Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE · RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 5, No.2
July 2016

Jo Boaler's Mathematical Mindsets

98

Jill Adler Mathematics Education Researcher

05

WOMEN IN THE FIELD OF MATH EDUCATION

BREAKING THE CLASS CEILING

PULLOUT
AREA AND PERIMETER

QUICK CHECKS!

How quickly can you check these answers?

Accountants of old, who had to tally large sets of numbers often used a very ingenious method to check their answers, read Anant Vyawahare's article on Digital Roots to find out in a jiffy which of these answers could be correct.

231457609 × 145784392 = 33742906801838728

231457609 + 145784392 = 377242001

8912587 × 11225577 = 10004893167699

$$
\begin{array}{r}
727381659231 \\
123456789101 \\
324356789201 \\
656575879311 \\
+\ 985432659851 \\
\hline
3354542324344
\end{array}
$$

1832 × 153 × 32 × 91 = 816221952

How long did you take?

Now, here's an interesting logical twist: this method will tell you if your answer is wrong.

Can it guarantee that your answer is right?

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From the Editor's Desk . . .

The quintessential '*maths master*' of folklore has since been often replaced in the Indian classroom by the equally legendary '*math teacher*'- that stern but loving disciplinarian who has inflexibly drawn many a student to success in mathematics (and very often, in study skills and perhaps even life skills). Gender has often played a role in the field of mathematics with studies being conducted on whether boys are better than girls in mathematics, and why there are so few female mathematicians in the field. Which is why *At Right Angles* has chosen to focus on the theme '*Women in the field of math education*' in this issue. Both Jill Adler and Jo Boaler are important figures in the field of math pedagogy. In Features, we bring to you an account of the sterling work done by Jill Adler in South Africa. The write up is by Ravi Subramaniam, HBCSE, who has interacted closely with her work. In Reviews, Prabhat Kumar has described Jo Boaler's latest book and the trendsetting work that she, along with Carol Dweck, has done on mindsets. Closer home, Parvin Sinclair who has been involved in preparing the B.Ed material for mathematics in the IGNOU B.Ed course. and Geetha Venkatraman, a well-known figure in college level math education efforts and our very own Jonaki Ghosh who has done sterling work in the field of technology in math education have all contributed significantly to the Indian mathematics education scene. May their tribe increase!

Also in Features, Shailesh Shirali wraps up his series on 3, 4, 5; J. Shashidhar talks about the concept of multiple infinities and Prithwijit De will take you on a fascinating journey in his succinctly titled article 'An Eye on Eyeball'. In ClassRoom, we have two articles on the Digital Root of a number, and Anant Vyawahare describes how it was used by accountants to tally their figures long before calculators were invented. We feature two groups of writers from outside India who talk about 'Three Means' and 'Inequality for the Area of a Quadrilateral'. Tech Space features the uses of a graphics calculator, increasingly being used by students to quickly generate graphs and study abstract concepts such as the derivative of a function. And we have an article sparked by the work of Agnipratim Nag, a class 8 student of Frank Anthony Public School, Bangalore; look for an unusual take on the 'Difference of Two Squares'. Students are clearly reading our articles with great interest, Jayaram Chandar not only provided an answer to our question on the smallest number which does not divide $1 \times 2 \times 3 \times \ldots\ldots \times 99 \times 100$ but also devised a Java program which gives the smallest number which does not divide any input number. The link has since been uploaded on our FaceBook page AtRiUM.

We conclude the article with Padmapriya Shirali's exciting PullOut on Area and Perimeter. Do try out the suggested activities. As always, we look forward to hearing from you; mail us at AtRiA.editor@apu.edu.in and don't hesitate to send us articles for publication.

Sneha Titus
Associate Editor

Design

Zinc & Broccoli
enquiry@zandb.in

Print

SCPL
Bangalore - 560 062
www.scpl.net

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
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Rishi Valley and Sahyadri School, Pune

Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Reviews

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Women Achievers

Jill Adler – A South African Mathematics Education Researcher

Practice Makes Perfect

Features

K. Subramaniam

Jill Adler, a mathematics education researcher from South Africa, received the Hans Freudenthal Medal for 2015. This is one of the top two medals given by the International Commission on Mathematics Instruction (ICMI) for achievement in mathematics education research. The award is given every two years and recognizes a major cumulative programme of research. The recipients of the award are leading researchers who have shaped the field of mathematics education. Adler is the seventh researcher to receive the Freudenthal medal.

Even among mathematicians, not many are aware that mathematics education has emerged as a robust academic discipline with its own community of researchers and set of research practices. Fewer still have an idea of the highly inter-disciplinary nature of mathematics education research, which requires not only a thorough understanding of mathematical content at the relevant educational level, but also draws on theories and methods in education, the social sciences and the humanities. The institution by the ICMI of the Freudenthal medal, along with the Felix Klein medal (awarded for lifetime achievement in mathematics education research), is a very significant step. It has served to give direction and momentum to the growth of the discipline of mathematics education.

Keywords: *Mathematics education, Mathematics education research, ICMI, Hans Freudenthal medal, multilingual education, PCK, mathematics teacher education*

Jill Adler's work is located squarely in a developing world context. South Africa freed itself from its apartheid past in 1994 and embarked on the transition to a more equitable society. During the apartheid era, access to education was segregated by race and quality education was denied to most of the population. With the commitment of the new government to provide all citizens better education, South Africa faced the problem of hugely inadequate numbers of qualified teachers. Unequal access to education and the shortage of capable and qualified teachers are problems that feed into each other. They require sound and long-term policies to be effectively implemented by a government that prioritises education. Many countries in the developing world, including India and South Africa, struggle to overcome these problems.

Adler's work is driven by a strong resolve to address the gaps in mathematics education in South Africa. In particular, she has grappled with the problem of enhancing the capacities of teachers, both pre-service and in-service. To quote the citation for her Freudenthal medal, "her work epitomizes what Wits University has called the 'engaged scholar', that is, doing rigorous and theoretically rich research at the cutting edge of international work in the field, which at the same time contributes to critical areas of local and regional need in education."[1] Adler began her career as a high school mathematics teacher in a so-called "coloured" school. She then spent several years in developing learning materials in mathematics for adult and youth learners who were excluded from learning mathematics due to the apartheid regime. She became a teacher educator in the 1980s, completing a PhD in 1996 on teaching and learning mathematics in multilingual classrooms. Her work, together with student colleagues, on multilingual education was pioneering and placed her as one of the leading researchers in mathematics education (Adler, 2001). Her subsequent work focused on studying the mathematical knowledge that is central to the work of teaching and designing and implementing teacher education programmes that sought to build strong mathematical capability among teachers. I will discuss these various aspects of her work.

The work of Adler and her colleagues on the challenges of teaching mathematics in multilingual environments was pioneering in two ways. First, it brought the crucial issue of language in mathematics teaching and learning to focus in the international mathematics education community, a focus that was unlikely to have emerged from research done in predominantly monolingual cultures. Second, it addressed a critical local issue, which was central in the South African context, to mathematics and science education, and to education generally. Adler's approach was sensitive to the specific contexts and challenges of South Africa, where language issues are complex and politically charged.

South Africa is a multilingual nation with 11 officially recognized languages. The earlier apartheid regime recognized only two official languages – English and Afrikaans.[2] It is common in urban and semi-urban schools to find multiple home languages even in a single classroom. Many South Africans learn to speak several languages. However, the language issues related to education are complex and difficult to resolve. As in many countries with a colonial history, it is not any of the African languages, but English which is recognized as the language of power and opportunity. Official education policy recommends beginning with education in the mother tongue, with the learners gradually acquiring capability in the language of teaching and learning, which is generally English. The current policy, in fact, requires children to learn three languages in school – the home language and two additional languages.

As Adler and her colleagues point out, despite the official policy that early education must be in the home language, in practice, education starting from primary school is almost invariably in the English medium, except for those whose

[1] *The citation is available at http://www.mathunion.org/icmi/activities/awards/the-hans-freudenthal-medal-for-2015/*

[2] *Afrikaans, derived from Dutch, is the language of the Dutch settlers, who came to South Africa before the English. According to Wikipedia, it is the mother tongue of about 13.5% of the South African population, which includes white and coloured (mixed racial descent) South Africans.*

mother tongue is Afrikaans. This is due to the overwhelming demand for English education among black South Africans. (Parents have the freedom to choose the medium of education – a freedom won through the historic struggles against the apartheid regime.) In fact, despite the official policy of mother-tongue-based education in the primary grades, there are hardly any science and mathematics textbooks in the African languages. This is perhaps because of the demand for English medium education – maths textbooks in African languages may have no takers. Part of the reason for such a demand lies in South Africa's recent political history, in the language-in-education policy that the apartheid regime tried to force on the black population. It decreed that the first eight years of schooling would be in the home language and that secondary school education would compulsorily be half (i.e., half the subjects) in English and half in Afrikaans. The policy, which was interpreted by the majority of the black population as a way of denying their access to English language and education, was one of the triggers for opposition to the regime.

The research studies by Adler and her colleagues contended with the reality of English being the language of learning and teaching in most schools. Adler proposed the concept of the "English language infrastructure" in a school environment, which refers to the kinds of English language resources available to the learners both in and outside school. She distinguished between environments where students had minimal or no exposure to English outside the school and those where students had exposure to spoken and written English outside the classroom. The former environment, Adler found, was typical of rural areas in some South African provinces. In such environments, she argued, English functioned essentially like a foreign language. In contrast, in urban and semi-urban contexts, where students were exposed to English outside school, English functioned like an "additional" language (i.e., a second or third language). The educational contexts in these two kinds of English learning environments were very different.

By official policy, and in actual practice, teaching in many classrooms in South Africa is

multilingual. Adler and her colleagues studied the practices adopted by teachers in a range of multilingual classrooms. One of the practices that she studied was code-switching, which refers to the switching between languages while speaking in the classroom. In many South African classrooms, teachers switch between English, which is the medium of instruction, and the home language of the children in the classroom. Indeed, code-switching in classroom teaching is not uncommon in English medium schools in India. Adler studied the prevalence of code-switching as well as the function that it served. It is natural to expect that code-switching would be more frequent in "English as a foreign language" environments, where students had little or no outside exposure to English. One of the surprising findings of her study was that code-switching was far less prevalent in classrooms where English functioned like a foreign language in comparison to classrooms where English functioned like an additional language. The reason was that in an environment where there was very limited English infrastructure in the surrounding community, it was the responsibility of the teachers to provide exposure to English. The students needed to learn English and the classroom was the only place where they had exposure to it. So teachers, usually guided by the school policy, tended to maximise their use of English in the lesson time available.

This finding pointed to the challenge faced by mathematics teachers in complex multilingual environments. They had responsibility for their students learning not only the mathematics in the curriculum, but also the English language in which mathematics was taught and learnt. In her analysis of mathematics lessons, Adler distinguished classroom talk that was exploratory in nature from discourse that was more formal and mathematical in character. The former allowed for exploring the meaning of the mathematical concepts and ideas through a two-way discussion and interaction. In the context of educational reforms that stressed the importance of exploratory talk for learning mathematics, Adler pointed out that the subject-specific mathematical language is equally important for students to acquire. In other words, formal mathematical discourse is as important as exploratory talk. Many teachers in her study recognized this and explicitly articulated the dilemmas that they faced in managing more than one language. They were trying to carefully balance the use of home language to facilitate exploration and understanding with the need, to learn the English language on the one hand, and the discourse and language of mathematics on the other.

Adler pointed out that the dilemma of code-switching faced by teachers is also an opportunity for the teachers' professional development, for crafting approaches to teaching mathematics that are context-specific, that use the resources of multiple languages in a thoughtful and explicit manner. It is such approaches that are more likely to be effective in classrooms in which teachers address several challenges at the same time. The other dilemmas faced by teachers that Adler identified in her work have to do with how much scaffolding to provide to students as they struggled to solve mathematical problems, and how explicit the teachers' explanations of concepts and procedures should be (Adler, 2001). These are dilemmas for the teacher because there are good reasons for both offering and withholding support. Similarly, too much or too little of explicit telling may inhibit learning.

The work of Adler and her colleagues on multilingual classrooms shaped this area of research internationally. The theoretical perspectives that she introduced have been useful for subsequent researchers. In the words of the award committee, Adler's work shows a "strong theoretical grounding that has served to advance the field's understanding of the relationship between language and mathematics in the classroom."

Alongside her research studies, Adler was active in shaping new approaches to the preparation of teachers. The segregated education policies of the apartheid era had led to a majority of black teachers entering the profession without adequate preparation. Most of them had a three-year teacher education degree, instead of a four-year degree which was required of teachers from the more advantaged communities. The post-apartheid South African government called for educational programmes that would allow under-qualified teachers to acquire the extra year of qualification. Many of these teachers did not have a strong background in terms of subject content. Adler stepped in to meet this challenge. In the mid-1990s, she co-ordinated the curriculum development for a one-year diploma programme at the University of Witwatersrand in teaching mathematics, science and English language. The challenge in the programme was to provide opportunities for teachers to gain knowledge and confidence in mathematics in a way that would positively impact their teaching. A few years later, Adler initiated and developed a curriculum for a post-Bachelor's honours programme in science and mathematics education. The programme is now a decade and half old and has produced a few hundred graduates, many of whom have played a leadership role in their schools. In both these programmes, central place was given to enhancing the mathematical knowledge that teachers needed to teach effectively.

Adler was part of the movement in mathematics education research that brought the issue of teachers' mathematical knowledge into central focus. This strand of work stems largely from Lee Shulman's work in the 1980s, in which he pointed out the neglect of subject matter (or content) knowledge in teacher education. Shulman introduced the now popular term "Pedagogical content knowledge" or PCK to signify "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of understanding" (Schulman, 1987). The majority of school (and college) teachers are subject teachers, typically teaching one of these subjects – science, mathematics, social science or language. Teacher education programmes usually assume that their student-teachers already have the required subject knowledge, since they have done a Bachelor's or Master's degree in the subject, and hence only focus on the pedagogical aspects. As a result, pre-service teachers, who specialize in mathematics as the subject, have little opportunity to revisit and strengthen their understanding of mathematics itself. The emphasis on teachers' subject matter knowledge, developed through the work of Shulman and many others including Adler, seeks to correct this trend. A good analysis of what is meant by deep understanding of school mathematics is presented in the famous book by Liping Ma, Knowing and Teaching Elementary Mathematics (Ma, 1999).

Adler's work takes a grounded approach, seeking to identify and describe mathematical aspects of classroom interaction, both in teacher education classrooms and school classrooms. What knowledge resources does the teacher draw upon and how does it shape the mathematics that emerges in the classroom? A central insight that underlies her analysis of classroom interaction is the understanding that pedagogic discourse involves the transmission of criteria. Teachers are continuously striving to pass on to students criteria for what is acceptable as a valid response, for what counts as mathematics, for what is acceptable as a justification for a given response and so on. The teacher's own judgement underlies the criteria that she chooses to transmit implicitly or explicitly to the students. Adler observed that teachers draw on four broad domains of knowledge to support their judgements: mathematical knowledge, everyday knowledge, professional knowledge and curriculum knowledge (Adler, 2012). She cautioned that when extra-mathematical domains are used to support judgements, the integrity of the mathematical idea must not be compromised.

In her recent work, Jill Adler has revisited the question of mathematical discourse in the classroom. In typical style, she has combined this research with intervention. She is leading a project aimed at improving the mathematics teaching and learning in an identified group of schools serving traditionally disadvantaged communities. In 2009, Adler was awarded a prestigious grant to carry out this project. The intervention was at multiple levels – providing opportunities to the teachers to strengthen their mathematical knowledge, evolving tools to track changes in teaching and learning gains, and developing a community of researchers and teachers engaged with the project.

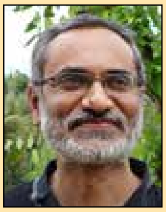
In this work, Adler, along with a group of colleagues, is shaping the tools and the framework to capture the mathematics in classroom interaction and discourse. Our own work at the Homi Bhabha Centre has shown that teachers do not simply repeat what is stated in the textbook; they do not merely articulate definitions, procedures or theorems in the classroom. Mere telling is generally ineffective in producing learning. Teachers should present examples, interpret the mathematical idea or concept using situations or contexts, design and assign tasks for students to complete, ask questions, design and use representations, moderate discussion, respond to students' utterances or writing, push certain lines of thinking, etc. In the course of doing this, the teacher unpacks the mathematics that is presented in the textbook, in a manner that is appropriate for her/his group of learners. If one examines the transcript (a text version usually prepared from a video recording) of an actual lesson, where there is a reasonable level of interaction between teachers and students, one gets an idea of the complexity of the activity of classroom teaching and learning. The more one pores over the transcript, the more one discovers of what may be going on in the lesson in terms of the teachers' goals, the students' thinking, the teachers' responses to this thinking and the dynamically evolving classroom context. Is there a systematic way of analysing the transcript for an understanding of what is occurring in the lesson? Can this understanding lead to a judgement of the mathematical quality of the lesson? Answering these questions calls for not only an adequate description of what is said, but also a principled interpretation of what remains implicit. Because what is implicit is important in understanding the teacher's and the students' utterances and actions. Adler's work is aimed at developing a framework for precisely such purposes, to understand the "mathematical discourse in instruction" (Adler & Ronda, 2015).

Like in her previous work, Adler brings powerful theoretical resources to this research. Using an eclectic approach, she combines perspectives from the Russian social psychologist, Lev Vygotsky, and the British sociologist of education, Basil Bernstein. This ongoing work promises to yield insights and tools that serve to better understand classroom teaching of mathematics and thereby design more effective professional development for teachers.

Adler's contributions to mathematics education go well beyond those of a researcher. I have already described her interventions in teacher education. She has contributed significantly to building the mathematics education community not only in South Africa but also in the Southern African countries. She chaired the programme committee of the 22nd Psychology of Mathematics Education (PME) conference in 1998. This is one of the most important annual conferences in mathematics education research and it was hosted in Africa for the first time in 1998. In South Africa, she has developed and guided teams of researchers – PhD students and post-docs, who have gone on to become established researchers making major contributions of their own. She oversaw the activities of ICMI as Vice-President for two terms. In this period, she initiated the African Congress in Mathematics Education (AFRICME), which is now held every four years and is emerging as a nucleating point for mathematics education research in Southern and East Africa. Adler has visited India several times, interacting with mathematics education researchers from the country. She was an invited speaker in the mathematics education section of the International Congress of Mathematicians (ICM) in Hyderabad in 2010. She has visited the Homi Bhabha Centre three times and has supported the research work at the Centre. She played an important role as a member of the committee that comprehensively reviewed the work of the Homi Bhabha Centre. To quote the award citation again, "she has played an outstanding leadership role in growing mathematics education research in South Africa, Africa, and beyond."

References

1. Adler, J. (2001). Teaching mathematics in multilingual classrooms. Kluwer Academic Publishers, Dordrecht.
2. Adler, J. (2012). Knowledge resources in and for school mathematics teaching. In G Gueudet, B., Pepin, B., and Trouche, L. (Eds.) Mathematics Curriculum Materials and Teacher Development: From text to 'lived resources' (pp. 3--22). Springer, The Netherlands.
3. Adler, J. & Ronda, E. (2015). A framework for describing Mathematics Discourse in Instruction and interpreting differences in teaching. African Journal for Research in Mathematics Science and Technology Education (AJRMSTE), 19(3), 237--254.
4. Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. Lawrence Erlbaum Associates Inc., Mahwah, NJ.
5. Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. Harvard Educational Review, 57(1), 1–23.



K. SUBRAMANIAM is professor of mathematics education at the Homi Bhabha Centre for Science Education (HBCSE), Mumbai. His areas of research are characterising learning strands for topics in middle school mathematics, like fractions and algebra, and developing models for the professional development of mathematics teachers. He has interest in cognitive science and philosophy, especially in relation to education and to maths learning. He has contributed to the development of the national curriculum framework in mathematics (NCF 2005), and to the development of mathematics textbooks at the primary level. He may be contacted at subra@hbcse.tifr.res.in. (See also: http://mathedu.hbcse.tifr.res.in.)

Through the Symmetry Lens

Part II - Wallpaper Patterns and Symmetry Around Us

Features

Geetha Venkataraman

This is the second part of a two-part article whose aim is to familiarise the reader with both the mathematical concept and an intrinsic idea of symmetry. The first part of the article concentrated on a 'working definition' of symmetry and also laid the mathematical base to understand symmetry. It discussed symmetries of figures that can be drawn on a sheet of paper and of a particular type of infinite pattern called a *strip pattern* or *a frieze pattern.*

In this part we will concentrate on another infinite two-dimensional pattern called the *wallpaper pattern* and also explore aspects of symmetry in the everyday objects around us. For ease, we reiterate the 'working definition' of symmetry here.

Intuitively, symmetry can be thought of as an action performed on an object, which leaves the object looking exactly the same and occupying the exact same space as before. If a person closes her eyes while the action is being performed, she will not know that any action has been performed.

Objects that can be drawn on a sheet of paper (finite planar objects) can have only two kinds of symmetries: rotations and possibly

Keywords: *symmetry, reflection, rotation, translation, glide, frieze pattern, wallpaper pattern, tessellation, Alhambra, Escher*

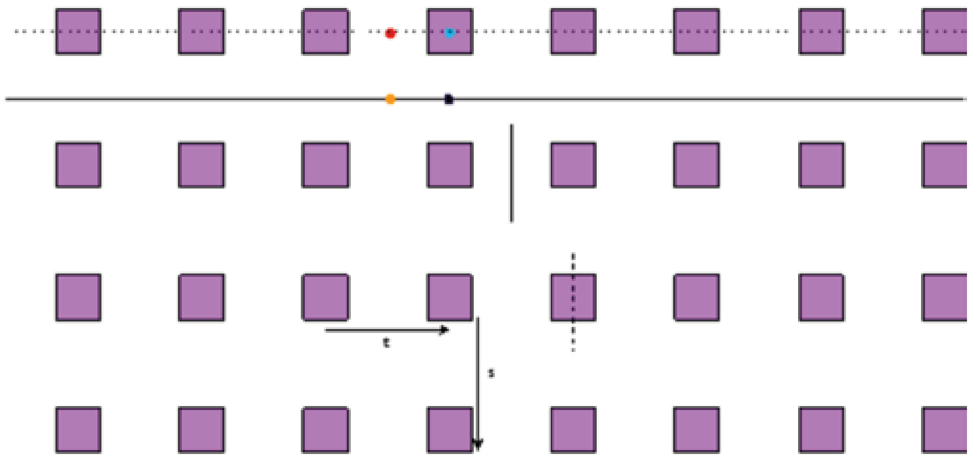


Figure 1

reflections. A strip pattern has a new kind of symmetry, called *translation symmetry*. It may also have symmetries of the following type: 180-degree rotations, reflections and glide-reflections. Part I of the article discusses all this in some detail and the reader is advised to consult the same. We now explore symmetries of wallpaper patterns or tessellations.

Wallpaper Patterns or Tessellations

As the name suggests this will be an infinite pattern that will spread both in the left-right and top-down directions. One way of creating a wallpaper pattern is by choosing a strip pattern and stacking it at equal intervals on top as well as below. An example of a wallpaper pattern is Figure 1.

