape. In the top figure, we see different orientations of Z.

Why are the Z angles equal to each other?

Notice what happens when we extend two of the lines. The lines now make an X angle.

Students know that the vertically opposite angles in X are equal.

Can they also see the F (corresponding angles) here?

From these facts, can we see that the two angles of the Z are equal?

Let students practise many problems that require them to identify Z (alternate) angles.

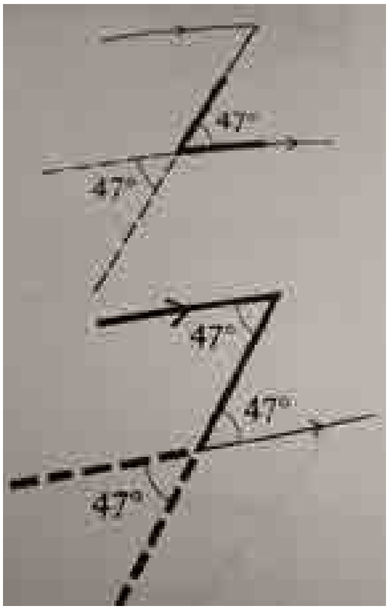


Figure 35

Show them the difference between alternate exterior angles and alternate interior angles.

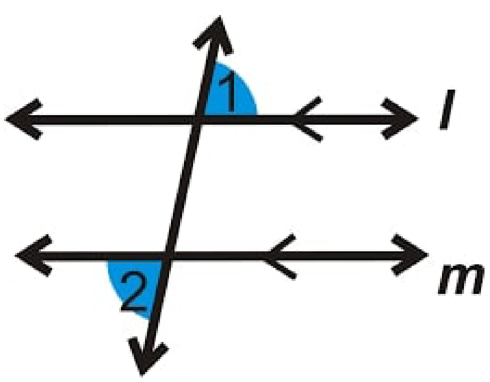


Figure 36

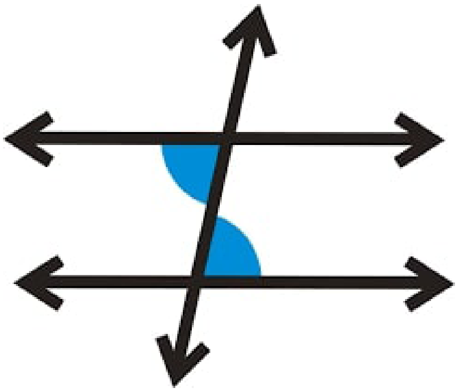


Figure 37

They can use a transparency to verify that the alternate angles match.

Rotate the transparency by 180° to get the alternate interior angles to match.

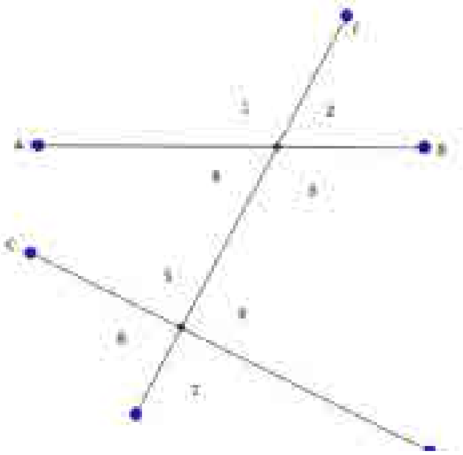


Figure 38

Let students draw a pair of non-parallel lines with a line cutting across and measure the angles. How does it compare with the earlier findings?

Will the F angles be equal here? What about Z angles?

GAME 1

Objective: Vocabulary practice

Players: 2 to 4

Vocabulary: Corresponding angles, alternate angles, alternate interior angles, alternate exterior angles, linear pair, vertically opposite angles

Make a drawing on the ground and let students play hopscotch, responding to the orders given.

Student 1 gives the orders, and student 2 has to place his/her two feet in the appropriate places. (Figure 39)

Student 2 is initially at the start position outside the drawing.

Order 1: Corresponding (Student 2 jumps onto a corresponding pair by placing one foot in one angle and the other in its corresponding angle)

Order 2: Linear pair (Student 2 now hops onto a linear pair)

In case of a mistake, he/she loses and it is the turn of Student 1 to play.

The pace of the game needs to be fast, i.e., the orders have to come without any delay so that the responses are also performed quickly.

Figure 39

ACTIVITY 12

Objective: To explore whether line *k* is parallel to line *m* when line *k* is parallel to line *n* and line *n* is parallel to line *m*

Materials: Dot paper

Will two straight lines that are parallel to a third line be parallel to each other?

Let students draw a straight line and draw two parallel lines to the given straight line.

Let them now check if the second and third lines are parallel to each other.