The wallpaper pattern in Figure 1 has translations in two different directions indicated by **s** and **t**. It has two types of vertical lines of reflection, namely the dotted type and the solid line, and similarly two different types of horizontal lines of reflection. There are four different types of rotocentres indicated about each of which a 180-degree rotation is possible. (Note that since the length of **t** is different from **s**, the grid created is basically rectangular and not square, so reflections about diagonal lines, or 90-degree rotations about the rotocentres will not give us a symmetry of the wallpaper pattern.) Since we have symmetries that are translations and reflections we will also have glide reflections. The reader may try to mark glide reflections on the wallpaper pattern above.

Classifying wallpaper patterns is much more complicated. There are 17 wallpaper patterns[1] in all. It is believed that the 14th-century Alhambra Palace in Granada, Spain, has all 17 patterns exhibited. The classification in a sense is based on the underlying grid (or lattice) in a wallpaper pattern.

For example, consider the example of the wallpaper pattern illustrated and discussed above. We can think of it being created not only as a strip being repeated at equal intervals in the top-down direction but also in the following way. Think of the plane being covered by a rectangular grid or lattice as shown in Figure 2. (Black lines with circles marking the vertices.) Then replace the

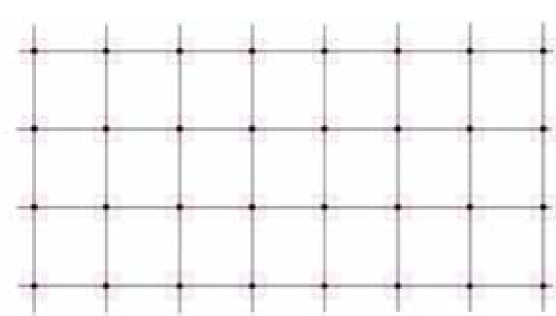


Figure 2

[1] *For more on classification of the wallpaper patterns please see Contemporary Abstract Algebra by Joseph Gallian. The chapter on Frieze Groups and Crystallographic Groups provides algorithms for classifying strip or frieze patterns and also for classifying wallpaper patterns or tessellations.*

vertices of the lattice with squares as shown. So the wallpaper is created by the light pink squares, replacing the black dots and then removing all the grid lines.

It turns out that in any wallpaper pattern, there are only five possible underlying grids or lattices, namely, square, rectangle (non-square), parallelogram (non-square, non-rectangular), equilateral triangle and regular hexagon. Using these and the different positioning of certain basic motifs, 17 wallpaper patterns are created.

Some more examples of wallpaper patterns using the different underlying lattices are given below. The reader is invited to discover which of the following symmetries exist in the examples of wallpaper patterns given below: translations, rotations, reflections and glide-reflections. Note that in the case of translations, the basic distance with directions should be marked. For rotations, rotocentres as well as the angle of rotation should be specified. For reflections, the lines of reflection and for a glide-reflection, the glide line and reflection line ought to be marked.

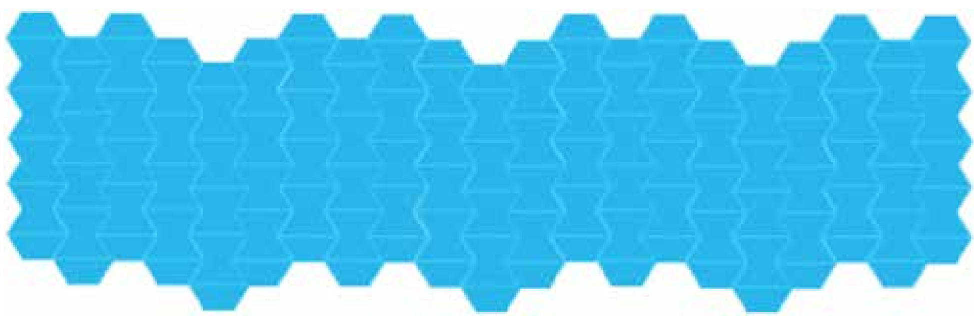


Figure 3 - Wallpaper Pattern based on a Regular Hexagon Lattice

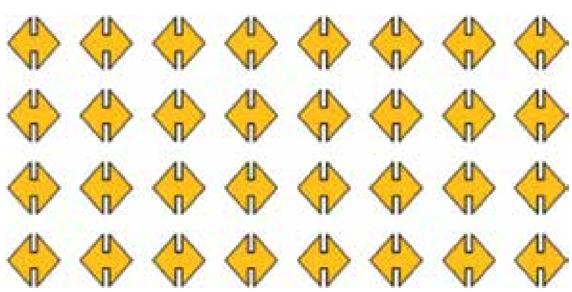


Figure 4 - Wallpaper Pattern based on a Square Lattice

Figure 5 - Wallpaper Pattern based on a Non-rectangular Parallelogram Lattice

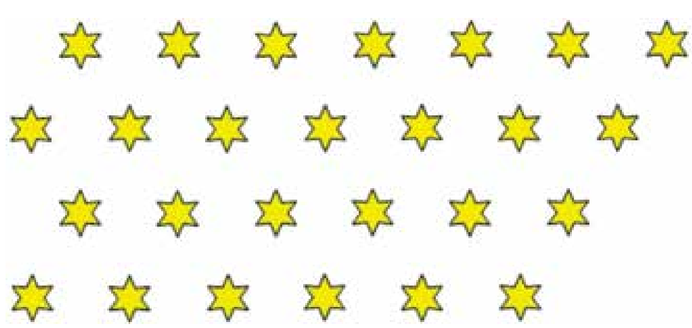


Figure 6 - Wallpaper Pattern based on an Equilateral Triangle Lattice

Exploring Symmetry Around Us

Now that we have an idea of the different types of symmetries that finite planar objects, strip patterns and wallpaper patterns have, we can use our knowledge to view the world around us through the symmetry lens. Of course, we have the ability so far only to make sense of the symmetries of planar objects. Symmetry of three-dimensional objects can also be studied in a similar manner but that is material for a different article.

We provide below examples of objects that we encounter around us often in our everyday lives and explore the underlying symmetry in some of those. Other examples are provided for the reader to peruse and analyse at leisure. One caveat, when exploring symmetry in real life, is that we may need to ignore imperfections or some parts of the scenery in order to appreciate the beauty of symmetry.

Nature abounds in plentiful examples of symmetry. Below are two flowers (Figure 7). The flower on the left has only rotational symmetries and no reflection symmetry. Consider the purple flower. We have marked the flower with *5* brown lines that have an angle of 72-degrees each between successive lines. As can be seen from the figure, the respective lines do not bisect the petals exactly. Indeed if we join the mid-point of the upper tips of the petals, we find that the figure formed is *5*-sided but not a regular pentagon. Using the brown lines we can reason that there are 'approximate' rotations of 0, 72, 144, 216 and 288-degrees. Thus we could classify this flower as of type C_5. (Please refer to Part I of the article in the March 2016 issue of *At Right Angles* for the meanings of these symbols.)

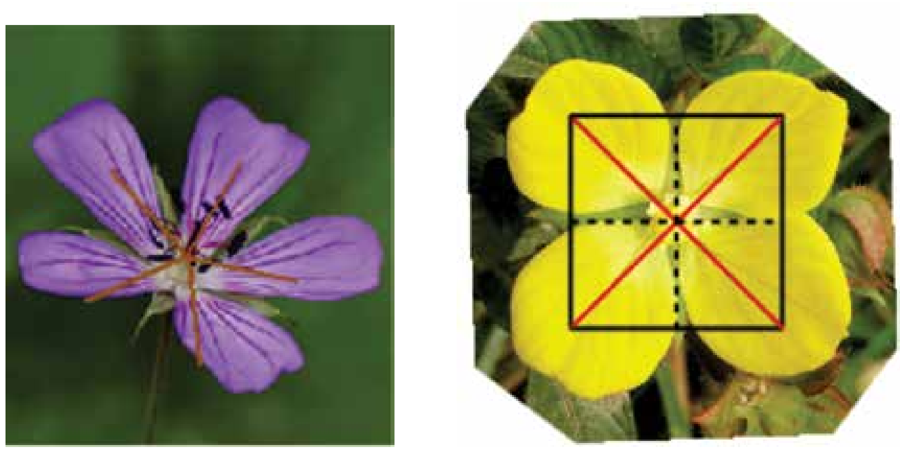


Figure 7

Figure 8

Figure 9

Figure 11

Figure 10

In the case of the yellow flower, the square grid indicates the possibility of 4 rotations of 0, 90, 180 and 270-degrees and 4 reflections; it is therefore of D_4 type. The butterfly in Figure 8 shows reflection symmetry and the 0-degree do-nothing rotational symmetry; it is thus of D_2 type.

We can find examples of strip patterns and wallpaper patterns in the designs on clothes that we wear. Sari borders are often fine examples of strip patterns while the interior provides examples of wallpaper patterns. Some examples are given below.

The pattern shown in Figure 9, which is part of a sari, can be regarded as a wallpaper pattern based on a rectangular grid having translations in two different directions, two different types of vertical lines of reflection and glide reflections. The only rotation possible is the 0-degree or the do-nothing rotation.

The strip in Figure 10 is part of a sari border. On analyzing it we note that there are translations, reflection symmetries in vertical and horizontal direction, 180-degree rotations and glide reflection. It is therefore a Type VII strip pattern.

The four leaves taken together as a motif are part of a cushion cover. It can be seen that this motif will have 4 rotations and 4 reflections and so it has classification type D_4.

The collage in Figure 11 has been made with four images of artwork from the Crafts Museum in Delhi. The top and bottom images should be analysed as strip patterns, the image in the centre right as a wallpaper pattern and the centre left image as a finite planar motif.

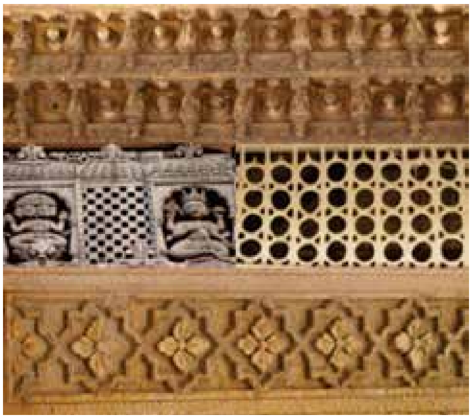


Figure 12

Figure 13

Old monuments, newer buildings, grills on windows, balcony railings, fences, and so on too provide for interesting examples to view via symmetry. The collage in Figure 12 provides examples of this and can be analysed in a manner similar to the earlier collage.

It would be a grave lacuna not to mention the artwork of M C Escher in an article on symmetry. However due to copyright restrictions we cannot reproduce photographs of Escher's work here. It is recommended that the reader peruse his art on the official website www.mcescher.com. We however leave the reader with an image of a floor puzzle inspired by Escher to enjoy at leisure and to discover its secrets via symmetry; see Figure 13.

Bibliography

1. M. A. Armstrong, *Groups and Symmetry,* Springer Verlag, 1988.
2. David W. Farmer, *Groups and Symmetry*: A Guide to Discovering Mathematics, American Mathematical Society, 1995.
3. Joseph A. Gallian, *Contemporary Abstract Algebra*, 7th edition, Brooks/Cole Cengage Learning, 2010.
4. Kristopher Tapp, *Symmetry*: *A Mathematical Exploration*, Springer, 2012.
5. Herman Weyl, *Symmetry*, Princeton University Press, 1952.

GEETHA VENKATARAMAN is a Professor of Mathematics at Ambedkar University Delhi. Her area of research is in finite group theory. She has coauthored a research monograph, Enumeration of finite groups, published by Cambridge University Press, UK. She is also interested in issues related to math education and women in mathematics. She completed her MA and DPhil from the University of Oxford. She taught at St. Stephen's College, University of Delhi from 1993 to 2010. Geetha has served on several curriculum development boards at the school level, undergraduate level and postgraduate level. She was Dean, School of Undergraduate Studies at Ambedkar University Delhi during 2011-2013. She is currently Dean, Assessment, Evaluation and Student Progression at Ambedkar University Delhi.

The Magical World of Infinities

Part II

Features

Shashidhar Jagadeeshan

Introduction

In the previous article we encountered the strange world of infinities, where a lot of our intuitive sense of how infinite sets should behave started breaking down. We saw for example that infinite sets can have subsets which have as many elements as the original set; we also saw our intuition about length breaking down. No matter what the lengths of the two lines are, they ended up having the same number of points. Moreover, all the examples of infinite sets we encountered ended up having the same number of elements. You might naturally assume that there is only one kind of infinity – which is what perhaps you had assumed right from the beginning?

In the last section of Part I of this article (*AtRiA*, March 2016), we had hinted at the possibility of there being different kinds of infinities. If there are, can we mathematically prove they exist? How many different infinities are there really? In this article we will answer these questions.

Recall that the cardinality of a set counts the number of elements it contains. We denote the cardinality of a set X by $|X|$. In some cases we have special symbols denoting the cardinality of sets.

Keywords: *Infinity, set, cardinality, countable, natural number, real number, unit interval, continuum, one-one correspondence, injective function*

For example, $\aleph_0$ represents the cardinality of the set $\mathbb{N}$ of natural numbers, and **c** (called the **continuum**) represents the cardinality of the set $\mathbb{R}$ of real numbers. If an infinite set has cardinality $\aleph_0$, then we say that this set is *countable*. $\aleph_0$ and **c** are examples of cardinal numbers.

At this stage it would be good to introduce some ideas and techniques that we use to compare two infinite sets, since we will be using them quite often. In the previous article we introduced the idea of 1-1 correspondence between two sets X and Y, and said that $|X| = |Y|$ if and only if we can find a 1-1 correspondence between X and Y. In a 1-1 correspondence we have a function which associates every element of X with a unique element of Y, and by inverting this association, every element of Y is associated with a unique element of X. A slightly weaker notion than 1-1 correspondence is the idea of an *injective function* (often referred to as a '1-1 function' as opposed to '1-1 correspondence', but we will use the term injective function to avoid confusion). An injective function $f : X \to Y$ is a function that satisfies the property that if $f(a) = f(b)$, then $a = b$. Notice that in an injective function we cannot be sure that every element in the set Y has a partner in X; however, if an element in the set Y does have a partner in X, then that partner is unique. Figure 1 illustrates an injective function.

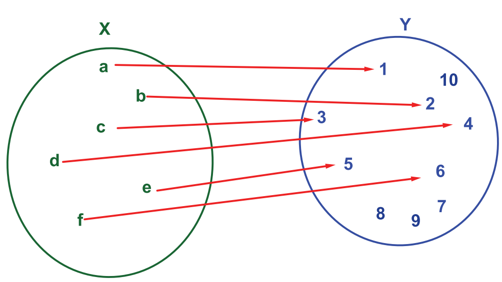


Figure 1

It is clear from Figure 1 that X and Y do not have the same number of elements and $|X| \leq |Y|$, and moreover there is a 1-1 correspondence between X and some subset of Y. Whenever we have $A \subseteq B$, we can construct the obvious injective function $i : A \to B$, which takes each element of A to itself, that is, $i(a) = a$, $\forall a \in A$.

If there is an injective function from a set X to a set Y, we can see that $|X| \leq |Y|$. The **Schroeder-Bernstein Theorem** (sometimes Cantor's name is also added) states that if we have an injective function $f : X \to Y$ and another injective function $g : Y \to X$ (note that f and g need not be inverses of each other), then $|X| = |Y|$.

We are now ready to compare the set $\mathbb{R}$ of real numbers with the set $\mathbb{N}$ of natural numbers and to show that $\aleph_0 < \mathbf{c}$.

Are real numbers countable?

Let us start by comparing the set of all real numbers between 0 and 1 (we denote this set by $(0, 1)$) and the set $\mathbb{N}$ of natural numbers.

We remind readers of the fact that the real numbers have decimal representations. Furthermore, by inserting a string of zeros, we can make it an infinite decimal representation. For example, $\frac{1}{4} = 0.25000\ldots$ (with infinitely many trailing zeros) and $\frac{1}{7} = 0.142857142857142857142857142857714\ldots$. The question is, are these representations unique? You might have come across the curious fact that $0.99999\cdots = 1$ (this is really fascinating, if you have not already done so, see if you can prove it for yourself). So it appears that we have two possible decimal representations for some real numbers. It turns out that if we can take care of the case of repeating nines, we can then have unique decimal representations for all real numbers. So, if we decide that we will choose to represent numbers like $0.2999\ldots$ by $0.3000\ldots$, then every member of our set has a unique decimal representation.

Here is Cantor's proof that there are more real numbers between 0 and 1 than there are natural numbers.

It is relatively easy to see that $\aleph_0 \leq |(0, 1)|$. For, consider the following injective function:

$$g : \mathbb{N} \to (0, 1), \quad g(n) = \frac{1}{n + 1};$$

then g is clearly an injective function, and from our discussion above we get $\aleph_0 \leq |(0, 1)|$. What we want to show is that the equality is not possible; that is, $\aleph_0 < |(0, 1)|$.

Remember, in order to do this we need to establish that it is *impossible* to have a 1-1 correspondence between the set $(0,1)$ and $\mathbb{N}$. We do so by assuming the contrary; that is, we assume that there does exist a 1-1 correspondence between these two sets and keep arguing logically, step by step, until something goes wrong! The only reason for something to go wrong could then be that we made an erroneous assumption in the beginning.

If there is a 1-1 correspondence between the set $\mathbb{N}$ of natural numbers and the set $(0,1)$, we can assign a natural number to each element in $(0,1)$. Let us denote the number in $(0,1)$ associated with 1 as a_1, the number associated with 2 as a_2 and so on, allowing us to enumerate the elements of the set $(0,1)$, using natural numbers thus:

$$(0,1) = \{a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots\}.$$

Let us further denote each element in the above list as a decimal expansion, and let us do it in a manner in which a clear pattern emerges.

$$\begin{aligned}
a_1 &= 0.a_{1,1}\ a_{1,2}\ a_{1,3}\ \ldots\ a_{1,n}\ \ldots\\
a_2 &= 0.a_{2,1}\ a_{2,2}\ a_{2,3}\ \ldots\ a_{2,n}\ \ldots\\
a_3 &= 0.a_{3,1}\ a_{3,2}\ a_{3,3}\ \ldots\ a_{3,n}\ \ldots\\
&\ \vdots\\
a_n &= 0.a_{n,1}\ a_{n,2}\ a_{n,3}\ \ldots\ a_{n,n}\ \ldots\\
&\ \vdots
\end{aligned}$$

Now here is where Cantor's brilliance can be seen again. He defines a new element

$$b = 0.b_1\ b_2\ b_3\ \ldots\ b_n\ \ldots$$

in the following manner. Let b_1 be any integer (between 1 and 8) other than $a_{1,1}$; let b_2 be any integer (between 1 and 8) other than $a_{2,2}$; let b_3 be any integer (between 1 and 8) other than $a_{3,3}$; and so on. So b_n is any integer between 1 and 8 other than $a_{n,n}$. Notice that the decimal expansion of b differs from the decimal expansion of a_1 in at least one place (namely, $a_{1,1}$) and similarly from the decimal expansion of a_2 in at least one place and in this way from every element in our list above. (The reason why we do not allow the integers 0 or 9 is to make sure that b does not have all zeros or nines in its decimal expansion.) By this clever construction (which is called Cantor's *diagonalization* technique) we have found an element b such that

$$b \in (0,1) \text{ and } b \notin \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

But this is a contradiction—because we had assumed that every element in the set $(0,1)$ is accounted for the list $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$! Where did we go wrong? If you go back and check all the steps in our argument, you will find that the mistake was in assuming that there is a one-to-one correspondence between $(0,1)$ and $\mathbb{N}$. In fact, what Cantor managed to show was that no matter how clever you are, you cannot come up with a 1-1 correspondence between the above two sets, because the moment you do, and you enumerate the elements of $(0,1)$ using the natural numbers, the diagonalization process guarantees that you will always come up with an element in $(0,1)$ which is not in the list that you had made! This establishes the fact that the set of real numbers contains more elements than the set of natural numbers, and therefore that $\aleph_0 < |(0,1)|$ or in other words $(0,1)$ is not countable.

What about the real numbers, are they countable? In Part I of this article we showed that there is a 1-1 correspondence between the set $(-1,1)$ and $\mathbb{R}$. We can use a similar argument to show that there is a 1-1 correspondence between $(0,1)$ and $\mathbb{R}$ and, in fact, between any open interval in the set of real numbers and $\mathbb{R}$. I hope this amazing fact has not slipped by the reader, that the set of real numbers $\mathbb{R}$ and any open interval contained in $\mathbb{R}$ have the same cardinality; namely, the continuum $\mathbf{c}$. Clearly, since $(0,1)$ is not countable, and $\mathbb{R}$ has the same cardinality as $(0,1)$, $\mathbb{R}$ is not countable and $\aleph_0 < \mathbf{c}$.

Cantor thus managed to introduce a new infinity! He showed that infinite sets are not all of the same size, that there are different types of infinite sets, which differ because of their sizes. This unleashes a whole set of questions about how many different kinds of infinities there are. It turns out that it is not so straightforward to generate new infinities. In order to illustrate this, we now compare the number of points in a square with the number of points on one of its edges and compare the number of points in a cube with the number of points on one of its edges. Be prepared to be surprised!

Edges, Squares and Cubes

We now compare the number of points in a square with the number of points on one of its edges. Our intuition tells us that since we are comparing objects in different dimensions, clearly the number of points in the square should be far larger. But, wait and see. ...

Let us take our square to be the unit square, that is, all points on the coordinate plane whose x- and y-coordinates satisfy the inequalities: $0 < x < 1$ and $0 < y < 1$. For the edge, we consider the unit interval, that is all points t such that $0 < t < 1$. From now on we will be a bit lazy, and when we say 'unit square' or the 'unit interval' we will mean the 'set of all points in the unit square' and the 'set of all points in the unit interval'. This laziness will hopefully allow for a more succinct expression, without loss of clarity! Let us denote the unit square as $(0, 1) \times (0, 1)$, and the edge (i.e., the unit interval) as before by $(0, 1)$.

Notice that every point in the square has coordinates (x, y) satisfying the conditions above, so we can represent x and y in terms of their decimal expansions to get $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$. Again assuming that we will consider decimal numbers like $0.5999\ldots$ and $0.6000\ldots$ as being the same, both x and y have unique decimal representations. We then define the function $f\colon (0, 1) \times (0, 1) \to (0, 1)$ by:

$$f(0.x_1x_2x_3\ldots,\ 0.y_1y_2y_3\ldots) = 0.x_1y_1x_2y_2x_3y_3\ldots.$$

You can see that every point in the square has been mapped onto a unique point on the edge, and it is not hard to see that f is an injective function. In other words, starting from the image of a point in the square we can unravel the above process and land back at the original point in the square. Does every point in the unit interval have a partner in the square? No. Consider the following point in the unit interval: $0.909090\ldots$. When we unravel this point using the procedure described above we end up with the point $(0.999\ldots, 0.000\ldots)$. Now we agreed to represent $0.9\ldots$ as 1. So we end up at $(1, 0)$, which is not in the unit square! Similarly the point $0.191919\ldots$ will be mapped onto $(0.1\ldots, 1)$ and the point $0.010101\ldots$ will be mapped onto $(0, 0.1\ldots)$, both of which do not belong to the unit square.

We illustrate this via Figure 2.

While f is injective, it is not a 1-1 correspondence. To prove our result that the unit square and the unit interval have the same cardinality, we resort to the Schroeder-Bernstein Theorem. Since we have an injective function $f\colon (0, 1) \times (0, 1) \to (0, 1)$, we may infer that the cardinality of the unit square is smaller than that of the unit interval. From the fact that we can easily find an injective function from the unit interval to the unit square, the cardinality of the unit interval is smaller than that of the unit square. From Schroeder-Bernstein, we now infer that the unit square and the unit interval have the same cardinality.