How are the students checking?

Are they using F angles? or Z angles?

Straight lines that are each parallel to a common straight line are parallel to one another.

Can two intersecting straight lines ever be parallel to a third straight line?

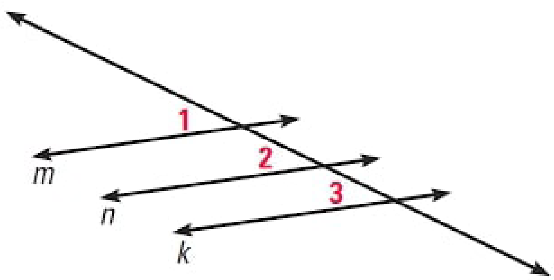


Figure 40

ACTIVITY 13

Objective: Problem solving

Students should now be able to use their understanding of the properties of parallel lines and a transversal to solve problems. They can look for F and Z angles to spot corresponding and alternate angles.

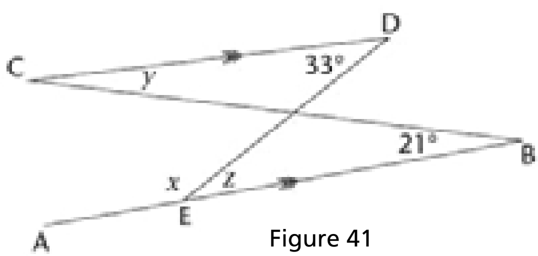


Figure 41

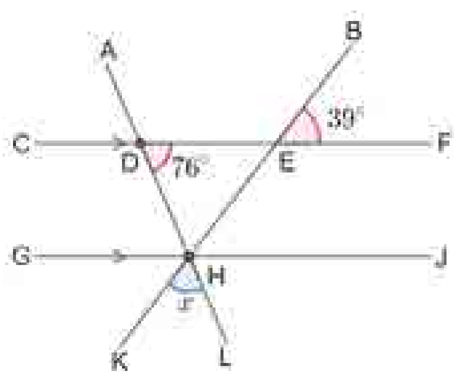


Figure 42

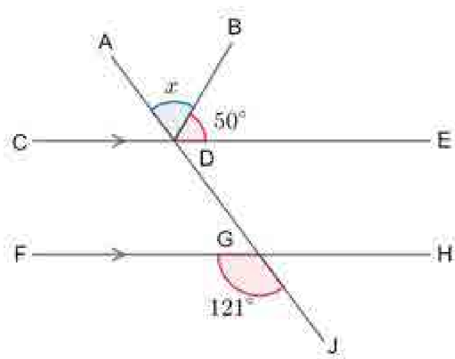


Figure 43

ACTIVITY 14

Objective: Parallel lines in polygons, designs, and 3D shapes

Materials: Various quadrilateral and Polygon stencils

Students can study the various shapes and verify if their opposite sides are parallel. They can use stencils of these shapes to draw their outlines, and check whether pairs of opposite sides are parallel to each other.

Figure 44

Students can study the various polygons and mark the parallel sides of the various polygons.

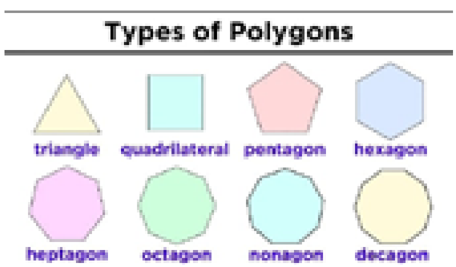


Figure 45

Can a triangle have parallel sides? Why? What happens if a kite has parallel sides?

Let the students draw regular polygons with different numbers of sides. What do they notice?

Do all regular polygons have pairs of parallel sides?

Let the students discover that polygons with an even number of sides have pairs of parallel sides while the rest don't.

They can also draw the diagonals of regular polygons to check whether they are parallel.

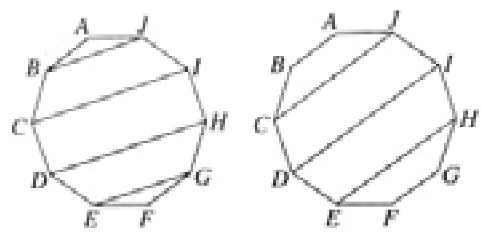


Figure 46

Now let the students draw regular polygons with different numbers of sides, and let them also draw their diagonals. What is the smallest number of sides required for a regular polygon to have parallel diagonals?

In regular polygons with an even number of sides, is there a shape which has a diagonal with no other diagonal parallel to it? Are there any sides which are parallel to it?

Is there a polygon which has two sets of parallel diagonals?

They can also study various 3D objects, drawn on isometric paper, and name the parallel lines.