Cantor himself was amazed by this result. In fact he spent the years 1871–1874 trying to prove it was false, and finally when he came upon this result, he wrote to his friend, the German mathematician Richard Dedekind, "I see it, but I don't believe it."

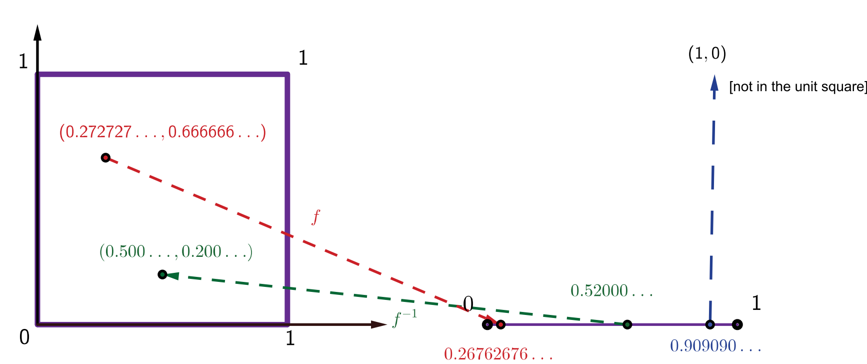


Figure 2

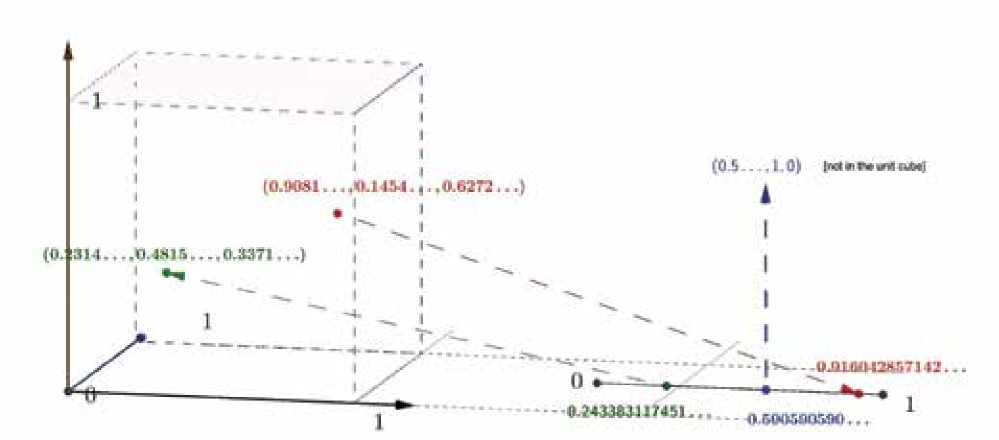


Figure 3

The same technique can be used to show that the cardinality of the unit cube is the same as the cardinality of any one of its edges, or in fact the cardinality of an n-dimensional unit cube is the same as the cardinality of any one of its edges! We illustrate a similar injective function between the 3-dimensional cube and the unit interval in Figure 3.

Cantor once again established, using the Schroeder-Bernstein Theorem, that the cardinality of the unit square is the continuum **c**, as is that of the unit cube. He went on to establish that the cardinality of the points in a 2-dimensional plane, in 3-dimensional space and in n-dimensional space are all the continuum!!

So you can see how hard it is to find a larger infinity. Even going to higher dimensions does not seem to produce a set with greater cardinality. Cantor did a lot of arithmetic with cardinal numbers and showed that if you combine sets by taking unions or cross products, you do not get a new set with larger cardinality. Even if you take the infinite union of infinite sets (remember the infinitely many buses each with infinitely many passengers arriving at the Hilbert hotel), you still do not get a larger infinity! These ideas are rather technical to get into in this article, but one can ask the question: Have we hit the end of the road as far as infinities are concerned? That is, are $\aleph_0$ and **c** the only cardinal numbers that exist? Trust Cantor to prove our intuition wrong yet again. He showed that not only are there more cardinal numbers, there are in fact infinitely many of them!

The hierarchy of infinities

In order to understand the hierarchy of infinities, we need to introduce the idea of a **power set**. Given a set A, the power set of A consists of all the subsets of A. For example if $A = \{1, 2, 3\}$ then

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{A\}\}.$$

Notice that both the empty set ($\{\}$) and the whole set A are elements of $P(A)$. It is a nice exercise to show that if a finite set A has n elements then $P(A)$ has 2^n elements.

It took a Cantor to look at power sets of infinite sets to find larger infinities! In what is now called simply *Cantor's Theorem*, he showed that for any infinite set X, the power set $P(X)$ has a larger cardinality than the set X.

Cantor's theorem
For any infinite set X*,* $\lvert P(X)\rvert > \lvert X\rvert$.

Proof. $X \subseteq P(X)$, therefore $|X| \leq |P(X)|$. What we now need to show is that it is impossible to have a 1-1 correspondence between X and $P(X)$. As before we will do this by assuming the contrary. Let us begin by assuming that there **is** a 1-1 correspondence between X and $P(X)$. This means that for every element of X we have managed to associate a unique element of $P(X)$, or in other words a unique subset of X, and vice versa.

Let us for the sake of illustration assume that elements of X will be denoted by lowercase letters and elements of $P(X)$ will be denoted by uppercase letters. So:

$$X = \{a, b, c, d, e, f, g, t, \dots\} \text{ and}$$
$$P(X) = \Big\{X, \{\}, A, B, C, D, E \dots\Big\}.$$

Suppose (for example) that $C = \{b\}$ and $E = \{a, d, g, t\}$. Let us assume that our 1-1 correspondence associates elements from X to $P(X)$ in the following manner:

$$\begin{array}{ll} a \to X, & b \to \{\}, \\ c \to A, & d \to B, \\ e \to C, & g \to D, \\ t \to E, & \dots . \end{array}$$

Cantor's genius lay in his ability to realize that there are two kinds of elements in the set X. One kind of element is associated with a set that contains that element itself. For example, a is associated with X, and X contains a. Similarly, t is associated with E, and E contains t. We will call all such elements **Insiders**. The other kind is where the element does not belong to the set with which it is associated. For example, b is associated with the empty set $\{\}$, and clearly $b \notin \{\}$; and e is associated with C, and $e \notin C$. We call all such elements **Outsiders**. Now clearly, every element in X is either an Insider or an Outsider.

Let us collect all the Outsider elements of the set X and call this the *Outsider Subset* of X; we denote it by O. (Notice that O is not empty, because it contains at least β.) Now, if we have a 1-1 correspondence between X and $P(X)$, there must be some element from X associated with O. Let us assume that this element is s. The natural question is, is s an Insider or an Outsider? Suppose that s is an Insider; then by definition, $s \in O$. But O contains all those elements that do not belong to the set with which they are associated, so s cannot belong to O; hence s is an Outsider! Now, suppose that s is an Outsider, then by definition, it should not belong the set with which it is associated, in our case O. But O is the *Outsider Subset* of X and contains all the Outsider elements, so that forces s to be in O, making it (s) an Insider! So you see, we cannot win! If s is an Insider, then it must be an Outsider; and if it is an Outsider, then it must be an Insider!!

This absurd situation arose because we had assumed there was a 1-1 correspondence between X and $P(X)$. It follows that $|P(X)| > |X|$ for any infinite set X. $\square$

I hope the reader appreciates the fact that every time we think we can set up a 1-1 correspondence between a set and its power set, we will produce the *Outsider Subset* and run into the same absurdity we just did.

Cantor not only showed how to produce larger infinities, but also showed how to produce infinitely many infinities! For if we start with X and denote $P(X)$ to be X_1, then by this logic we denote $X_2 = P(X_1)$ and so on. Therefore we then get the following nested sequence of infinite cardinal numbers:

$$|X| < |X_1| < |X_2| < |X_3| \dots$$

Much more than we bargained for!

The continuum hypothesis

We end this two-part article by explaining where $\aleph_0$ belongs in the hierarchy of infinities and introducing the famous continuum hypothesis.

In this world of infinities, does it even make sense to ask, is there a smallest infinity? Let us start with any infinite set S and take a rather naive approach of creating a subset which is countable. Choose any element s_1 from S. Since S is infinite, $S - \{s_1\}$ is not empty. Choose another element s_2 in this set; clearly, $s_2 \neq s_1$. Again, since S is infinite, $S - \{s_1, s_2\}$ is not empty. Continuing in this vein and choosing an element s_n for each natural number n, we can produce an infinite countable set $T = \{s_1, s_2, s_3, \dots, s_n, \dots\}$. Since $T \subseteq S$, we have $|T| \leq |S|$. What our naive approach tells us is that no matter what infinite set S we start with, we have $\aleph_0 \leq |S|$. This tells us that $\aleph_0$ *is the smallest cardinal number*.

Cantor decided to denote the cardinal number just bigger than by $\aleph_0$ by $\aleph_1$, and so on, producing what we would technically call a partially ordered set of cardinal numbers $\{\aleph_0 < \aleph_1 < \aleph_2 \dots\}$. For Cantor the next obvious question was how to do you get $\aleph_1$ from $\aleph_0$?

What if you look at the power set of the natural numbers. Here is what he found (the modern notation for the cardinality of $|P(\mathbb{N})|$ is $2^{\aleph_0}$):

Another amazing result by Cantor

$$2^{\aleph_0} = \mathbf{c}.$$

That is, there is a 1-1 correspondence between the set of real numbers $\mathbb{R}$ and the set of all possible subsets of $\mathbb{N}$. The proof of this result is beyond the scope of this article, but we urge the interested reader to look up one of the references.

So is $\aleph_1 = \mathbf{c}$? In other words, is $\mathbf{c}$ the next infinity after $\aleph_0$? That, my friend, is the famous continuum hypothesis.

The continuum hypothesis

$$2^{\aleph_0} = \aleph_1.$$

The continuum hypothesis has attracted some of the greatest minds in mathematics. Cantor himself spent the rest of his life trying to prove it. The German mathematician David Hilbert, who was a keen admirer of Cantor (recall the quote from Part I) proposed 23 famous unsolved problems in 1900 during the International Congress of Mathematicians in Sorbonne. These problems have influenced the growth and direction of mathematics to this day. The continuum hypothesis was the first on his list!

The continuum hypothesis has been settled in a strange way, perhaps not to the satisfaction of all. The approach to resolving the continuum hypothesis has been somewhat akin to how the question of whether Euclid's fifth postulate was really needed or not. Remember that in that case, assuming the negation of the postulate led to new non-Euclidean geometries.

In the case of the continuum hypothesis, the famous logician Kurt Gödel showed in 1940 that assuming that the continuum hypothesis is true does not lead to any contradictions assuming the 'standard' axioms of set theory, and Paul Cohen showed in 1963 that assuming that the continuum hypothesis is not true also does not lead to any contradictions. In a sense what Gödel and Cohen showed was that the continuum hypothesis is independent of the standard axioms of set theory.

I hope that you have had a taste of the infinite, and will now be lured to pursue many of its other attributes on your own.

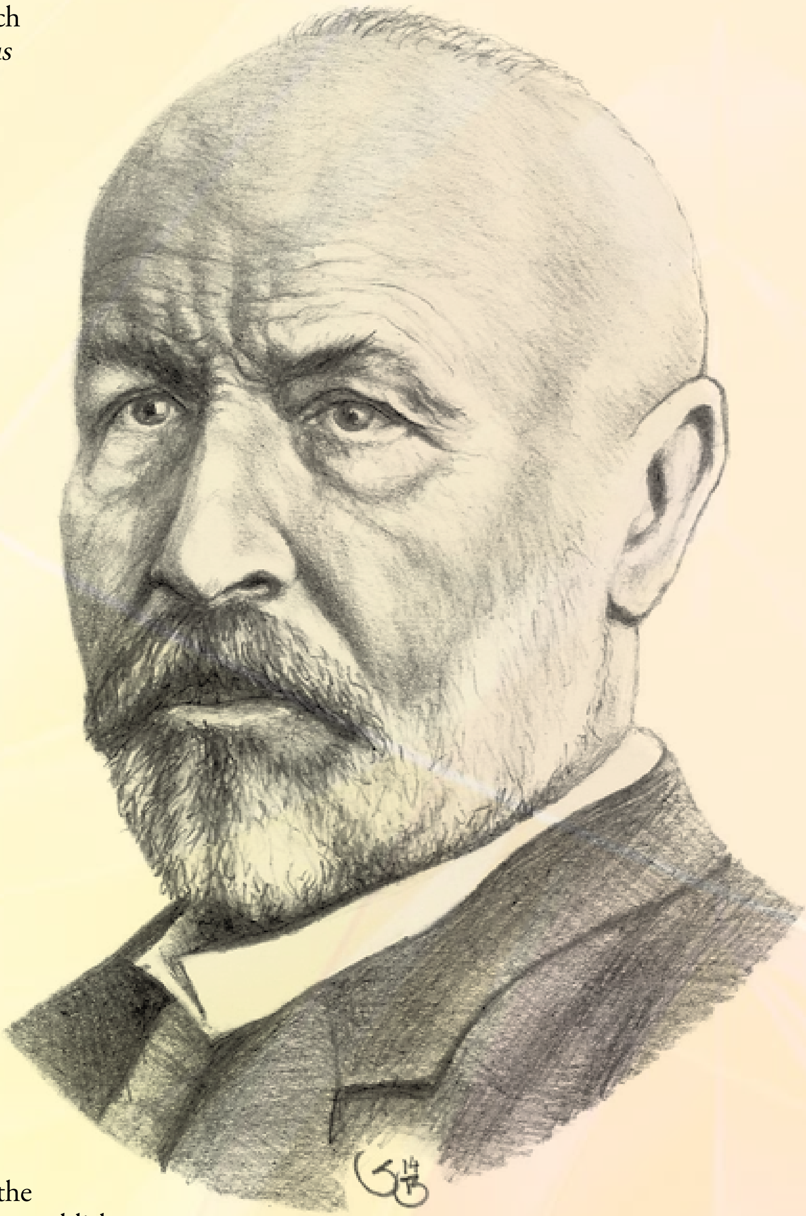
References

1. Dunham, William. *Journey Through Genius: The Great Theorems of Mathematics*. John Wiley & Sons, 1990.
2. Lang, Serge. *Math! Encounters with High School Students*. Springer-Verlag,1985.
3. Lipschutz, Seymour. *Set Theory and Related Topics*, Schaum's Outline Series, McGraw-Hill Book Company, Singapore, 1981.
4. Strogatz, Steven. *The Hilbert Hotel*, http://opinionator.blogs.nytimes.com/2010/05/09/the-hilbert-hotel/
5. Suber, Peter. *A Crash Course in the Mathematics of Infinite Sets*, http://legacy.earlham.edu/ peters/writing/infapp.htm.

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He may be contacted at jshashidhar@gmail.com.

GEORG CANTOR

Georg Cantor was born in 1845 in St Petersburg, Russia, and when he was eleven he moved with his family to Germany. Cantor's father was a devout Christian and had a strong influence on his son. In a letter to 15-year-old Cantor, he wrote: *". . .How often the most promising individuals are defeated after a tenuous, weak resistance in their first struggle following their entry into practical affairs. Their courage broken, they atrophy completely. . . "* He goes on to say that such people lacked *"that steady heart"* and the *"truly religious spirit"* which one obtained by a *"humble feeling of the most reverence for God."* Cantor's father seemed to have an uncanny prescience for the enormous difficulties that Cantor would face. This strong religious influence would shape Cantor's world-view, and perhaps explains how he dealt with the severe criticism he experienced in his later life as a mathematician.



Cantor had an urge to study mathematics from a very early age. In 1862 he wrote to his father: *". . . I hope that you will still be proud of me one day, dear Father, for my soul, my entire being lives in my calling. . . "* In 1866, Cantor completed his study in the University of Berlin, studying with some of the great mathematicians of the time: Weierstrass, Kummer and Kronecker. In 1867, he began working at the University of Halle, living there till his death in 1916.

Cantor's early work was in number theory. In Halle he studied trigonometric series, which led him to explore the real numbers (the continuum) very deeply. He worked on the construction of irrational numbers from an infinite sequence of rational numbers, and this might have made him begin to suspect that there are more irrational than rational numbers. In an 1873 letter to Dedekind, he described the notion of using one-to-one correspondence to compare two infinite sets. Beginning with an 1874 paper in which he established the nondenumerability of the continuum, Cantor went on to publish ground-breaking results in set theory and the theory of transfinite numbers for the next 25 years.

From the outset, Cantor's work on infinities invited the ire of mathematicians, philosophers and theologians. Cantor had ventured into the theory of the infinite which was the realm of God, and hence was by its very definition not the business of man to comprehend. There was a tradition (perhaps more in the west than in the east) that one must not deal with infinities as a whole, because the concept of a 'completed infinity' would lead to strange paradoxes. Gauss for example writes to a friend: *"... I protest above all against the use* *of an infinite quantity as a completed one, which in mathematics is not allowed. The infinite is only a fason de parler (form of speech), in which one properly speaks of limits."* From the time of Aristotle, thinkers had realized that infinities lead to strange results. For example, they 'annihilate numbers,' because $\infty + a = \infty$, for any positive finite number *a*, which contradicts properties of positive numbers. Galileo had stumbled upon the question of comparing the number of points in circles of different radii. He also discovered that the set of natural numbers can be put into a one-to-one correspondence with their squares and hence have the same cardinality (this is now known as Galileo's paradox). Thinkers before Cantor had decided that the best way to avoid these paradoxes was to avoid treating infinities as a whole. Cantor's response to these was that one could not impose the properties of finite numbers on infinite numbers. He expresses this eloquently: *". . . the infinite numbers, if they are to be considered in any form at all, must (in contrast to the finite numbers) constitute an entirely new kind of number, whose nature is entirely dependent upon the nature of things and is an object of research, but not of our arbitrariness or prejudices."* Cantor deeply believed that the essence of mathematics was its freedom.

However, some mathematicians, Kronecker in particular, were 'against' the infinite and even irrational numbers! They felt that only entities constructed by finite processes should be allowed into mathematics. Kronecker's ideology and personality were so strong that he seems to have worked very hard to belittle Cantor's work. He made sure that Cantor did not get a position in the more prestigious institutions in Germany, and tried to prevent the publication of his work. In contrast, Poincaré, who also believed quite strongly that set theory was a "disease from which mathematics should be freed", seems to have been on good terms with Cantor.

It was not just Cantor's infinite quantities and irrational numbers that created a stir. His 'naive' definition of sets as *any collection into a whole of definite and separate objects of our intuition or our thought* would later lead to the famous paradoxes in set theory, which in turn unleashed a whole crisis in the foundation of mathematics and engaged the best minds in mathematics for nearly half a century.

In spite of the repeated doubts expressed about his work, Cantor himself had no doubts. In 1888, he said: *"My theory stands firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things."* Cantor deeply believed that he had discovered the transfinite numbers with the help of God, and that they would lead to the *"one true infinity"* which was incomprehensible by man.

No note on Cantor can fail to mention his struggle with mental health. Cantor's first breakdown came in 1884, and here we find some possible misrepresentation. One story propagated by many historians is that Kronecker's vicious attacks drove Cantor to madness; another story is that Cantor's dabbling with abstruse concepts like transfinite numbers did it. Examination of his medical records suggests that he suffered from manic depression. Perhaps no matter what career Cantor had pursued, he would have suffered from mental illness. Certainly the hostility that he encountered did nothing to help.

But it is also true that Cantor had many friends and supporters throughout his life. He was awarded the Sylvester medal in 1904, the highest honour of the Royal Society of London. His work also generated positive interest among mathematicians in his lifetime. Minkowski referred to him as one of the deepest mathematicians of the time, and Bertrand Russell believed that he was one of the greatest intellects of the 19th century. No greater compliment can be paid to his work than Hilbert's proclamation *"No one will drive us out of this paradise that Cantor has created for us!"*

References

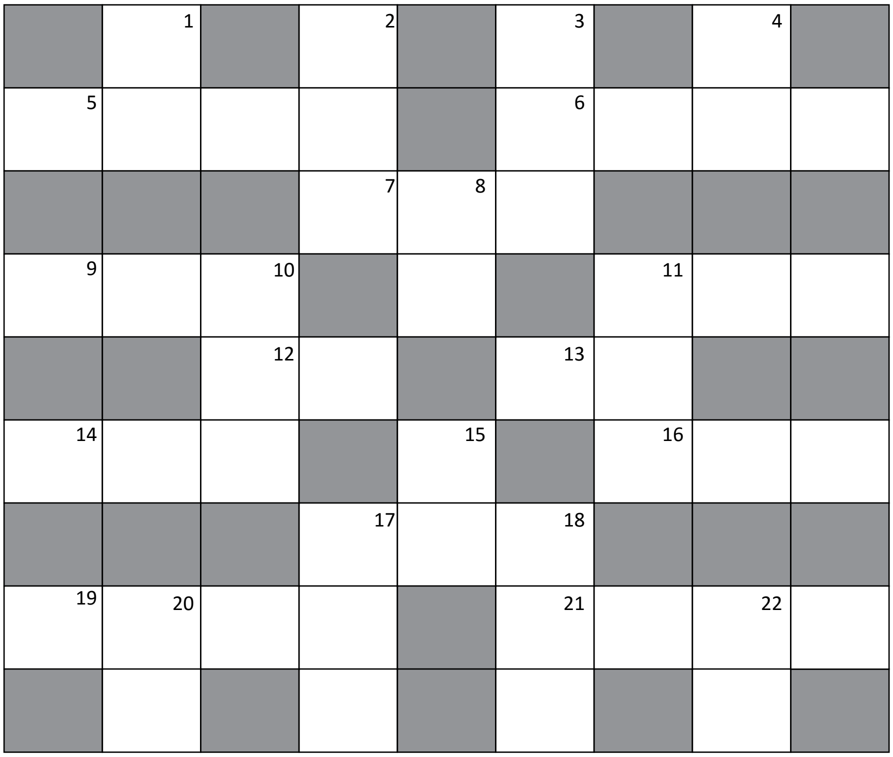
i. Duaben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Princeton University Press, 1979.

ii. Reid, Constance. *Hilbert*. Copernicus,1996.

NUMBER CROSSWORD

Solution on Page 41

D.D. Karopady



CLUES ACROSS

5: Number of seconds in a week divided by 100

6: Volume of a cylinder with a radius of four units and height of 100 units

7: Four times (4D – 10)

9: Average of largest three digit perfect square and three digit perfect cube

11: Product of first four prime numbers

12: Number of diagonals in a 12 sided polygon

13: Number is 5 times the sum of the digits

14: 3D minus 1 divided by 3

16: Narcissistic number (equal to sum of cubes of its digits)

17: Last digit is 4 times the first digit

19: 3618, 4728, ……., 6954

21: Product of the Pythagorean triplet with 2 values given as 8 and 15

CLUES DOWN

1: Smallest number divisible by 1 through 6

2: A palindrome

3: Sum of first two digits is the third digit

4: Reverse of this number can be written as sum of two squares

8: Difference of roots of largest and smallest four digit perfect squares

11: 7 less than 17A

15: 12A in reverse

17: 10% of 21A

18: 9A minus reverse of 4D

20: Number is the square of sum of its digits

22: 5A minus 6A divided by (2 raised to 8) is the sum of the digits

3 4 5 ...

And Other Memorable Triples

Part IV

Features

Shailesh Shirali

In Parts I, II and III of this series of articles we identified triples of consecutive positive integers with striking geometrical properties; for each triple, the triangle with the three integers as its sides has some special geometrical property. The triples $(3, 4, 5)$, $(4, 5, 6)$ and $(2, 3, 4)$ all turned out to be special in this sense.

The 'special property' possessed by $(3, 4, 5)$ is well known: there is just one triple of consecutive integers with the feature that the triangle with these integers as its sides is right-angled; namely, $(3, 4, 5)$. But the triple $(3, 4, 5)$ has a further feature which can easily be missed: the area of the triangle with sides $3, 4, 5$ is $(3 \times 4)/2 = 6$, an integer. This property is not possessed by either of the triples $(2, 3, 4)$, $(4, 5, 6)$. Indeed, the area of the triangle with sides $2, 3, 4$ is

$$\sqrt{\frac{9}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2}} = \frac{3\sqrt{15}}{4},$$

while the area of the triangle with sides $4, 5, 6$ is

$$\sqrt{\frac{15}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2}} = \frac{15\sqrt{7}}{4};$$

Keywords: *Triangles, consecutive integers, area, integers, perfect squares, Heronian, induction*

and both of these numbers are not even rational. What about $(5, 6, 7)$? The semi-perimeter now is 9, so the area is

$$\sqrt{9 \times 4 \times 3 \times 2} = 6\sqrt{6},$$

which too is not rational. The same is true of the triples $(6, 7, 8)$ and $(7, 8, 9)$. So the triple $(3, 4, 5)$ scores above all these triples in this regard.

The following question now poses itself naturally: *Is there another triple of consecutive integers with the property that the triangle with the three integers as its sides has integral area?*

A triangle whose sides and area are both integers is known in the literature as a **Heronian triangle**. Thus the triangle with sides $3, 4, 5$ is Heronian; so are the triangles with sides $5, 5, 6$ (made by joining two 3-4-5 triangles along the side of length 4, so its area is $2 \times 6 = 12$); sides $5, 5, 8$ (made similarly); and sides $4, 13, 15$ (with area 24). The above question may thus be framed: *Is there another triple of consecutive integers which yields a Heronian triangle?* (See Box 1.)

A Heronian query

The area of the $3, 4, 5$ triangle is 6, which is an integer.

Are there other triangles whose sides are consecutive integers and whose area is also an integer?

Box 1

Stated this way, the answer is easily found, simply by trying out more possibilities in sequence (and, obviously, making use of a computer). The next such triple we find after $(3, 4, 5)$ is $(13, 14, 15)$. The semi-perimeter is $(13 + 14 + 15)/2 = 21$, so the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = 84,$$

an integer. We see that the triple $(3, 4, 5)$ is not unique in possessing the stated property.

Now we ask: How many such triples are there? Remember that our question is limited to triples of consecutive integers. The answer comes as a pleasant surprise: *There are infinitely many such triples.* Moreover, they come in a beautiful pattern which enables us to enumerate them, as we now show.

Let $n > 2$ be an integer such that the triangle with sides $n - 1$, n and $n + 1$ has integer area. The semi-perimeter s is given by $2s = 3n$, so by Heron's formula the area of the triangle is

$$\sqrt{\frac{3n}{2} \times \frac{n-2}{2} \times \frac{n}{2} \times \frac{n+2}{2}} = \frac{n\sqrt{3(n^2-4)}}{4}. \tag{1}$$

It follows that n must be such that $n\sqrt{3(n^2-4)}/4$ is an integer.

We first show that under this requirement, n must be even. Suppose that n is odd. Then n^2 leaves remainder 1 under division by 4, hence $3(n^2 - 4)$ leaves remainder 3 under division by 4. But no square integer is of this form. It follows that if n is odd, then $\sqrt{3(n^2-4)}$ is an irrational quantity, implying that $n\sqrt{3(n^2-4)}/4$ cannot be an integer. Hence n must be even.

In the paragraph below, it is assumed that n is an even integer. We shall suppose that n is such that $\sqrt{3(n^2-4)}$ is an integer and show that under this supposition, the area too is an integer.

Suppose that $\sqrt{3(n^2-4)}$ is an integer; then $\sqrt{3(n^2-4)}$ is an *even* integer (since n itself is even), hence $n\sqrt{3(n^2-4)}$ is a multiple of 4, implying that $n\sqrt{3(n^2-4)}/4$ is an integer. That is, the area of the triangle is an integer, as claimed.

On the other hand, if the area is an integer, then $\sqrt{3(n^2-4)}$ obviously must be an integer.

Therefore the exploration reduces to the following question:

Find all even integers $n > 2$ such that $3(n^2-4)$ is a perfect square.

Write $n = 2x$ where x is an integer. Then $3(n^2-4) = 4 \times 3(x^2-1)$. For this to be a perfect square, $3(x^2-1)$ must be a perfect square, hence x^2-1 must be of the form $3y^2$ for some integer y. So the problem further reduces to finding all pairs (x, y) of positive integers such that $x^2-1 = 3y^2$, i.e.,

$$x^2 - 3y^2 = 1. \tag{2}$$

For each x belonging to such a pair, the triangle with sides $2x-1$, $2x$, $2x+1$ has integer area. See Box 2.

An equation that generates the triangles we seek

The pair of positive integers (x, y) where $x^2-3y^2 = 1$ defines a triangle with sides $2x-1$, $2x$, $2x+1$ (i.e., three consecutive integers, the middle one being even). The area of this triangle is an integer.

Box 2

Equation (2) has a familiar form; it is an instance of the *Brahmagupta-Bhaskara-Fermat equation* (also known in the literature as the 'Pell equation'), $x^2 - ny^2 = 1$, with $n = 3$. Readers may recall that we dwelt on this kind of equation in the November 2014 issue of the magazine and studied an algorithm to solve it: the Chakravāla method.

In this article we do not use the Chakravāla. Instead we present a way of enumerating the solutions of the equation which makes use of the irrational quantity $\sqrt{3}$.

We start by noting that $x = 2$, $y = 1$ is a solution of (2); it is the smallest possible integral solution. Using these two numbers we form the following number α which is an irrational surd:

$$\alpha = 2 + 1 \cdot \sqrt{3} = 2 + \sqrt{3}.$$

We now compute the square of α:

$$\alpha^2 = \left(2+\sqrt{3}\right)^2 = 2^2 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}.$$

Note the numbers which have appeared in the above expression: 7 and 4. If we try out the values $x = 7$ and $y = 4$ in the expression $x^2 - 3y^2$, we find that it equals 1:

$$7^2 - 3 \times 4^2 = 49 - 3 \times 16 = 1.$$

So $x = 7$, $y = 4$ is a solution of (2). Now we have two solutions to the equation, $(2, 1)$ and $(7, 4)$.

Next, let us find the cube of α. We have:

$$\begin{aligned}\alpha^3 = \left(2+\sqrt{3}\right)^3 &= \left(2+\sqrt{3}\right)^2 \times \left(2+\sqrt{3}\right) \\ &= \left(7+4\sqrt{3}\right) \times \left(2+\sqrt{3}\right) \\ &= 14 + (7+8)\sqrt{3} + 12 = 26 + 15\sqrt{3}.\end{aligned}$$

Is $x = 26$, $y = 15$ a solution of (2)? Let's check:

$$26^2 - 3 \times 15^2 = 676 - 3 \times 225 = 676 - 675 = 1.$$

It is!

It is natural to try the fourth power now:

$$\begin{aligned}\alpha^4 = \left(2+\sqrt{3}\right)^4 = \left(\left(2+\sqrt{3}\right)^2\right)^2 &= \left(7+4\sqrt{3}\right)^2 \\ &= 49 + 2 \times 7 \times 4\sqrt{3} + (16 \times 3) = 97 + 56\sqrt{3}.\end{aligned}$$

We may check that $x = 97$, $y = 56$ is yet another solution of (2):

$$97^2 - 3 \times 56^2 = 9409 - 3 \times 3136 = 9409 - 9408 = 1.$$

We seem to have found a way of generating an unlimited number of solutions of (2)! See Box 3.

Generating Heronian triangles whose sides are consecutive integers

If $\alpha = u_1 + v_1\sqrt{3}$ where u_1, v_1 are positive integers, then positive integral powers of α yield all possible triangles which have sides as consecutive integers and whose area is an integer as well.

Box 3

Two questions pose themselves:

Question 1: Does every positive integral power of α yield a solution to the equation? More specifically, for each positive integer n let

$$\alpha^n = u_n + v_n\sqrt{3}, \tag{3}$$

where u_n and v_n are integers. Is the following relation true for all n?

$$u_n^2 - 3v_n^2 = 1. \tag{4}$$

Question 2: Assuming that the answer to Question 1 is 'Yes', does the list (u_1, v_1), (u_2, v_2), (u_3, v_3), …enumerate *all* the positive integral solutions of (2)?

We shall show that the answer for both questions is 'Yes'. In other words, the procedure we have described does yield every possible positive integral solution to (2).

Answer to Question 1

Suppose that $x = a$, $y = b$ is a solution to the equation $x^2 - 3y^2 = 1$, that is, $a^2 - 3b^2 = 1$. Consider the following product:

$$\left(a + b\sqrt{3}\right) \cdot \left(2 + \sqrt{3}\right) = (2a + 3b) + (a + 2b)\sqrt{3}. \tag{5}$$

We now show that $x = 2a + 3b$, $y = a + 2b$ is a solution to the equation. To do so, we must verify that $(2a + 3b)^2 - 3(a + 2b)^2 = 1$. Here is the verification:

$$\begin{aligned}(2a + 3b)^2 - 3(a + 2b)^2 &= \left(4a^2 + 12ab + 9b^2\right) - 3\left(a^2 + 4ab + 4b^2\right)\\ &= a^2 - 3b^2 = 1 \quad \text{(as was required).}\end{aligned}$$

So $x = 2a + 3b$, $y = a + 2b$ is a solution to (2), as claimed.

Inductively, it follows that $x = u_n$, $y = v_n$ as defined by (3) is a solution to equation (2) for every positive integer n. So relation (4) is true for every positive integer n. This answers Question 1.

Do you see how this is a proof by induction? What we have shown is: If $x = u_n$, $y = v_n$ is a solution to equation (2), then so is $x = u_{n+1}$, $y = v_{n+1}$. This is the inductive step. The anchor had already been established, i.e., checking that $x = u_1$, $y = v_1$ is a solution.

Remark. Let us denote the solution $x = a$, $y = b$ by the pair (a, b). Then the generation of the solution $x = 2a + 3b$, $y = a + 2b$ from the solution $x = a$, $y = b$ may be written in the form of a *map* which we call f:

$$(a, b) \xmapsto{f} (2a + 3b, a + 2b). \tag{6}$$

Starting with the solution $(2, 1)$, we may use f repeatedly to generate infinitely many solutions:

$$(2, 1) \xmapsto{f} (7, 4) \xmapsto{f} (26, 15) \xmapsto{f} (97, 56) \xmapsto{f} (362, 209) \xmapsto{f} (1351, 780) \cdots. \tag{7}$$

The x-values of these pairs are the following:

$$2,\ 7,\ 26,\ 97,\ 362,\ 1351,\ \ldots. \tag{8}$$

These yield the following triples of consecutive integers with the property that a triangle with those three integers as side lengths has integer area (recall the rule: associated with the pair (x, y) is the triangle with sides $2x - 1$, $2x$, $2x + 1$ and area $3xy$):

$$\begin{array}{ccc}(3, 4, 5), & (13, 14, 15), & (51, 52, 53),\\ (193, 194, 195), & (723, 724, 725), & (2701, 2702, 2703).\end{array}$$

We can continue applying the rule and get indefinitely many such triples.

Remark. The sequence of x-values,

$$2,\ 7,\ 26,\ 97,\ 362,\ 1351,\ \ldots,$$

has a striking and beautiful pattern which you may have spotted:

$$\begin{aligned}26 &= 4 \times 7 - 2,\\ 97 &= 4 \times 26 - 7,\\ 362 &= 4 \times 97 - 26,\\ 1351 &= 4 \times 362 - 97,\end{aligned}$$

and so on. So if c and d are two successive x-values, with $d > c$, the next one is $4d - c$. We invite the reader to show that this pattern continues indefinitely. (Hint: Use induction.)

Answer to Question 2

Now we tackle the more ambitious question: show that the above procedure captures every solution to (2). The key to the analysis is to find a way of reversing the move from (a, b) to $(2a + 3b, a + 2b)$; in other words, to find the inverse map $f^{-1} = g$, say. Since the map f is based on multiplication by $\alpha = 2 + \sqrt{3}$, the inverse map g must involve division by that number, which is the same as multiplication by $1/\alpha$. But we have:

$$\frac{1}{\alpha} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3},$$

so division by $2 + \sqrt{3}$ is the same as multiplication by $2 - \sqrt{3}$. Next, observe that:

$$\left(a + b\sqrt{3}\right) \cdot \left(2 - \sqrt{3}\right) = (2a - 3b) + (2b - a)\sqrt{3}. \tag{9}$$

We infer that if (a, b) is a solution to the equation $x^2 - 3y^2 = 1$, then so is $(2a - 3b, 2b - a)$. This is easy to verify:

$$\begin{aligned}(2a - 3b)^2 - 3(2b - a)^2 &= \left(4a^2 - 12ab + 9b^2\right) - 3\left(4b^2 - 4ab + a^2\right) \\ &= a^2 - 3b^2 = 1.\end{aligned}$$

We depict this map as follows:

$$(a, b) \stackrel{g}{\longmapsto} (2a - 3b, 2b - a). \tag{10}$$

Note that f and g are a pair of inverse maps.

Now we ask: *Given that (a, b) is a positive integral solution to $a^2 - 3b^2 = 1$, under what conditions on a and b will it be true that $(2a - 3b, 2b - a)$ is strictly smaller than (a, b)? In other words, under what conditions does it happen that both the following are true?*

$$0 < 2a - 3b < a, \qquad 0 < 2b - a < b. \tag{11}$$

The two conditions are equivalent respectively to:

$$1.5b < a < 3b, \qquad b < a < 2b, \tag{12}$$

and these together imply that:

$$1.5b < a < 2b. \tag{13}$$

Now if $a^2 - 3b^2 = 1$ and $b > 1$, then we certainly have:

$$a^2 = 3b^2 + 1 < 4b^2, \qquad \therefore\ a < 2b,$$

and:

$$3b^2 = a^2 - 1 < a^2, \qquad \therefore\ 9b^2 < 3a^2 < 4a^2, \qquad \therefore\ 3b < 2a,$$

i.e., $1.5b < a$. So if $b > 1$, then $0 < 2a - 3b < a$ and $0 < 2b - a < b$. Therefore:

> *If (a, b) is a solution to $x^2 - 3y^2 = 1$ and b exceeds 1, then the solution $(2a - 3b, 2b - a)$ is **strictly smaller** than (a, b).*

For example, the rule g applied to the solution $(26, 15)$ yields $(7, 4)$, which is smaller than $(26, 15)$; and if we apply g to $(7, 4)$, we get $(2, 1)$, which is smaller still.

If we apply the rule g to $(2, 1)$, we get $(1, 0)$. Though this solution satisfies the relation $x^2 - 3y^2 = 1$, and $(1, 0)$ is certainly smaller than $(2, 1)$, we do not accept it as a solution as we want solutions in positive integers only.

Now let us start with any solution (a, b) to $x^2 - 3y^2 = 1$ with $b > 1$, and let us apply the rule g to it. As already explained, we will get a strictly smaller solution. If the second coordinate of this new solution exceeds 1, we can apply g to that solution and thus obtain a still smaller solution. And so we can continue, obtaining steadily smaller solutions.

Can this process continue forever? Clearly not; we cannot have a strictly decreasing sequence of positive integers which continues forever (this is the 'descent principle': the positive integers are bounded below, by 1; they cannot go below 1). So at some point we will be forced to a halt.

When will this happen?—precisely when the second coordinate of the solution is 1. If we ever reach that stage, it means that the first coordinate is equal to $\sqrt{3 \times 1^2 + 1} = 2$. This means that we have landed on the solution $(2, 1)$. But then the solution *prior* to that must have been $(7, 4)$, for it is the image of $(2, 1)$ under f (and therefore the preimage of $(2, 1)$ under g). And the solution prior to *that* must have been $(26, 15)$. And so on.

We infer that the solution we started with must be part of the sequence enumerated in (7), in other words, it is equal to (u_n, v_n) for some n. This is another way of saying that the list (u_1, v_1), (u_2, v_2), (u_3, v_3), ..., that is:

$$(2, 1), \ (7, 4), \ (26, 15), \ (97, 56), \ (362, 209), \ \ldots \tag{14}$$

is a complete enumeration of the positive integral solutions of the equation $x^2 - 3y^2 = 1$.

This fully answers Question 2. We have thus succeeded in enumerating every triple of consecutive integers such that the triangle with those three integers as side lengths has integral area.

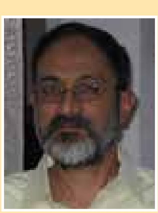
Exercises.

(1) Prove that if x, y are positive integers such that $x^2 - 3y^2 = 1$, then the triangle with side lengths $2x - 1$, $2x$ and $2x + 1$ has area $3xy$.

(2) Show that x is any member of the following sequence,

$$2, \ 7, \ 26, \ 97, \ 362, \ 1351, \ \ldots,$$

then the triangle with side lengths $2x - 1$, $2x$ and $2x + 1$ has integer area. The defining property of the sequence is this: its initial two terms are 2 and 7, and if a, b, c are any three consecutive members of the sequence, then $c = 4b - a$.



SHAILESH SHIRALI is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

An Eye on Eyeball

PRITHWIJIT DE

Euclidean Geometry is fascinating. It has captured our imagination for centuries. Many beautiful theorems have been discovered and proved, and myriad mind-boggling problems have been posed and solved, yet we haven't got tired of it. To the creative mind, geometry is a source of immense pleasure and contentment. We look for some more in a little-known result in plane geometry called "The Eyeball Theorem" and uncover some of its geometrical features.

The Eyeball Theorem

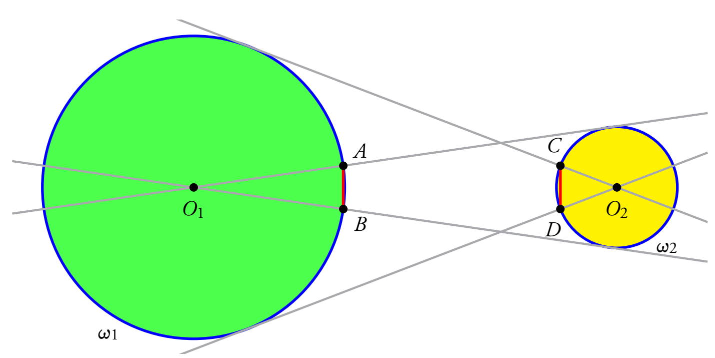


Figure 1

Keywords: *Geometry, circles, tangents, chords, angles, concyclic*

Consider two non-overlapping circles ω_1 and ω_2 in the plane; neither circle is contained in the other. Let O_1 and O_2 be their respective centres. Draw tangents to ω_2 from O_1 and to ω_1 from O_2. Let the tangents to ω_2 intersect ω_1 at A and B. Let the tangents to ω_1 intersect ω_2 at C and D. The **Eyeball Theorem** now states that $AB = CD$. (See Figure 1.)

There are several ways to prove the assertion. To start with, let us mark a few more points in the configuration. Let X and Y be the respective points of intersection of O_1O_2 with AB and CD; see Figure 2. Let P_1 and P_2 be the points of contact of the tangents from O_1 to ω_2, and let Q_1 and Q_2 be the points of contact of the tangents from O_2 to ω_1.

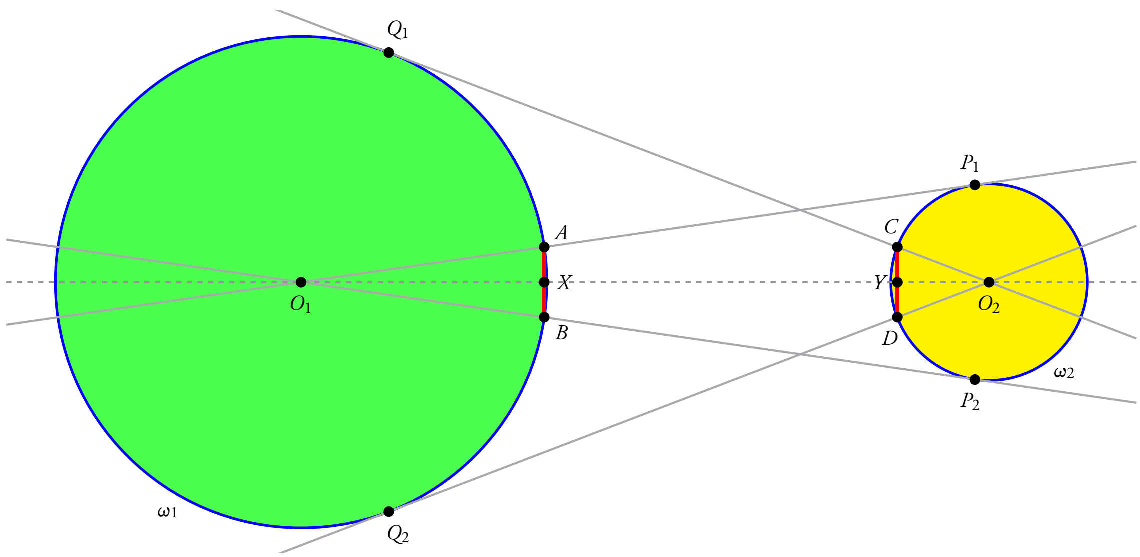


Figure 2

Here is a simple argument which shows that $AB = CD$. Let r_1 and r_2 be the radii of ω_1 and ω_2 respectively. Observe that line O_1O_2 is an axis of symmetry of the configuration. Therefore $AB = 2AX$ and $CD = 2CY$. The triangles O_1XA and $O_1P_1O_2$ are similar. Hence:

$$\frac{AX}{P_1O_2} = \frac{O_1A}{O_1O_2}, \quad \therefore\ AB = 2AX = \frac{2r_1r_2}{O_1O_2}. \tag{1}$$

A similar argument leads to

$$CD = \frac{2r_1r_2}{O_1O_2}, \tag{2}$$

and we see that $AB = CD$.

If the circles touch each other externally, then

$$AB = CD = \frac{2r_1r_2}{r_1 + r_2}, \tag{3}$$

which is the harmonic mean of the radii of the two circles.

This configuration abounds in sets of four or more concyclic points. Let us find as many such sets as we can. The missing ones may be reported by perceptive readers. As the figure is symmetric about O_1O_2, it suffices to look for concyclic sets on one side of O_1O_2, say on the same side of the line as A. See Figure 3, which is the same as Figure 2; we have reproduced it only for the readers' convenience.

Observe that $\measuredangle O_1Q_1O_2 = \measuredangle O_1P_1O_2 = 90^\circ$, which shows that O_1O_2 subtends the same angle at two points P_1 and Q_1 on the same side of it. Therefore the four points O_1, O_2, P_1 and Q_1 are concyclic.