Note the definition that parallel lines lie on the same plane.

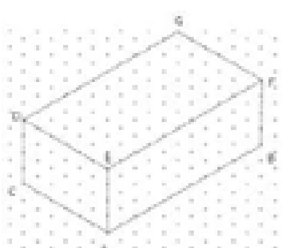


Figure 47

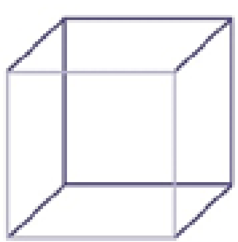


Figure 48

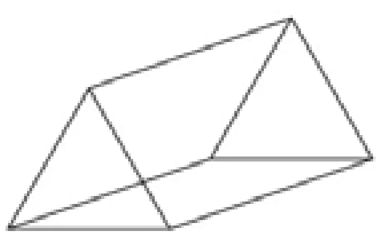


Figure 49

ACTIVITY 15

Objective: Art designs with parallel lines

Materials: Dot paper and some Rangoli drawings, border drawings and celtic drawings.

Many rangoli designs, celtic designs and border designs make use of parallel lines. Students can study and learn to make such designs.

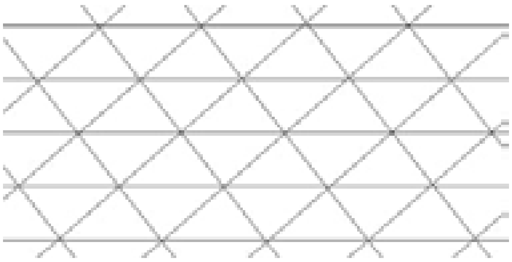


Figure 50

Figure 51

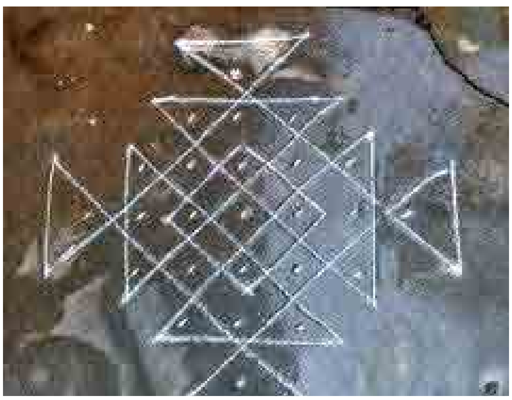


Figure 52

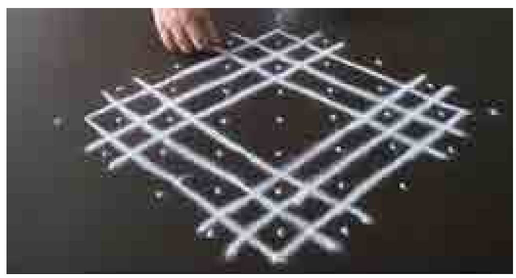


Figure 53

ACTIVITY 16

Objective: To spot places according to information

Materials: Design of a locality map

Group Project: There are many projects that require students to make use of their understanding of parallel lines and transversal to create a map based on some guidelines.

Design a map of a locality with three roads that are parallel to each other and one road that intersects all the three roads.

Name your roads ex. M G Road, Ramanujam Road.

Figure 54

Now give some clues. Ex.

The school and the hospital lie in congruent alternate exterior angles.

The medical shop and the hospital lie in congruent corresponding angles.

The gas station and the bus stop are at vertically opposite locations.

The hospital and the bookshop are on the same side of the transversal at exterior locations.

Students work in groups to create a map which fits the given clues. Are their maps identical?

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Keywords: Parallel lines, visualisation, patterns, symmetry, angles, transversal

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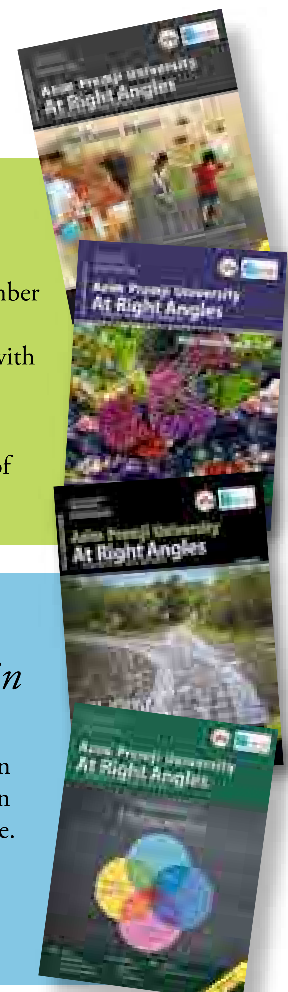

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