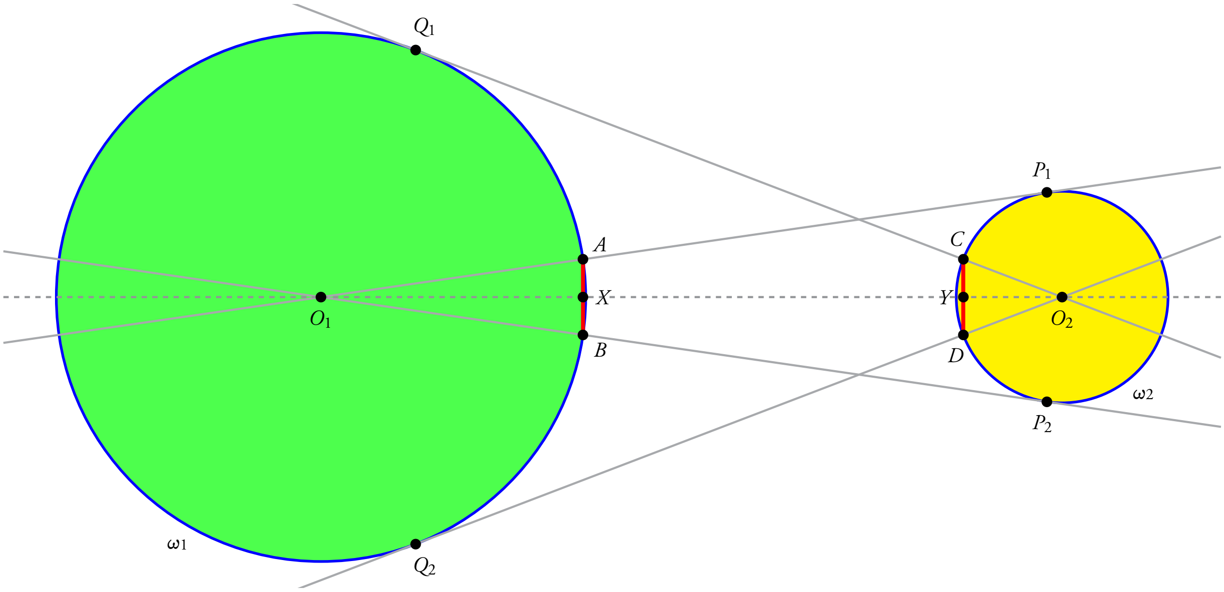


Figure 3

Moreover, O_1O_2 is a diameter of the circle. By symmetry, P_2 and Q_2 lie on the same circle. Thus we have six points on the same circle. Call this circle Ω. We have more in store. Observe that

$$\measuredangle AQ_1O_2 = \frac{1}{2}\measuredangle AO_1Q_1 = \frac{1}{2}\measuredangle CO_2P_1 = \measuredangle CP_1O_1, \tag{4}$$

showing that the points A, Q_1, P_1, C are concyclic. By symmetry the same is true for the points B, Q_2, P_2, D.

Quadrilateral $ABDC$ is a rectangle because $AB = CD$ and segments AB and CD have a common perpendicular bisector, namely, line O_1O_2. Therefore, points A, B, D, C are concyclic. The reader may easily deduce that quadrilaterals AP_1P_2B and CQ_1Q_2D are isosceles trapezoids and therefore their vertices form concyclic sets of points.

The centres of the circles ω_1, ω_2 and $(O_1Q_1P_1O_2P_2Q_2)$ all lie on O_1O_2; so also for the circles (AP_1P_2B) and (CQ_1Q_2D). What can be said about the centre of the circle containing the points A, Q_1, P_1, C? Let us investigate. If O_3 is the centre of this circle, then observe that it is the point of intersection of the perpendicular bisector of AQ_1 and that of CP_1. But the perpendicular bisector of AQ_1 passes through O_1 and that of CP_1 passes through O_2. Now

$$\measuredangle O_3O_1O_2 = \measuredangle O_3O_1P_1 + \measuredangle P_1O_1O_2 = \frac{1}{2}\measuredangle Q_1O_1P_1 + \measuredangle P_1O_1O_2, \tag{5}$$

and

$$\measuredangle O_3O_2O_1 = \measuredangle O_3O_2Q_1 + \measuredangle Q_1O_2O_1 = \frac{1}{2}\measuredangle P_1O_2Q_1 + \measuredangle Q_1O_2O_1. \tag{6}$$

But we also have

$$\measuredangle Q_1O_1P_1 + \measuredangle P_1O_1O_2 + \measuredangle Q_1O_2O_1 = 90^\circ, \tag{7}$$

and

$$\measuredangle Q_1O_2P_1 + \measuredangle Q_1O_2O_1 + \measuredangle P_1O_1O_2 = 90^\circ. \tag{8}$$

It follows that

$$\measuredangle O_3O_1O_2 + \measuredangle O_3O_2O_1 = 90^\circ. \tag{9}$$

Therefore $\measuredangle O_1O_3O_2 = 90^\circ$ and so O_3 lies on Ω. By symmetry, the centre of the circle containing B, Q_2, P_2, D also lies on Ω. (See Figure 4.)

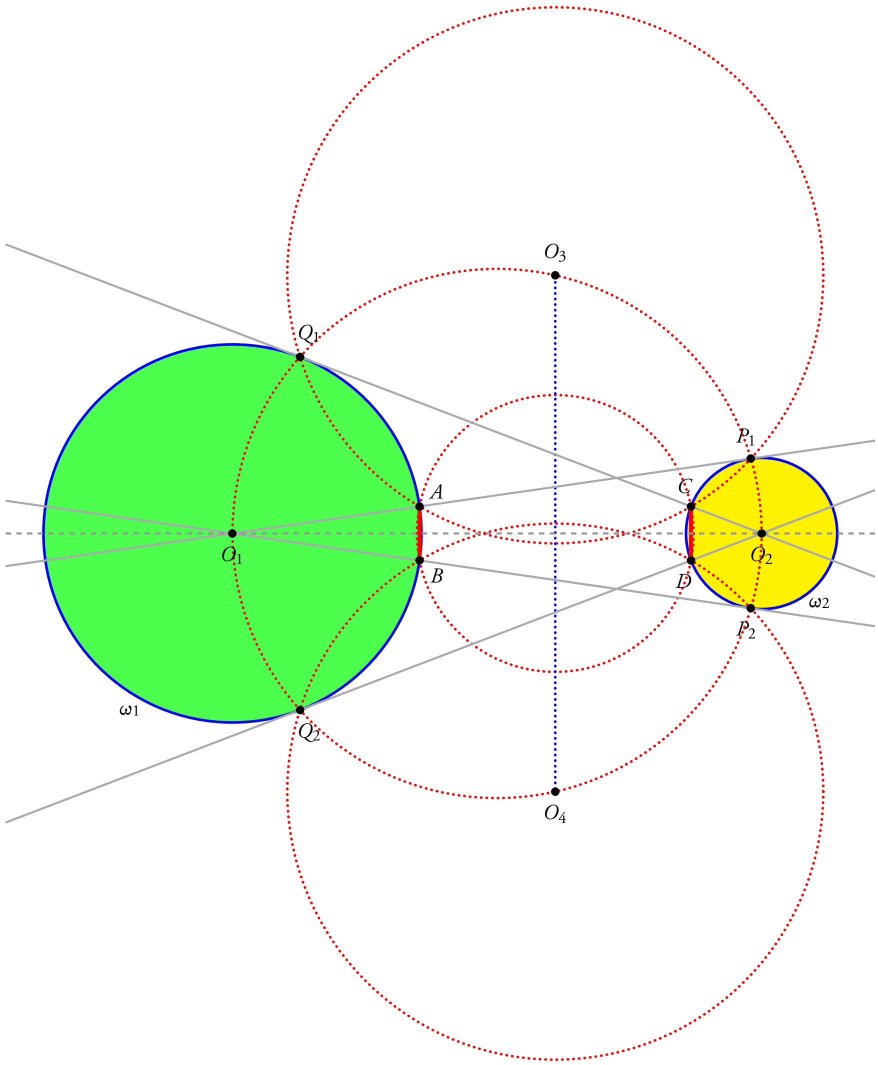


Figure 4

So we have found eight points on a circle. That's exciting, isn't it? Here is something even more exciting. Let O_4 be the centre of the circle passing through B, Q_2, P_2, D. The line O_3O_4 passes through the centre of the circle passing through A, B, C, D. How does one prove it? We leave that as an exercise for you!

References

1. The Eyeball Theorem. http://nrich.maths.org/1935
2. The Eyeball Theorem. http://www.cut-the-knot.org/Curriculum/Geometry/Eyeball.shtml

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.

ClassRoom

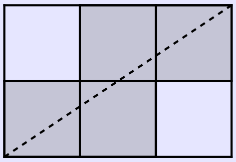
A Diagonal Investigation

Tanuj Shah

In this issue, we will look at an investigation which at first seems as though it is not going to yield much in terms of patterns. See Figure 1.

Count the squares!

Shown below is a 2×3 rectangle. We see that the diagonal of the rectangle (shown as a dashed line) passes through 4 squares (shown shaded).



Investigate for rectangles of other shapes.

Figure 1

This is a suitable investigation for class 8/9 level and easily accessible for all students. The teacher should provide 1 cm square paper and emphasise the need to use a sharp pencil for this investigation; some children may find it useful to shade the squares through which the

Keywords: Rectangle, grid, diagonal, pattern, investigation, exploration, GCD, HCF

diagonal passes. A few students will ask if passing through a vertex of a square is considered as 'passing through that square'. The teacher can preempt this by mentioning at the start that the line has to pass through some region of the square (including the boundary) for it to be counted.

With this minimal instruction the teacher should allow the children to proceed with the investigation in whichever way they prefer. Most will begin by drawing different rectangles and counting the number of squares the diagonal is passing through. Some will do it in a random fashion, while others will adopt some kind of a system. Those who are very systematic may start by looking at rectangles that are 1×1, 2×1, 3×1, 4×1, etc., and will quickly make a conjecture that the number of squares that a diagonal passes through is the same as the longer side of a rectangle. *It is important that the students get into the habit of writing down any conjectures or patterns that they observe during the course of an investigation.* Here, as the students look at a different set of rectangles, they will realise that their conjecture is not true; they should also write this down, and if they have any thoughts on how they would now proceed, they should state that as well.

The teacher needs to be in touch with the pulse of the class and must create an ambience free of competition. If required, the teacher could spend some time reiterating that in an investigation, what is important is the thinking process, and the justifications provided for the conjectures, irrespective of whether the conjecture is invalidated with new evidence that may emerge later.

After the students have worked on the investigations for some time, the teacher could conduct a brief discussion on how students could record their results in a precise manner. (See Figure 2.) Some students may suggest that the results could be tabulated, with the width being constant in each table. Others may decide to have one large table with the widths on one side and lengths on the other as shown in Figure 3.

Those who are more adventurous may adopt a symbolic way of presenting the findings, like $s(2,3) = 4$, which states that for a 2×3

Pedagogic strategies

This task is designed to give students a feel of how mathematicians 'work'. It develops the skills of documentation, communication, reasoning and conjecture.

Figure 2

$L\backslash W$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2			4		6		8	
3				6	7		9	10
4					8		10	
5						10	11	12
6							12	
7								14
8								

Figure 3

rectangle, the number of squares a diagonal passes through is 4. If time permits, one could have a discussion on how length is defined sometimes as the longer side and at other times as the horizontal distance. This is one of the rare instances where mathematicians have not created a convention, a standard way of defining something precisely; and this creates confusion for students and teachers in elementary school. This is probably because it occurs in application-oriented questions, where the everyday usage is also ambiguous.

As the investigation proceeds, some students are able to see that $s(W, L) = W + L - 1$ in some instances. A teacher can nudge them towards sorting rectangles that follow the above rule and those that do not. Those who are good at spotting visual patterns will see that in those rectangles that do not fit the above rule, the diagonal sometimes passes through the vertices of some squares, whereas in the rectangles that follow the rule, the diagonal never passes through the vertices, except at the start and the end. Most children will also come across the more specific case of $W = L$, i.e., the square, and find that in this case the number of squares a diagonal passes through is the same as a side of a square; also, the diagonal passes through the vertices of all the squares that it touches. One may also find students following a more analytical approach, for example starting with a rectangle and seeing what happens when the rectangle is enlarged by different integral scale factors e.g. $s(2,3)$, $s(4,6)$, $s(6,9)$. The recording sheet could also have space for them to document their approach; this will help students to formalise their thinking.

This is an investigation that can be started in class, and then left for students to continue at home. The teacher can spend a few minutes towards the end of the class to discuss some of the approaches and conjectures that students have come up with, to enable everyone to proceed with some degree of confidence. If there are students who have seen the general case or are close to it, the teacher could ask them to justify their answer and if they are keen, to extend the investigation to three dimensions by looking at cuboids. The teacher should, however, not reveal the general case to students who are still grappling with the original problem, and should also advise those who have come upon the answer to not give it to the others.

After a week, the teacher could follow up on the investigation by giving students an opportunity to share their findings with the rest of the class. The teacher needs to allow this to happen in a calibrated way, so that a range of different approaches and patterns are revealed before coming to the general case. There will be students who will have explored rectangles with dimensions that are only prime numbers and would state that in those cases the $W + L - 1$ formula works. The teacher can encourage this to be stated in symbolic form, e.g., $s(W, L) = W + L - 1$ where both W, L are prime numbers; this could also be extended for W, L being coprime. Some students would have seen that if the rectangle is of the kind where one side is a multiple of the other, i.e., $L = kW$ where k is some positive integer, then $s(W, L) = L$ (this also works for the case when $k = 1$, a square). These formulas then lead to the general case $s(W, L) = W + L - \text{GCD}(W, L)$. (Here, GCD means *greatest common divisor* which is the same as *highest common factor*.)

The first step in seeing why this relation holds lies in spotting the relation between $s(W, L)$ and $s(kW, kL)$ where k is any positive integer. For example, consider the case when $(W, L) = (2, 3)$ and $k = 2$. Examining a 4×6 rectangle, we see that its diagonal passes through two 2×3 rectangles arranged corner-to-corner as shown in Figure 4. We infer from this that $s(4, 6) = 2 \times s(2, 3)$.

In much the same way we see that for any positive integer k,

$$s(kW, kL) = k \times s(W, L).$$

The second step is to understand how to deduce the general formula from the conjectured formula for the special case when W, L are coprime, i.e., $s(W, L) = W + L - 1$. Suppose that W and L are not coprime; say $\text{GCD}(W, L) = k$ (where $k > 1$). Then W/k and L/k are coprime positive integers, hence:

$$s\left(\frac{W}{k}, \frac{L}{k}\right) = \frac{W}{k} + \frac{L}{k} - 1.$$

Multiplying through by k we get:

$$k \times s\left(\frac{W}{k}, \frac{L}{k}\right) = W + L - k.$$

But we know that

$$k \times s\left(\frac{W}{k}, \frac{L}{k}\right) = s(W, L).$$

Hence $s(W, L) = W + L - k$, i.e.,

$$s(W, L) = W + L - \text{GCD}(W, L).$$

So the proof entirely hinges on showing that if W, L are coprime, then

$$s(W, L) = W + L - 1.$$

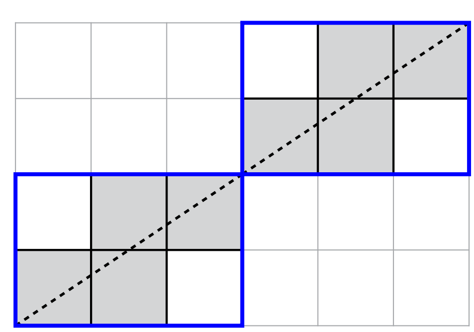


Figure 4

Devising the proof that this is so is a lovely exercise. With careful prompting, the teacher should be able to coax the proof from the class.

Remarks.

(1) For those pursuing the investigation for cuboids, straws and connectors could be provided and also multi-link cubes to help in visualising the problem. The 3D problem has a lot in common with the 2D version.

(2) Students study the notion of highest common factor (GCD) in school, but rarely come across it elsewhere; the unexpected appearance of it in this context should create an *AHA!* moment for the students.

TANUJ SHAH teaches Mathematics in Rishi Valley School. He has a deep passion for making mathematics accessible and interesting for all and has developed hands-on self learning modules for the Junior School. Tanuj did his teacher training at Nottingham University and taught in various Schools in England before joining Rishi Valley School. He may be contacted at shahtanuj@hotmail.com.

SOLUTIONS

NUMBER CROSSWORD

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	[1] 6		[2] 1		[3] 3		[4] 5	
[5] 6	0	4	8		[6] 5	0	2	4
			[7] 1	[8] 6	8			
[9] 8	4	[10] 5		7		[11] 2	1	0
		[12] 5	4		[13] 4	5		
[14] 1	1	9		[15] 4		[16] 1	5	3
			[17] 2	5	[18] 8			
[19] 5	[20] 8	4	0		[21] 2	0	[22] 4	0
	1		4		0		0	

ClassRoom

The Digital Root

ANANT VYAWAHARE

Introduction

The *digital root* of a natural number n is obtained by computing the sum of its digits, then computing the sum of the digits of the resulting number, and so on, till a single digit number is obtained. It is denoted by $B(n)$. In Vedic mathematics, the digital root is known as *Beejank*; hence our choice of notation, $B(n)$. Note that the digital root of n is itself a natural number. For example:

- $B(79) = B(7+9) = B(16) = 1+6 = 7$; that is, $B(79) = 7$.
- $B(4359) = B(4+3+5+9) = B(21) = 3$.

The concept of digital root of a natural number has been known for some time. Before the development of computer devices, the idea was used by accountants to check their results. We will examine the basis for this procedure presently.

Ten arithmetical properties of the digital root

The following arithmetic properties can be easily verified. Here $n, m, k, p, \ldots$ denote positive integers (unless otherwise specified).

Property 1. *If $1 \le n \le 9$, then $B(n) = n$.*

Property 2. *The difference between n and $B(n)$ is a multiple of 9; i.e., $n - B(n) = 9k$ for some non-negative integer k.*

To prove this, it suffices to show that for any positive integer n, the difference between n and the sum $s(n)$ of the digits of n is a multiple of 9. This step, carried forward recursively, will prove the claim. But the claim is clearly true, for if

$$n = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + \cdots,$$

Keywords: *Digital root, remainder, Beejank, sum of digits, natural number, triangular number, square, cube, sixth power, modulo*

then $s(n) = a_0 + a_1 + a_2 + a_3 + \cdots$, and therefore:

$$n - s(n) = (10 - 1)a_1 + \left(10^2 - 1\right) a_2 + \left(10^3 - 1\right) a_3 + \cdots.$$

Since the coefficients $10 - 1, 10^2 - 1, 10^3 - 1, \ldots$ are all multiples of 9, it follows that $n - s(n)$ is a multiple of 9.

Corollary: *The difference between n and $B(n)$ is a multiple of 3.*

Property 3. *If n is a multiple of 9, then $B(n) = 9$. If n is not a multiple of 9, then $B(n)$ is equal to the remainder in the division $n \div 9$.*

This follows from Property 1 and Property 2.

Property 4. *For all pairs m, n of integers, the following relations are true:*

$$B(m+n) = B\left(B(m) + B(n)\right),$$
$$B(m\,n) = B\left(B(m)\,B(n)\right).$$

For example, let $m = 12$, $n = 17$. Then $B(m) = B(12) = 3$ and $B(n) = B(17) = 8$. Also, $m + n = 29$, $mn = 204$, so $B(m+n) = 2$, $B(mn) = 6$. Now observe that:

$$B\left(B(m) + B(n)\right) = B(3+8) = B(11) = 2 = B(m+n),$$

and

$$B\left(B(m)\,B(n)\right) = B(3 \times 8) = B(24) = 6 = B(mn).$$

The two relations may be proved in general using Property 3. They will follow from the following more general assertion: If integers m' and n' are such that m, m' leave the same remainder under division by 9, and n, n' likewise leave the same remainder under division by 9, then

$$B(m+n) = B\left(m' + n'\right),$$
$$B(m\,n) = B\left(m'\,n'\right).$$

We leave the proof of this to you; it follows once again from Property 3.

Comment. It is Property 3 which forms the basis of the traditional method of "checking a calculation by casting out nines." Thus, we can quickly show that the computation

$$34567 \times 23456 = 810802552$$

must be incorrect, because the digital root of the expression on the left side is $B(7 \times 2) = 5$, while the digital root of the expression on the right side is 4.

Property 5. *A prime number exceeding 3 cannot have a digital root equal to 3, 6 or 9.*

For, since $n - B(n)$ is a multiple of 3, if $B(n)$ is a multiple of 3, then so must be n; and the only prime number which is a multiple of 3 is 3 itself.

Remark: The following statement looks plausible but is **not** true: "The greatest common divisor of n and $B(n)$ is equal to the greatest common divisor of $B(n)$ and 9." We invite you to find a counterexample for this claim.

Property 6. *The digital root of a triangular number is one of the following numbers:* $1, 3, 6, 9$.

Recall that a **triangular number** has the form $m(m+1)/2$ where m is a positive integer. To prove the above claim, we consider the different forms that a triangular number $n = m(m+1)/2$ can have, depending on the remainder that m leaves under division by 6.

The different forms for m are: $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, $6k+5$. If $m = 6k$ or $6k+2$ or $6k+3$ or $6k+5$, then the product $m(m+1)/2$ is a multiple of 3; hence the digital root of $m(m+1)/2$ will be one of the numbers $3, 6, 9$. If $m = 6k+1$ or $6k+4$, then $m(m+1)/2$ is of the form $9m+1$; hence its digital root is 1.

Expressed negatively, the above result yields a useful corollary.

Corollary. A triangular number which is not a multiple of 3 has digital root equal to 1.

Property 7. *If n is a perfect square, then* $B(n) \in \{1, 4, 7, 9\}$.

We consider the different forms that a perfect square $n = m^2$ can have, depending on the remainder that m leaves under division by 9. The different forms for m are: $9k$, $9k \pm 1$, $9k \pm 2$, $9k \pm 3$, $9k \pm 4$. By squaring the expressions and discarding multiples of 9, we find that $B(n) \in \{1, 4, 7, 9\}$ in each case.

Property 8. *If n is a perfect cube, then* $B(n) \in \{1, 8, 9\}$.

The same method may be used as in the case of the squares.

Property 9. *If n is a perfect sixth power, then* $B(n) \in \{1, 9\}$.

If n is a perfect sixth power, then it is a perfect square as well as a perfect cube; hence $B(n) \in \{1, 4, 7, 9\}$ as well as $B(n) \in \{1, 8, 9\}$. This yields: $B(n) \in \{1, 9\}$. (A nice application of set intersection!)

Corollary. A perfect sixth power which is not a multiple of 3 has digital root equal to 1.

Property 10. *If n is an even perfect number other than* 6, *then* $B(n) = 1$. Recall that a **perfect number** is one for which the sum of its proper divisors equals itself. The first few perfect numbers are: 6; 28; 496; 8128; 33550336. The digital roots of these numbers are:

$$B(28) = B(10) = 1,$$

$$B(496) = B(19) = 1,$$

$$B(8128) = B(19) = 1,$$

$$B(33550336) = B(28) = 1,$$

and so on.

This result is far from obvious and will need to be proved in stages. The full proof is given in the addendum at the end of this article.

Concluding remark. The notion of digital root has been known for many centuries. As described in this article, there is a simple number theoretic basis for the notion. The simplicity of the topic makes it an attractive one for closer study by students in middle school and high school. It certainly needs to be better known than it is at present.

ANANT W. VYAWAHARE received his Ph.D in Number Theory. He retired as Professor, M.Mohota Science College, Nagpur. At present, he is pursuing research on the History of Mathematics. He has a special interest in Vedic Mathematics and in the mathematics in music.

Digital Roots of Perfect Numbers

C⊗MαC

It was claimed in the article by Anant Vyawahare on *Digital Roots* that a perfect number other than 6 has digital root 1 (Property 10 of that article). We now provide a proof of this claim.

But, first things first; let us start by defining the basic motions and related concepts.

Perfect numbers. Many children discover for themselves the following property of the integer 6: the sum of its proper divisors is equal to the number $(1 + 2 + 3 = 6)$. Noticing such a property, they may naturally wonder about the existence of more such integers. Two millennia back, the Greeks decided that such a property indicated a kind of perfection, and called such numbers **perfect**. (This is of course the English translation of the word used by the Greeks; other translations could be: *complete*, *ideal*.) So 6 is a perfect number (and it is obviously the smallest perfect number).

Students will naturally ask what we should call numbers which are not perfect, i.e., other than simply calling them imperfect! This line of thinking gives rise to the following definitions.

Keywords: *Perfect number, deficient number, abundant number, Mersenne prime, factor theorem*

Let $d(n)$ denote the sum of the divisors of the positive integer n. We include among the divisors the number n itself (that is why we did not use the word 'proper'). For example, $d(10) = 1 + 2 + 5 + 10 = 18$, and $d(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. In terms of the d-function, we arrive at the following definitions:

- If $d(n) < 2n$, we say that n is **deficient**.
- If $d(n) = 2n$, we say that n is **perfect**.
- If $d(n) > 2n$, we say that n is **abundant**.

For example:

- 10 is deficient, because $1 + 2 + 5 + 10 < 20$;
- 6 is perfect, because $1 + 2 + 3 + 6 = 12$; and
- 12 is abundant, because $1 + 2 + 3 + 4 + 6 + 12 > 24$.

This categorisation of the natural numbers has considerable antiquity; indeed it goes back to the first century AD! According to Wikipedia: "The natural numbers were first classified as either deficient, perfect or abundant by Nicomachus in his *Introductio Arithmetica* (circa 100)."

Number crunching. Write D, P and A to denote (respectively) the sets of deficient, perfect and abundant numbers. Computer-assisted experimentation (using any computer algebra system, for example, *Mathematica*) yields the following data for the positive integers below 100:

$$\begin{aligned}D = \{&1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33,\\ &34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 61, 62, 63,\\ &64, 65, 67, 68, 69, 71, 73, 74, 75, 76, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94,\\ &95, 97, 98, 99, \ldots\},\end{aligned}$$

$$P = \{6, 28, \ldots\},$$

$$A = \{12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, \ldots\}.$$

Examining these figures, we are struck by the following: there appear to be very few perfect numbers; there appear to be many more deficient numbers than abundant numbers; abundant numbers appear to be all even. But …these being generated by computer experimentation, we must not assume immediately that they are all true; data can deceive us. (Indeed, one of the statements turns out to be incorrect; but we will leave it to you to find out which one!)

Some statements however are easy to conjecture and also easy to prove. For example:

(1) *There exist infinitely many deficient numbers.*

For, any prime p is deficient, since $d(p) = 1 + p < 2p$. Since there are infinitely many primes, there must also be infinitely many deficient numbers.

(2) *There exist infinitely many abundant numbers.*

For, if p is a prime number, then $12p$ is necessarily an abundant number, because

$$d(12p) = p + 2p + 3p + 4p + 6p + 12p = 28p > 24p.$$

Since there are infinitely many primes, there must also be infinitely many abundant numbers.

It is tempting now to conclude: *There exist infinitely many perfect numbers.* But as of today, mathematicians do not know the truth concerning this question!

One can frame the above as a question rather than a conjecture: *Is there a way of generating as many perfect numbers as one wants?* Long back, the Greeks discovered a partial answer to this question by connecting it with a question about prime numbers. Here is the connection. Within the sequence of prime numbers,

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ \dots$$

one can identify various subsequences of interest. One which we will be concerned with is the following:

$$3,\ 7,\ 31,\ 127,\ 8191,\ \dots .$$

These are the prime numbers which are 1 less than powers of 2:

$$3 = 2^2 - 1, \quad 7 = 2^3 - 1, \quad 31 = 2^5 - 1, \quad 127 = 2^7 - 1, \quad 8191 = 2^{13} - 1, \quad \dots .$$

Today, such primes are called **Mersenne primes**, after the French mathematician-scientist-theologian Marin Mersenne (1588–1648), because he made a close study of such primes ([1]). But much before Mersenne, the Greeks had identified these primes as being of interest. Indeed, they discovered the following remarkable rule for generating perfect numbers using such prime numbers:

Theorem 1 (Euclid). *If P is a prime number which is 1 less than a power of 2, then the number $P(P+1)/2$ is perfect.*

Let us see how the rule given by this theorem acts.

- $P = 3$ yields the perfect number $3 \times 2 = 6$;
- $P = 7$ yields the perfect number $7 \times 4 = 28$;
- $P = 31$ yields the perfect number $31 \times 16 = 496$;
- $P = 127$ yields the perfect number $127 \times 64 = 8128$;

and so on. (Please check for yourself that the numbers 28 and 496 are perfect.) It may seem now that we have hit upon a perfect recipé for generating infinitely many perfect numbers. But wait, there is a catch! Simply put: we do not know if there are infinitely many Mersenne primes!

Remark. Identifying the precise set of values of n for which the number $2^n - 1$ is prime makes for a fascinating exploration. It is easy to see that n must be prime for $2^n - 1$ to be prime. Indeed, if a is a divisor of n, then $2^a - 1$ is a divisor of $2^n - 1$; for example, $2^3 - 1 = 7$ is a divisor of $2^9 - 1 = 511$.

What complicates the picture is that there are prime values of n for which $2^n - 1$ is not prime! The first such value is 11. Finding the factorisation of $2^{11} - 1 = 2047$ makes for a nice exercise.

Proof of Theorem 1. Proving Theorem 1 is fairly easy. Let $P = 2^n - 1$ be a Mersenne prime (here $n \geq 2$). Then $P(P+1)/2 = 2^{n-1}P$. What are the divisors of the number $2^{n-1}P$? As this number has only two distinct prime factors (2 and P), it is easy to list all its divisors:

$$\begin{cases} 1, & 2, & 2^2, & 2^3, & \dots, & 2^{n-1}, \\ P, & 2P, & 2^2P, & 2^3P, & \dots, & 2^{n-1}P. \end{cases}$$

The sum of the divisors is therefore

$$\begin{aligned}
&\left(1+2+2^2+\cdots+2^{n-1}\right)+\left(P+2P+2^2P+\cdots+2^{n-1}P\right)\\
&=(1+P)\left(1+2+2^2+\cdots+2^{n-1}\right)\\
&=(P+1)\left(2^n-1\right) \qquad \text{(by summing the GP)}\\
&=(P+1)P,
\end{aligned}$$

which is twice the number under study; hence $P(P+1)/2$ is perfect, as claimed. □

How the Greeks stumbled upon this result is not clear. It is noteworthy that they did so and also proved the correctness of the procedure in a pre-algebra age.

Now it is obvious that for $n>1$, the formula $\left(2^n-1\right)2^{n-1}$ will only generate even numbers; hence this rule generates only *even perfect numbers*. A full twenty centuries after the Greeks, the Swiss German mathematician Leonhard Euler (1707–1783) proved a sort of converse to Theorem 1:

Theorem 2 (Euler)**.** *Every even perfect number has the form $P(P+1)/2$ where P is a prime number of the form 2^n-1.*

Euler's theorem is rather more challenging to prove than Euclid's; we invite you to find a proof of your own.

Euclid's and Euler's results acting in tandem provide a complete characterisation of the even perfect numbers.

Remark. Nothing has been said till now about *odd perfect numbers*. There is a mystery here. It has been conjectured that there do not exist any odd perfect numbers. However, no proof has been found for this statement, though it is widely believed to be true (all the evidence points in its favour).

Digital root of a perfect number

We are now in a position to prove the statement made in the companion article: *All even perfect numbers other than 6 have digital root 1.*

From Theorem 2 we know that every even perfect number has the form $2^{n-1}\left(2^n-1\right)$ where n is such that 2^n-1 is a prime number.

Now if n is even, 2^n-1 is a multiple of 3 (this may be established using induction; please do so); and $2^n-1>3$ for $n>2$. Hence if $n>2$ and is even, 2^n-1 is not a prime number. Expressing this statement in a negative way, we infer that if $n>2$ and 2^n-1 is a prime number, then n is odd.

The claim that all even perfect numbers other than 6 have digital root 1 is implied by the following result, which actually establishes a much stronger statement.

Theorem 3. *If n is odd, then $2^{n-1}\left(2^n-1\right)$ leaves remainder 1 under division by 9.*

For example, for $n=3$, $2^{n-1}\left(2^n-1\right)=2^2\left(2^3-1\right)=28=(9\times 3)+1$; and for $n=9$, $2^{n-1}\left(2^n-1\right)=2^8\left(2^9-1\right)=130816=(9\times 14535)+1$.

Proof of Theorem 3. The statement of the theorem can be rewritten thus:

If n is odd, then $2^{2n-1}-2^{n-1}-1$ is divisible by 9.

Multiplication by 2 has no effect on divisibility by 9; so we may express this as:

If n is odd, then $4^n - 2^n - 2$ is divisible by 9.

We offer two proofs of this statement. The first one runs along lines which should be very familiar to class 11 and 12 students.

First proof. Let $f(n) = 4^n - 2^n - 2$; then $f(1) = 0$ and $f(3) = 54$. So $f(1)$ and $f(3)$ are multiples of 9. To prove the theorem, it suffices to show that $f(n+2) - f(n)$ is divisible by 9 for all odd positive integers n. Now we have:

$$\begin{aligned} f(n+2) - f(n) &= 4^{n+2} - 4^n - 2^{n+2} + 2^n \\ &= 16 \cdot 4^n - 4^n - 4 \cdot 2^n + 2^n = 15 \cdot 4^n - 3 \cdot 2^n \\ &= 3 \cdot 2^n \left(5 \cdot 2^n - 1\right). \end{aligned}$$

Therefore, to prove that $f(n+2) - f(n)$ is divisible by 9 for all odd positive integers n, it suffices to show that $5 \cdot 2^n - 1$ is divisible by 3 for all odd positive integers n. But:

$$5 \cdot 2^n - 1 = 3 \cdot 2^n + 2 \cdot 2^n - 1 = 3 \cdot 2^n + 2^{n+1} - 1.$$

Hence it suffices to show that $2^{n+1} - 1$ is divisible by 3 for all odd positive integers n. But this is easily seen to be true, for if n is odd, then $n+1$ is even, and we have already noted earlier that if m is even, then $2^m - 1$ is divisible by 3.

Traversing the chain of reasoning in the reverse direction, we see that we have shown that $2^{n-1}\left(2^n - 1\right)$ leaves remainder 1 under division by 9 for all odd n. $\square$

Second proof. Noting that $4^n - 2^n - 2$ is of the form $x^2 - x - 2$ (with the substitution $x = 2^n$), and this polynomial factorizes conveniently as $x^2 - x - 2 = (x-2)(x+1)$, we get:

$$4^n - 2^n - 2 = \left(2^n - 2\right)\left(2^n + 1\right).$$

Since n is an odd positive integer, $n - 1$ is an even nonnegative integer; therefore $2^{n-1} - 1$ is a multiple of 3, say $2^{n-1} - 1 = 3a$ where a is an integer. Hence $2^{n-1} = 3a + 1$, and so

$$2^n = 6a + 2 = 3b + 2$$

where b is another integer ($b = 2a$). From this we get $2^n - 2 = 3b$ and $2^n + 1 = 3b + 3$. So both $2^n - 2$ and $2^n + 1$ are multiples of 3, implying that their product is a multiple of 9. Therefore $4^n - 2^n - 2$ is a multiple of 9. $\square$

References

1. Marin Mersenne, https://en.wikipedia.org/wiki/Marin_Mersenne

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ClassRoom

Three Means

Marcus Bizony

Given two positive numbers u and v, their arithmetic mean (AM) is a, such that u, a, v are in arithmetic progression; this requires that

$$2a = u + v.$$

The geometric mean (GM) is g, such that u, g, v are in geometric progression, which means that

$$g^2 = uv.$$

The harmonic mean (HM) is h, such that the reciprocals of u, h, v are in arithmetic progression, and so

$$\frac{2}{h} = \frac{1}{u} + \frac{1}{v}, \qquad \therefore \; h = \frac{2uv}{u+v}.$$

Interestingly, this can be rearranged as

$$h \cdot \frac{u+v}{2} = uv,$$

from which it follows that the GM of u and v is also the GM of their AM and HM.

For each of these three means, there is a simple and well-known geometric construction that illustrates it, but I was curious to see whether one could find a single diagram that illustrated all three at the same time.

Keywords: *Arithmetic mean, geometric mean, harmonic mean, visualisation, geometry*

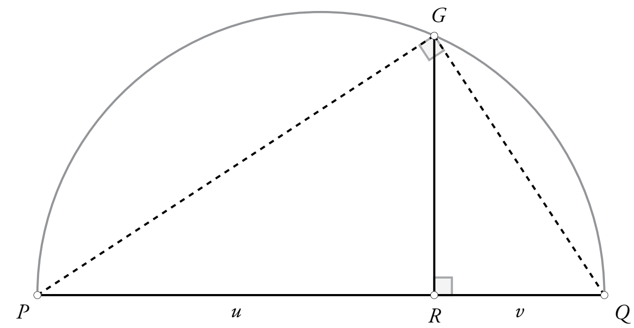


Figure 1

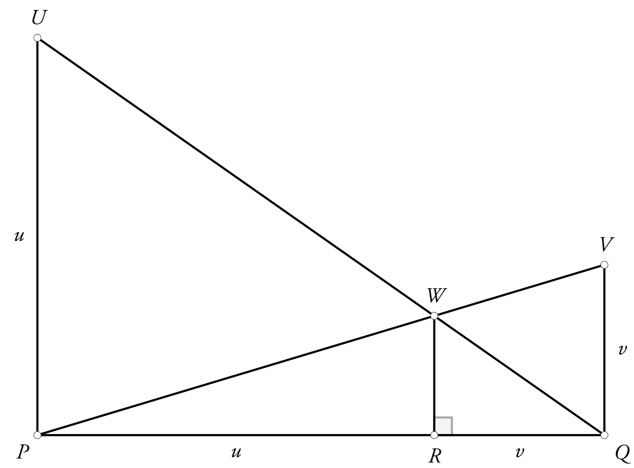


Figure 2

We begin by recalling two of the basic diagrams.

Figure 1 depicts a semicircle with diameter $u + v$; PQ is a diameter of the semicircle, and the perpendicular RG is erected at the point R so that $PR = u$ and $QR = v$. Then $\triangle PGR \sim \triangle GQR$ and so $GR^2 = PR \cdot RQ$, i.e. the length RG represents the geometric mean of the lengths PR and QR.

In Figure 2, PQ is the common perpendicular to PU and QV; UQ and VP are joined and meet at W, and R is the foot of the perpendicular to PQ from W. Using the proportional intercepts theorem for $\triangle QUP$, and then again for $\triangle PVQ$, we find that $PR : RQ = u : v$, and also

$$\frac{WR}{u} + \frac{WR}{v} = 1, \qquad \therefore \quad \frac{1}{WR} = \frac{1}{u} + \frac{1}{v},$$

which means that the length of RW is half the length of the harmonic mean of the lengths PU and QV.

Now, consider a line segment PQ with length $u + v$, and a point R on that segment such that $PR = u$, $QR = v$ (Figure 3). Erect perpendiculars PU and QV so that $PU = PR = u$, $QV = QR = v$, and erect the perpendicular to PQ at R. Draw the semicircle on PQ as diameter (centre O), meeting the perpendicular through R at G, and let W be the point of intersection of UQ and PV.

From our earlier remarks, it is clear that since $\measuredangle PGO$ is a right angle subtended by the diameter PQ, RG is the geometric mean of PR and QR, and therefore of u and v; moreover the vertical WR coincides with the vertical GR, i.e. W does indeed lie on RG.

We now add in the circle centred on W and passing through R, meeting RG again in H (Figure 4). RH, being twice RW, will be the harmonic mean of PR and QR.

Prettily, it seems that the smaller circle is tangent to the semicircle. We prove that this is indeed the case by verifying that the distance between the centres of the two circles is equal to the difference between their radii. Equivalently, we may consider the line OW and extend it till it meets the semicircle at

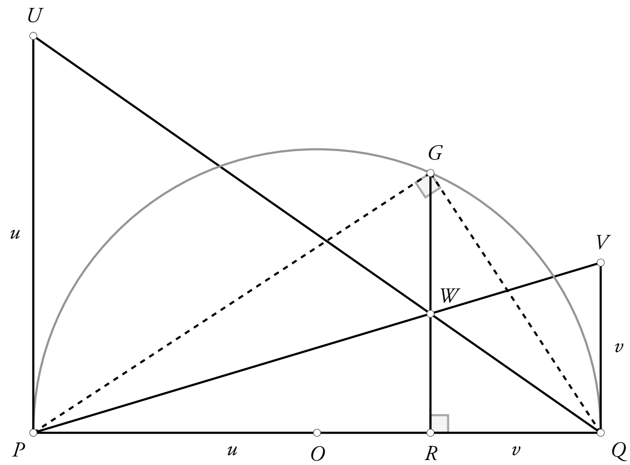


Figure 3

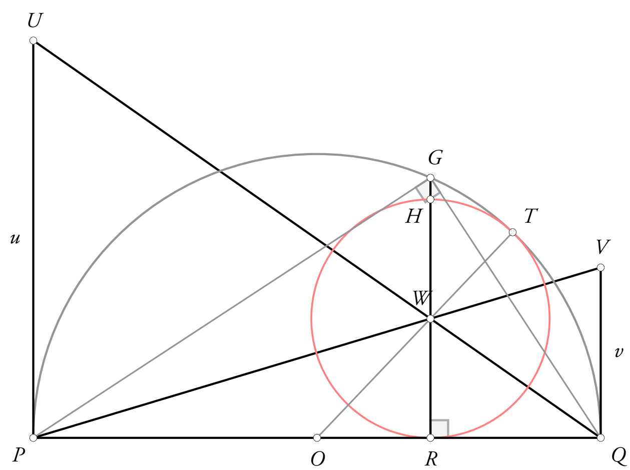


Figure 4

point T. If the length of WT is equal to the radius of the smaller circle, this claim will follow.

Let $WR = h$. We have now: $OR = \frac{1}{2}(u - v)$, so:

$$\begin{aligned} OW^2 = OR^2 + RW^2 &= \frac{(u-v)^2}{4} + h^2 \\ &= \frac{(u+v)^2}{4} - uv + h^2 \\ &= \frac{(u+v)^2}{4} - h \cdot \frac{u+v}{2} + h^2 \\ &= \left(\frac{u+v}{2} - h\right)^2. \end{aligned}$$

Hence the distance between the centres of the principles is equal to the difference between their radii, implying that the circles are internally tangent to each other as claimed.

Now the semicircle and the small circle have a common tangent at T; let this meet line $RWHG$ produced in A (Figure 5). It turns out that RA is the arithmetic mean of PR and RQ.

To prove this, we consider the triangles ORW and ATW: they are congruent (right-angled, vertically opposite angles equal and $RW = WT = h$), so that $WA = OW$; also $OW = OT - h$. This means that $AR = OT - h + WR$, but of course $WR = h$ and so $AR = OT = \frac{1}{2}(u + v)$.

We thus have a very elegant illustration of the three means of the lengths PR and QR in one diagram. Moreover, the standard result that $HM \leq GM \leq AM$ is visually confirmed, for clearly H must lie inside the semicircle, G on it and A outside it.

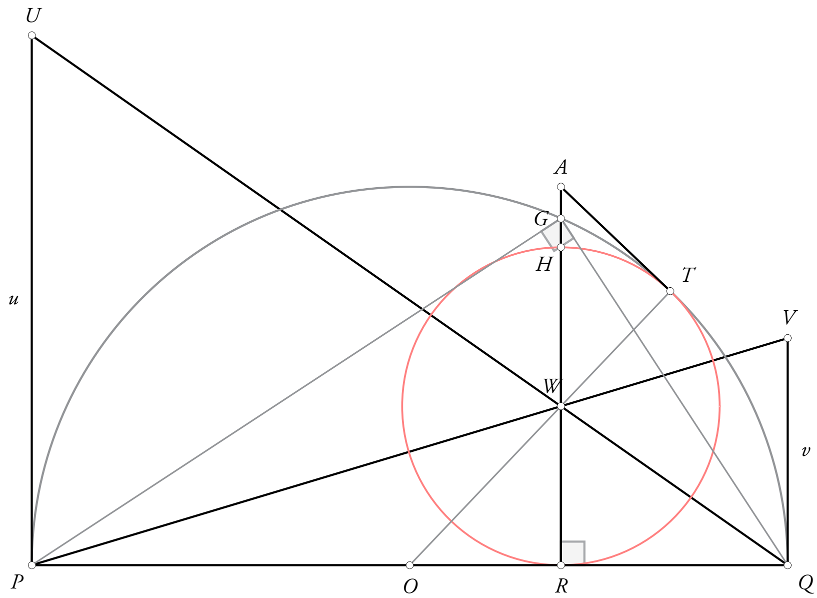


Figure 5

MARCUS BIZONY is Deputy Head (Academic) at Bishops in Cape Town, and before that was Head of Mathematics at Bishops, where he taught for over thirty years. He is Associate Editor of *Learning and Teaching Mathematics*, a journal published by the Association for Mathematics Educators of South Africa. He is also convenor of the panel that sets the South African Olympiad questions for juniors (Grades 8 and 9). Mr Bizony may be contacted at mbizony@bishops.org.za.

DIGGING DEEPER

See https://vimeo.com/27986708 for a convincing series of arguments that 25 ÷ 5 = 14

And https://www.youtube.com/watch?v=xkbQDEXJy2k to hear that 28 ÷ 7 = 13

What just happened? We'd like to hear from our student readers (teachers, please pass on this question to them if you happen to read it).

And now, we encourage you to send in more examples of such fallacies. You could start with looking for more examples of Odd number ÷ Odd number and Even number ÷ Odd number that is, the types already given.

Now stretch further! How about examples for Odd number ÷ Even number and Even number ÷ Even number. If you think, either of these cases is impossible, give a reason!

Do mail AtRiA.editor@apu.edu.in if you would like to share your thoughts!

ClassRoom

The Difference-of-Two-Squares Formula: A New Look

C⊗MαC

The difference-of-two-squares formula $a^2 - b^2 = (a - b)(a + b)$ is so basic that it would seem a difficult task to say anything new about it! But **Agnipratim Nag** of Frank Anthony Public School, Bangalore (Class 8) has done just this. In this short note we describe his interesting and innovative approach to prove the identity. It has particular relevance for those who teach at the middle school level.

The proof assumes throughout that a and b are integers. Pedagogically, this is very appropriate, as the difference-of-two-squares identity should be introduced to students as a relationship between integers. Interesting explorations involving integers can be designed which will lead students to the identity. The extension to arbitrary numbers can happen later. In the account below, a is assumed to be the larger number.

Keywords: *Algebra, identities, alternative proofs, number line, investigation, visualisation*

The case when a and b are consecutive integers, i.e., a = b + 1. We have:

$$\begin{aligned} ab &= ba, \\ \therefore\ a(a-1) &= b(b+1), \\ \therefore\ a^2 - a &= b^2 + b, \\ \therefore\ a^2 - b^2 &= a + b. \end{aligned}$$

This relation may be expressed in words as follows:

The difference between the squares of two consecutive integers is equal to the sum of the two integers.

The relation $a^2 - a = b^2 + b$ can be depicted in an attractive way (Figure 1) when we recall that a^2 and b^2 are a pair of consecutive squares.

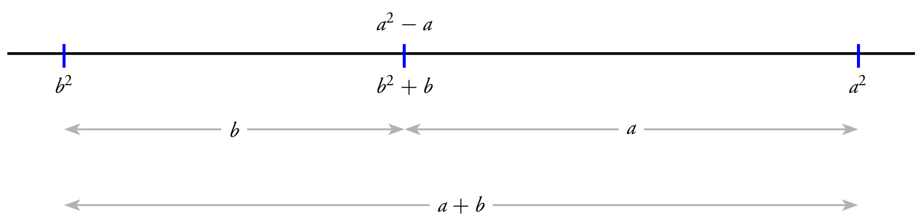


Figure 1

The case when a and b are not consecutive integers. There are two ways of proceeding. The more symbolic approach is this. Let $a - b = k$; then $a = b + k$, $b = a - k$, and:

$$\begin{aligned} ab &= ba, \\ \therefore\ a(a-k) &= b(b+k), \\ \therefore\ a^2 - ak &= b^2 + bk, \\ \therefore\ a^2 - b^2 &= ak + bk = k(a+b), \\ \therefore\ a^2 - b^2 &= (a-b)(a+b). \end{aligned}$$

A more interesting and also more colourful approach (as described by Agnipratim) is the following. The integers strictly between b and a are $b+1, b+2, \ldots, a-2, a-1$. Let these integers be denoted by $p_1, p_2, \ldots, p_{k-2}, p_{k-1}$ (if $a - b = k$, then there are $k - 1$ integers strictly between b and a); so we have $p_1 = b + 1, p_2 = b + 2, \ldots, p_{k-1} = a - 1$. The configuration may now be depicted as shown in Figure 2.

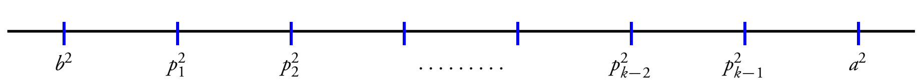


Figure 2

(Note: The figure is meant to be symbolic. It may give the impression that the gaps between $a^2, p_1^2, p_2^2, \ldots$ are all the same, but this is not actually the case. In fact, the gaps steadily get larger as we move from left to right.)

The total gap between a^2 and b^2 is clearly equal to the sum of the gaps between a^2 and p_{k-1}^2, between p_{k-1}^2 and p_{k-2}^2, …, between p_2^2 and p_1^2, and between p_1^2 and b^2:

$$a^2 - b^2 = \left(a^2 - p_{k-1}^2\right) + \left(p_{k-1}^2 - p_{k-2}^2\right) + \cdots + \left(p_2^2 - p_1^2\right) + \left(p_1^2 - b^2\right).$$

Since $\{a, p_{k-1}\}$, $\{p_{k-1}, p_{k-2}\}$, ..., $\{p_2, p_1\}$, $\{p_1, a\}$ are pairs of consecutive integers, the result established earlier applies ("The difference between the squares of two consecutive integers is equal to the sum of the two integers"). Hence

$$\begin{aligned} a^2 - p_{k-1}^2 &= a + p_{k-1}, \\ p_{k-1}^2 - p_{k-2}^2 &= p_{k-1} + p_{k-2}, \\ p_2^2 - p_1^2 &= p_2 + p_1, \\ p_1^2 - b^2 &= p_1 + b, \end{aligned}$$

which means that

$$a^2 - b^2 = (a + p_{k-1}) + (p_{k-1} + p_{k-2}) + \cdots + (p_2 + p_1) + (p_1 + b)\,.$$

The expression on the right may be rearranged to give:

$$a^2 - b^2 = (a + b) + 2\,(p_1 + p_2 + \cdots + p_{k-2} + p_{k-1})\,.$$

A further rearrangement is possible, but the behaviour is slightly different depending upon whether k is odd or even. We illustrate the behaviour using two specific values, $k = 5$ and $k = 6$. Consider first $k = 5$; we have:

$$\begin{aligned} a^2 - b^2 &= (a + b) + 2\,(p_1 + p_2 + p_3 + p_4) \\ &= (a + b) + 2\,(p_1 + p_4) + 2\,(p_2 + p_3)\,. \end{aligned}$$

Now we have, clearly:

$$p_1 + p_4 = a + b, \quad p_2 + p_3 = a + b.$$

Hence:

$$a^2 - b^2 = (a + b) + 2(a + b) + 2(a + b) = 5(a + b),$$

and since $a - b = 5$ in this instance,

$$a^2 - b^2 = (a - b)(a + b).$$

Next, consider $k = 6$; we have:

$$\begin{aligned} a^2 - b^2 &= (a + b) + 2\,(p_1 + p_2 + p_3 + p_4 + p_5) \\ &= (a + b) + 2\,(p_1 + p_5) + 2\,(p_2 + p_4) + 2p_3. \end{aligned}$$

As earlier, we have:

$$p_1 + p_5 = a + b, \quad p_2 + p_4 = a + b, \quad 2p_3 = a + b.$$

Hence:

$$a^2 - b^2 = (a + b) + 2(a + b) + 2(a + b) + (a + b) = 6(a + b),$$

and since $a - b = 6$ in this instance,

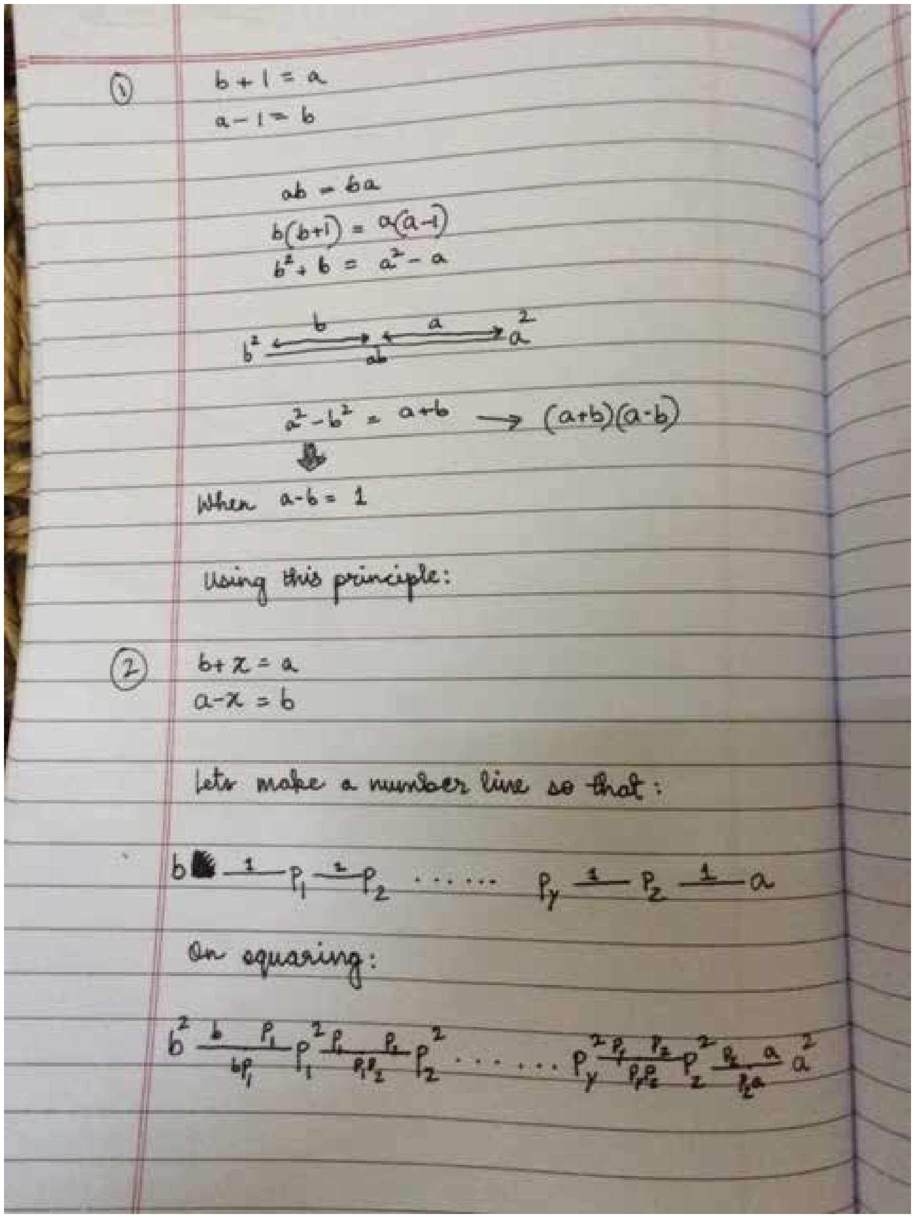
$$a^2 - b^2 = (a - b)(a + b).$$

This reasoning clearly holds for all values of k. We have not bothered to write out the argument for the general case in a formal manner, but the way the terms can be rearranged to add up to $a + b$ should be clear. We deduce that the relation

$$a^2 - b^2 = (a - b)(a + b)$$

is always true, i.e., it is an identity. $\square$

Figure 3 shows a photograph of a page from Agnipratim's notebook.



(1) $b + 1 = a$

$a - 1 = b$

$ab = ba$

$b(b+1) = a(a-1)$

$b^2 + b = a^2 - a$

$b^2 \xrightarrow[ab]{b} \xleftarrow{a} a^2$

$a^2 - b^2 = a + b \rightarrow (a+b)(a-b)$

When $a - b = 1$

Using this principle:

(2) $b + x = a$

$a - x = b$

Lets make a number line so that:

$b \xrightarrow{1} P_1 \xrightarrow{1} P_2 \;\cdots\cdots\; P_y \xrightarrow{1} P_z \xrightarrow{1} a$

On squaring:

$b^2 \xrightarrow[bP_1]{b \quad P_1} P_1^2 \xrightarrow[P_1P_2]{P_1 \quad P_2} P_2^2 \;\cdots\cdots\; P_y^2 \xrightarrow[P_yP_z]{P_y \quad P_z} P_z^2 \xrightarrow[P_za]{P_z \quad a} a^2$

Figure 3. Image sent to us by Agnipratim Nag

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ClassRoom

How to Prove It

In this episode of "How To Prove It", we prove a simple yet surprising collinearity associated with any triangle. We do so in two different ways and then contrast the two proofs. It would be instructive for a student to study these proofs and also the remarks made at the end.

Shailesh A Shirali

In $\triangle ABC$, let the feet of the altitudes from A, B, C be D, E, F respectively. Select any one altitude, say AD, and from its foot (D), drop perpendiculars to the other two sides (AB, AC) and to the other two altitudes (BE, CF). Let the feet of these perpendiculars be P, S and Q, R, as shown in Figure 1. Then the points P, Q, R, S lie in a straight line.

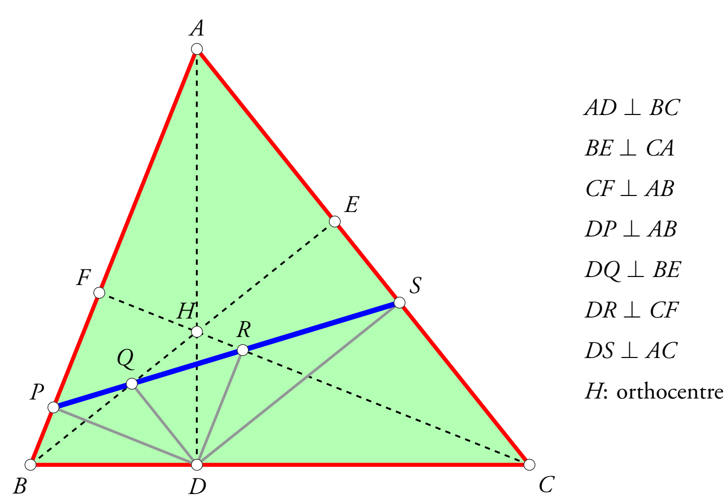


Figure 1

Keywords: *Altitudes, collinearity, cyclic quadrilateral, angle chasing, supplementary, coordinate, slope, linear equation, pure geometry, proof*

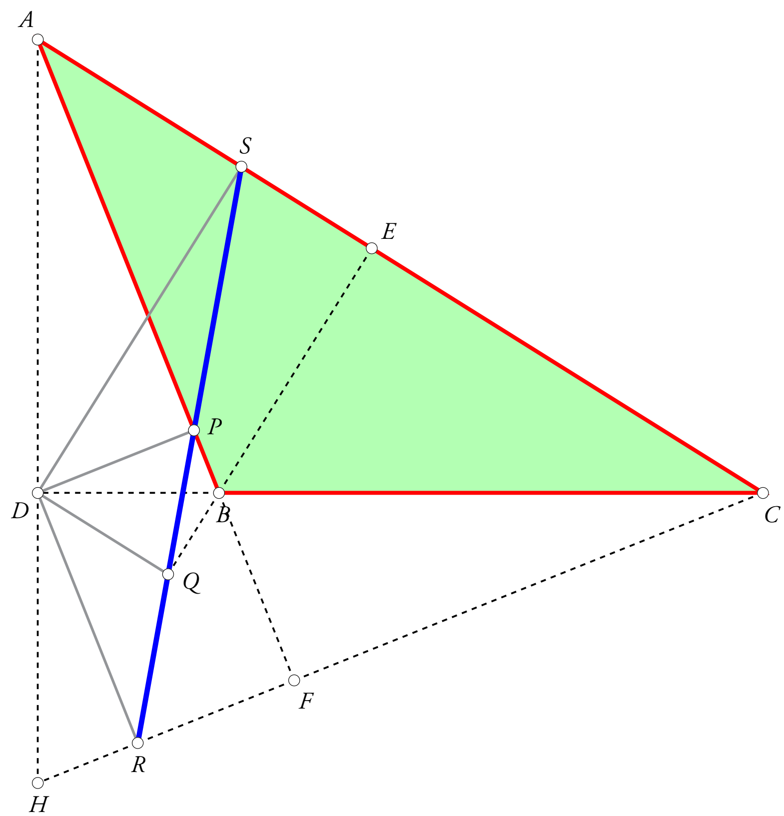


Figure 2

In case you are wondering how the configuration will change if the triangle is obtuse, please study Figure 2 (in which $\angle B$ is obtuse). Many of the points now fall outside the given triangle. The conclusion however stays intact.

We offer two proofs of the proposition. The reader is invited to judge the relative merits of each proof.

First proof

This proof uses nothing more than 'angle chasing'. We start by redrawing the figure, taking care to use different colours for segments PQ, QR and RS (see Figure 3). The purpose of doing this is simply to avoid the error of implicitly assuming the very thing you are trying to prove—an error that is particularly easy to make in geometry.

Note the presence of a number of cyclic quadrilaterals; this is natural, given the number of perpendiculars that have been drawn. Note also that we have coloured some of the quadrilaterals differently.

To prove that points P, Q, R lie in a straight line, it suffices to show that $\angle PQD$ and $\angle RQD$ are supplementary. We show this as follows. Since quadrilateral $PBDQ$ is cyclic, it follows that $\angle PQD = 180^\circ - \angle B$. Since quadrilateral $QDRH$ is cyclic, it follows that $\angle RQD = \angle RHD$. But $\angle RHD = \angle FBD$ (this follows because quadrilateral $FBDH$ is cyclic), i.e., $\angle RHD = \angle B$. It follows that $\angle PQD$ and $\angle RQD$ are supplementary. Hence points P, Q, R lie in a straight line.

In exactly the same way we show that points Q, R, S lie in a straight line. It follows that all four points P, Q, R, S lie in a straight line, as required. $\square$

Second proof

This time we shall use coordinates. Adopt a system of coordinates in which line BC represents the x-axis, and line AD represents the y-axis; so D serves as the origin. Let the coordinates of the vertices be as follows: $A = (0, a)$, $B = (b, 0)$, $C = (c, 0)$. Note that a, b, c do not represent the lengths of the sides of the triangle. We now compute the coordinates of the points P, Q, R, S in terms of a, b, c.

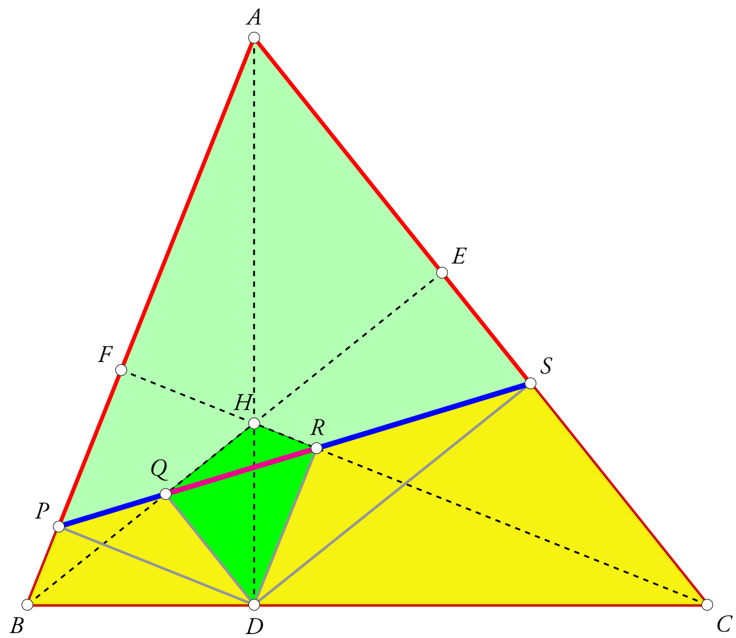


Figure 3

The slopes of the lines AB and AC are $-a/b$ and $-a/c$, hence the slopes of CF and DP are b/a, and the slopes of BE and DS are c/a. It follows that the equations of the various lines in the figure are as follows:

- Equation of AB: $ax + by = ab$
- Equation of AC: $ax + cy = ac$
- Equation of BE: $cx - ay = bc$
- Equation of CF: $bx - ay = bc$
- Equation of DP: $bx - ay = 0$
- Equation of DQ: $ax + cy = 0$
- Equation of DR: $ax + by = 0$
- Equation of DS: $cx - ay = 0$.

Solving appropriate pairs of equations, we get the coordinates of P, Q, R, S:

$$P = \left(\frac{a^2 b}{a^2 + b^2}, \frac{ab^2}{a^2 + b^2}\right),$$

$$S = \left(\frac{a^2 c}{a^2 + c^2}, \frac{ac^2}{a^2 + c^2}\right),$$

$$Q = \left(\frac{bc^2}{a^2 + c^2}, \frac{-abc}{a^2 + c^2}\right),$$

$$R = \left(\frac{b^2 c}{a^2 + b^2}, \frac{-abc}{a^2 + b^2}\right).$$

Now, by a routine calculation (but we omit the details), we find that the slopes of the segments PQ, QR and RS are all equal to the following expression:

$$\frac{a(b+c)}{a^2 - bc}.$$

Hence the points P, Q, R, S lie in a straight line. □

Remarks. The following remarks may be of interest and should be taken note of:

- The coordinates of S may be obtained from the coordinates of P by the switch $b \leftrightarrow c$, i.e., by uniformly switching the roles of b and c.
- Similarly, the coordinates of R may be obtained from the coordinates of Q by the switch $b \leftrightarrow c$, i.e., by uniformly switching the roles of b and c.
- The slope of line $PQRS$ is symmetric in b and c.

If you think about it for a minute, you will realise that each of these observations could have been anticipated before we started the computation. This would have lessened our work.

Some Remarks on Problem-Solving

Let us now critically examine what we have done. What lessons can we draw which will help us in problem-solving in general?

This particular problem viewed as a 'pure geometry problem' is not too difficult; it yields to elementary angle chasing. However, let us consider the matter more generally.

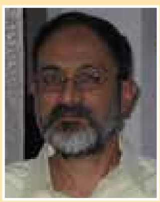
Problems in geometry can sometimes be very challenging, because the figure typically does not give any clue or hint as to the direction in which one must proceed. In general, the difficulty is that one is not able to 'see' the key elements contained in the figure. The solution may depend on seeing that a particular quadrilateral is cyclic; but the quadrilateral may be well-concealed within the figure, its sides not standing out in any way. Or the solution may depend on seeing that two particular angles are equal; but the arms of the angles may not stand out in any way. For such reasons, it helps if we mark the figure suitably, in advance: systematically look for pairs of angles which are equal to one another, and mark them so; systematically look for pairs of segments that are congruent to one another, and mark them so; similarly, mark pairs of lines which are parallel to one another, or perpendicular to one another, and mark them so; and so on. The judicious use of colour can help in carrying out these markings. Obviously, all these steps by themselves will not guarantee anything; but they can ease the path for us. Often they do, so it is worth taking these steps.

The use of coordinates to solve problems in geometry is generally not recommended (particularly by the problem-solving aficionado who would like to see problems in geometry solved by the methods of pure geometry); the purist tells us: "such approaches should be tried only when other approaches have failed." One reason for their asserting this is that the coordinate geometry approach when opted for too easily can start to leach away at our geometrical intuition. This happens because coordinate geometry is highly algebraic, and the symbols used tend to be driven by their own logic; the driver is the machinery of algebra and not our visual sense, and intuition tends to play a much diminished role. In consequence, our intuitive abilities can start to lose their muscle.

Though there is certainly some truth in the above comments, it is important to realise the nuances involved. For example, it is not true to say that no intuitive feel is involved in the application of coordinates. A very important first step when one uses coordinates is the choice of axes; they must be chosen in such a way as to minimise the number of symbols being used, and exploit to the maximum the symmetries implicit in the figure. In the above proof, note how D was made the origin of the coordinate system, with BC and AD as the axes; this has clearly been done keeping in mind the number of lines of the figure which pass through D. Next, note the symbols used for the coordinates of B and C, namely: $(b, 0)$ and $(c, 0)$. The beginner, noting that in the figure B lies to the 'left' or negative side of D while C lies to the 'right' or positive side of D, may be tempted to write: $B = (-b, 0)$ and $C = (c, 0)$. But this would be quite unnecessary, because b and c can be either positive or negative; no signs have been fixed as yet. Also, such a use of symbols would have spoiled the symmetry which we see at present. Writing $B = (-b, 0)$ would have been contrary to the very spirit of algebra. During the course of the solution we remarked: "The coordinates of S may be obtained from those of P by switching the roles of b and c." But this would not have been possible had we written $B = (-b, 0)$.

Another noteworthy point is the following. A proof based on purely geometrical considerations typically starts by drawing a diagram and marking various relationships on it. Now it is obvious that in drawing a diagram, one is going to end up with some angle larger than some other angle, the triangle may be acute-angled or obtuse-angled (one of them), and so on; without any particular intention in mind, one ends up making certain choices in drawing a diagram, perhaps without even being aware that one has made choices. A proof based on such a diagram may have the obvious defect that certain relationships which are true for the diagram may not be true for the diagram drawn in a different way. For example, in the pure geometry proof given above, we wrote at one point, with reference to Figure 3: Since quadrilateral $QDRH$ is cyclic, it follows that $\angle RQD = \angle RHD$. But a quick glance at Figure 2 (the obtuse-angled case) will show that the quadrilateral is $QDHR$ and not $QDRH$, and in this configuration $\angle RQD$ and $\angle RHD$ are supplementary and not equal to each other! This means that a diagram-based proof may have errors which we do not even suspect! In contrast, a proof based on the use of coordinates is typically diagram-independent and suffers from no such defect.

In summary, one may say that one must pause before applying the method of coordinates and choose the axes in a wise manner, exploiting to the maximum all the symmetries of the figure. And if one opts for a pure geometry approach, it is wise to check whether the reasoning one uses is valid for all possible diagrams.



SHAILESH SHIRALI is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

A CIRCULAR CHALLENGE

Shown in Figure 1 is a circle Γ, centre O. Within it are drawn a pair of perpendicular radii OP, OQ, and circles ω_1, ω_2 with OP, OQ as their diameters. The circles give rise to two regions as shown (shaded), with areas x and y respectively.

Problem: Find the ratio $x : y$.

Send in your solutions to AtRiA.editor@apu.edu.in

GENERALIZATION!

Suppose that $\measuredangle POQ = t$ (earlier we had $OP \perp OQ$, i.e., $t = \pi/2$). See Figure 2. Now find the ratio $x : y$ in terms of t.

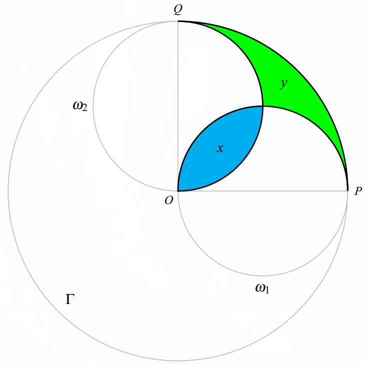


Figure 1

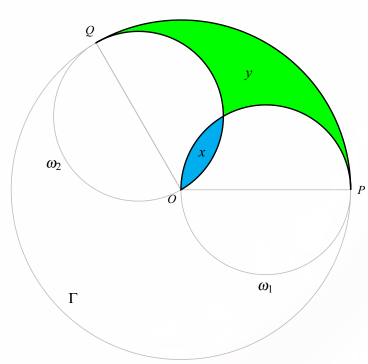


Figure 2

A 80-80-20 Triangle

ClassRoom

$\mathcal{C} \otimes \mathcal{M}\alpha\mathcal{C}$

In the previous issue of *At Right Angles*, we studied a geometrical problem concerning the triangle with angles of 130°, 20° and 30°. We made the comment that the problem belongs to a class of geometrical problems dealing with triangles with numerous lines drawn within them, intersecting at angles whose measures are an integer number of degrees; we are required to find the measure of some indicated angle. In this note, we study another problem of this genre—a particularly famous such problem. We present a trigonometric solution as well as a 'pure geometry' solution.

Problem. In Figure 1 we see $\triangle ABC$ with $\measuredangle A = 20^\circ$, $\measuredangle B = 80^\circ = \measuredangle C$. Points E and F are located on sides AC and AB by drawing rays from B and C, such that $\measuredangle EBC = 60^\circ$ and $\measuredangle FCB = 50^\circ$. Segment EF is then drawn. We are asked to find the measure of $\measuredangle BEF$.

Trigonometric solution

Numerous trigonometric identities are going to be used in the solution presented below. Please refer to page 70 of the March 2016 issue of AtRiA for a list of these identities.

Keywords: Integer degree, sine rule, addition formula for sine, difference formula for cosine, isosceles, equilateral, triangle

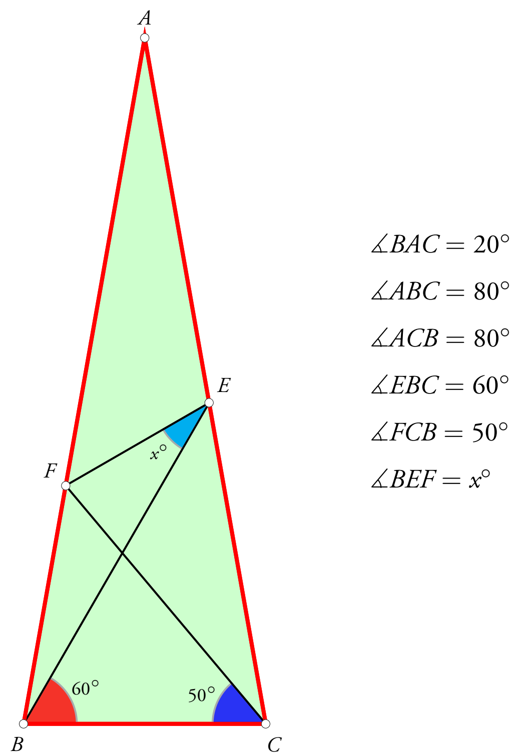


Figure 1

A DIY Invitation!

We can find $\measuredangle BEF$ using the sine rule and the cosine formula for the difference of two angles! See if you can Do-It-Yourself!

Let $\measuredangle BEF = x^\circ$. From $\triangle BEF$ (see Figure 1) we have:

$$\frac{BE}{BF} = \frac{\sin \measuredangle BFE}{\sin \measuredangle BEF} = \frac{\sin(x+20)^\circ}{\sin x^\circ} = \cos 20^\circ + \sin 20^\circ \cdot \cot x^\circ.$$

Also, from $\triangle BCE$ we have:

$$\frac{BE}{BC} = \frac{\sin 80^\circ}{\sin 40^\circ} = 2\cos 40^\circ.$$

Now we have $BC = BF$ (from $\triangle BCF$, in which $\measuredangle BCF = \measuredangle BFC = 50^\circ$). Hence:

$$\cos 20^\circ + \sin 20^\circ \cdot \cot x^\circ = 2\cos 40^\circ,$$

and from this equation we must find x. We have:

$$\begin{aligned}
\sin 20^\circ \cdot \cot x^\circ &= 2\cos 40^\circ - \cos 20^\circ \\
&= \cos 40^\circ + (\cos 40^\circ - \cos 20^\circ) \\
&= \cos 40^\circ - 2\sin 30^\circ \cdot \sin 10^\circ \\
&= \cos 40^\circ - \sin 10^\circ = \cos 40^\circ - \cos 80^\circ \\
&= 2\sin 20^\circ \cdot \sin 60^\circ = \sqrt{3} \cdot \sin 20^\circ.
\end{aligned}$$

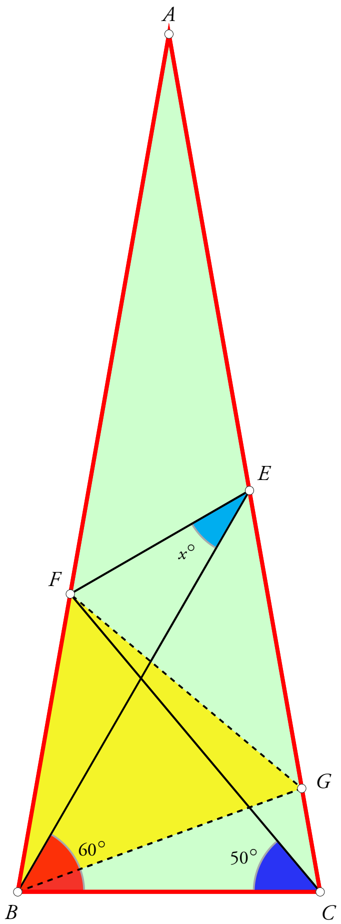
It follows that

$$\cot x^\circ = \sqrt{3},$$

and therefore that $x = 30$. Hence $\measuredangle BEF = 30^\circ$. $\square$

A 'pure geometry' solution

The fact that we have obtained such a neat answer to the problem (30°; what could be neater?) challenges us to find a solution to the problem that does not involve computation; in other words, a pure geometry solution. Let us now take up this challenge.



$\measuredangle BAC = 20°$

$\measuredangle ABC = 80°$

$\measuredangle ACB = 80°$

$\measuredangle EBC = 60°$

$\measuredangle FCB = 50°$

$\measuredangle BEF = x°$

Figure 2

Construction: Locate a point G on side AC such that $\measuredangle GBC = 20°$ (so $\measuredangle GBF = 60°$); join GF.

Since $BG = BC$ (this is so because $\measuredangle BCG = 80° = \measuredangle BGC$), and also $BF = BC$, it follows that $BF = BG$; and since $\measuredangle FBG = 60°$, it further follows that $\triangle BFG$ is equilateral. Hence $\measuredangle BGF = 60°$ and $\measuredangle EGF = 40°$.

Next, $GE = GB$, since $\measuredangle GBE = 40° = \measuredangle GEB$; hence $GE = GF$.

It follows that $\measuredangle GEF = \measuredangle GFE = 70°$. Therefore $\measuredangle BEF = 30°$. □

Remark 1. The solution presented above is only one of many pure geometry solutions of this justly famous problem. We invite you to look for one of your own!

Remark 2. Note the important (indeed, crucial) roles played by the equilateral triangle in the above solution and in the pure geometry solution for the 20-30-130 triangle problem (discussed elsewhere in this issue). You will find that this is a recurring theme in almost all such problems.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Addendum to "A 20-30-130 triangle"

In the March 2016 issue of AtRiA, we posed the following problem: *Triangle ABC has* $\measuredangle A = 130°$, $\measuredangle B = 30°$ *and* $\measuredangle C = 20°$. *Point P is located within the triangle by drawing rays from B and C, such that* $\measuredangle PBC = 10°$ *and* $\measuredangle PCB = 10°$. *Segment PA is drawn. Find the measure of* $\measuredangle PAC$. (See Figure 1.)

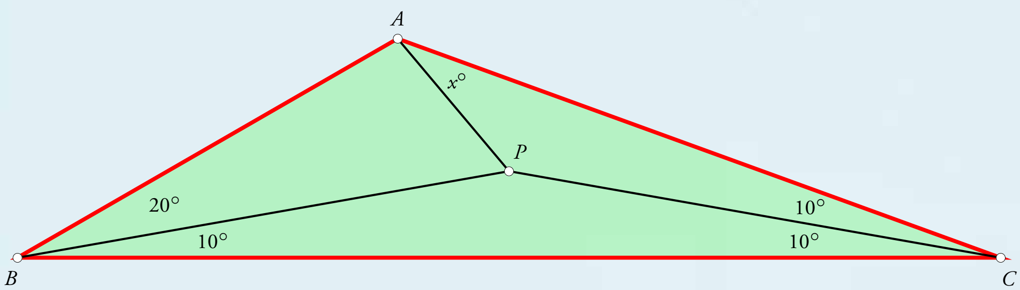


Figure 1

We had offered a trigonometric solution, making use of the sine rule and numerous standard trigonometric identities. At the end of the article we posed the question of finding a pure geometry solution.

We are happy to say that a reader (and contributor of several articles in earlier issues), **Ajit Athle**, has sent in a very elegant pure geometry solution—just as we had hoped! Here are the details.

Construction: Extend BA to E such that $AE = CE$ (see Figure 2; this is equivalent to saying: let the perpendicular bisector of segment AC meet BA extended at E); then $\measuredangle EAC = \measuredangle ECA = 50°$, and $\measuredangle AEC = 80°$.

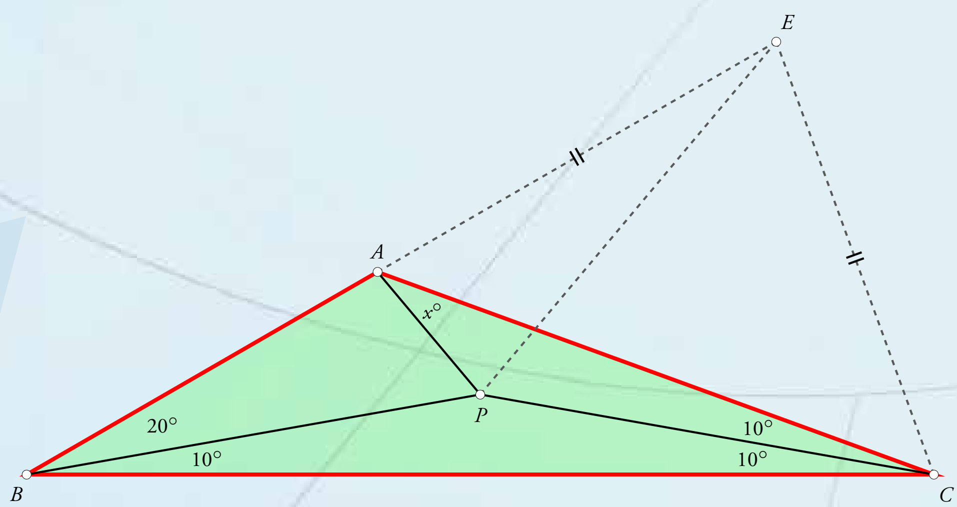


Figure 2. Solution to the 20-30-130 triangle problem by Ajit Athle

Keywords: *angle chasing, isosceles, equilateral, circum-circle, pure geometry*

Since $\measuredangle BPC = 2\measuredangle BEC$ and also $PB = PC$, it follows that P is the circumcentre of $\triangle EBC$. From this it follows that $\measuredangle EPC = 2\measuredangle EBC$, i.e., $\measuredangle EPC = 60°$.

This in turn implies that $\triangle EPC$ is equilateral, and hence that $\measuredangle PEC = 60°$. From this we infer that $\measuredangle AEP = 20°$. Again, $EA = EP$ (both sides are equal to EC), i.e., $\triangle EAP$ is isosceles. Hence $\measuredangle EAP = 80°$. Since $\measuredangle EAC = 50°$, it follows that $\measuredangle PAC = 30°$, i.e., $x = 30$. $\square$

The fact that P is the circumcentre of $\triangle EBC$ suggests an alternate way of presenting this proof. Namely: draw the circle centred at P and passing through B and C. Let it intersect the extension of BA at E. (See Figure 3.)

Then we have $PE = PC$ and $\measuredangle EPC = 2\measuredangle EBC = 60°$, hence $\triangle EPC$ is equilateral, so $\measuredangle PCE = 60°$ and $\measuredangle ACE = 50°$. We also have $\measuredangle EAC = 50°$ (since $\measuredangle BAC = 130°$); therefore $EA = EC = EP$. The rest of the solution is the same as earlier; we get $x = 30$. $\square$

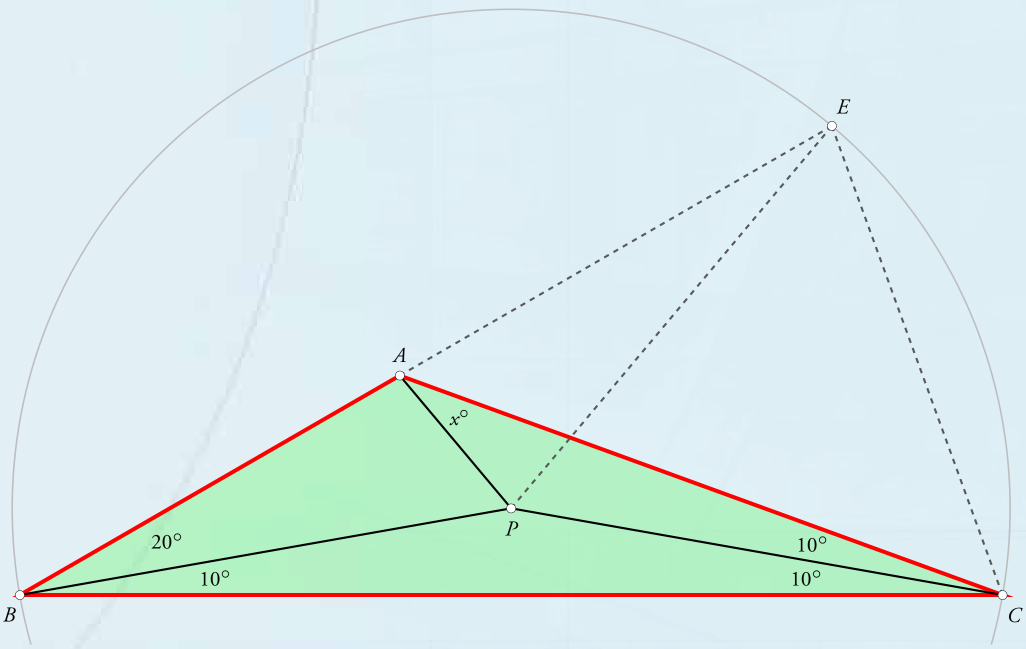


Figure 3

Remark. In hindsight, the idea of trying a circle centred at P and passing through B and C should have suggested itself to us right away; after all, we have $PB = PC$ as per the given data.

But, as they say, hindsight is the best sight of all!

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TechSpace

Introducing Differential Calculus on a Graphics Calculator

Barry Kissane

Calculus is fundamentally concerned with understanding and measuring change, which is why it has proved to be such a useful tool for more than three hundred years and has frequently been studied at the end of secondary school. The concept of a derivative is critical to the study of calculus, and is concerned with how functions are changing. In this article, we will outline how a modern graphics calculator can be used to explore this idea.

We will use a particular graphics calculator, the CASIO fx-CG 20, which does not have computer algebra capabilities. Too frequently, students focus on the symbolic manipulation aspects of calculus, which are appropriate for a later and more general treatment of ideas, but not as helpful for an introduction. Of course, similar explorations can be undertaken with other graphics calculators and with various kinds of computer software.

Keywords: *graphics calculator, graph, slope, linear, quadratic, derivative, function*

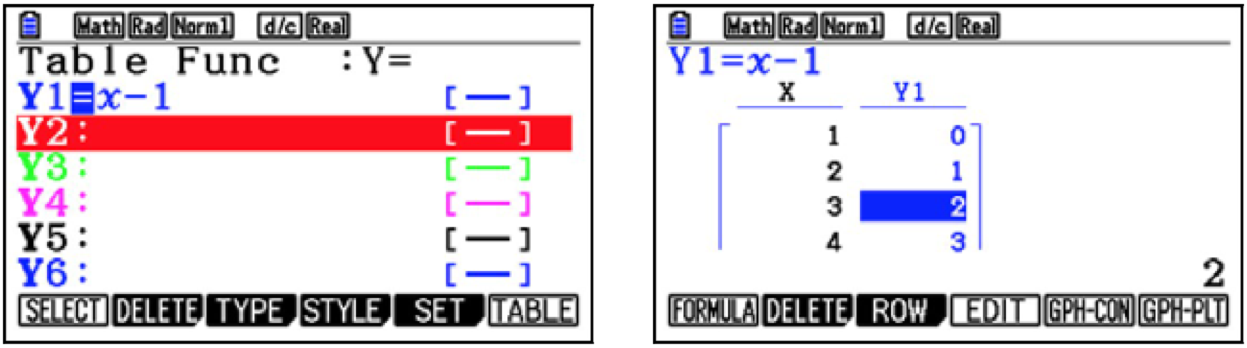


Figure 1

The choice of a graphics calculator is deliberate: it is the only technological tool that has been designed expressly for secondary school mathematics, and so includes a great deal of mathematical functionality designed for education. Kissane and Kemp (2008) described some affordances of graphics calculators for students learning calculus in Australian secondary schools. Graphics calculators have been in use in schools in many countries now for almost thirty years, and yet are frequently misunderstood in other countries (usually by those who have not used them) as devices merely useful to undertake calculations, rather than as educational tools. In many countries that measure student learning with examinations (such as Australia and the USA), students are expected to use their graphics calculators both for everyday learning of mathematics as well as for routine use in examinations. In the case of India, students enrolled in the International Baccalaureate are expected to use graphics calculators as an integral part of their learning, and are also expected to make use of them in official examinations.

Recently, Kissane and Kemp (2014) outlined a model for learning with calculators, suggesting that they can be used to represent mathematical ideas, to undertake computations, to explore mathematical situations and to affirm (or contradict) one's thinking. Of course, the graphics calculator is of value in many other parts of the mathematics curriculum as well, not only for the calculus, as illustrated in Kissane and Kemp (2014). In this article, examples of all of these aspects of calculator use for the particular context of calculus are provided.

Examining change

To see how a function is changing, it is convenient to draw a table of values and examine successive terms. For example, consider the simple case of a linear function such as $f(x) = x - 1$. A table of values (see Figure 1) shows that an increase of 1 in x results in an increase of 1 in the value of the function, $f(x)$. Each row of the table, which can be explored with a cursor, shows this steady increase.

Graphics calculators represent functions in three different ways, sometimes referred to as the 'Rule of Three': symbolically, numerically and graphically. Much is to be gained from students toggling between these representations. In this case, if the function is represented graphically, rather than numerically, a familiar linear function, with a steady increase (called the slope, in this case 1) is seen. The screen in Figure 2 shows that the graph can be explored by tracing, which helps to see the connection between the table of values and the graph.

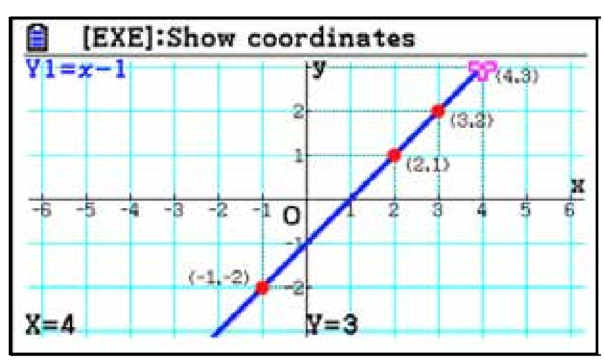


Figure 2

Linear functions are important, as they describe the most fundamental form of growth of a function. Students usually study these, and develop an understanding of slope, before they begin studying the calculus. Both negative and zero slopes and the meaning of larger and smaller slopes are readily explored. Thus, the screen in Figure 3 shows the function $f(x) = 3 - 2x$, which has a slope of -2. The change is still steady, but the values of the function now decrease instead of increasing and change by twice as much as the x-values.

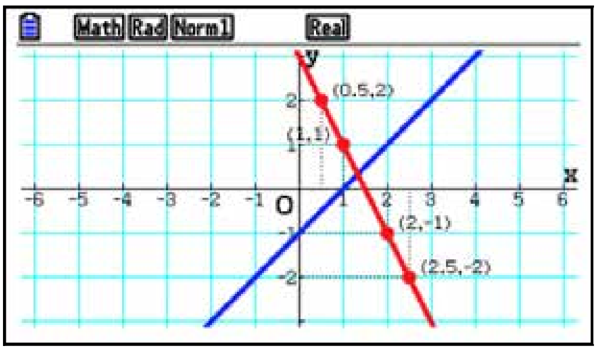


Figure 3

Most functions do not change in steady and easily predictable ways like this, however, and their graphs are not lines but are curves. A good example is the quadratic function, $f(x) = x^2$. Figure 4 shows a curve and not a line, while a table of values shows that the increase in the values of the function is not uniform. An increase in x from 0 to 1 results in an increase of 1 in $f(x)$, but the increase of x from 1 to 2 shows a much larger increase of 3 in $f(x)$. Furthermore, an increase in x from -2 to -1 results in a decrease in $f(x)$.

While describing how linear functions change is relatively accessible, and regularly studied by younger students, this example illustrates that it is clearly a more difficult and complicated matter to describe how non-linear functions change.

Local linearity

A very important concept in introductory calculus is local linearity: the idea that, when examined on a small enough (i.e., local) scale, most functions of interest to secondary school students become (almost) linear. This surprising and powerful result is difficult to see without using some technology, which is why it was not commonly discussed in introducing calculus many years ago. Yet it is relatively easy to see on a graphics calculator through a process of 'zooming' both on a graph or on a table of values. A graphics calculator typically has a zoom command, giving an efficient way for a small part of the graph to be seen more closely. The two screens in Figure 5 show this around the

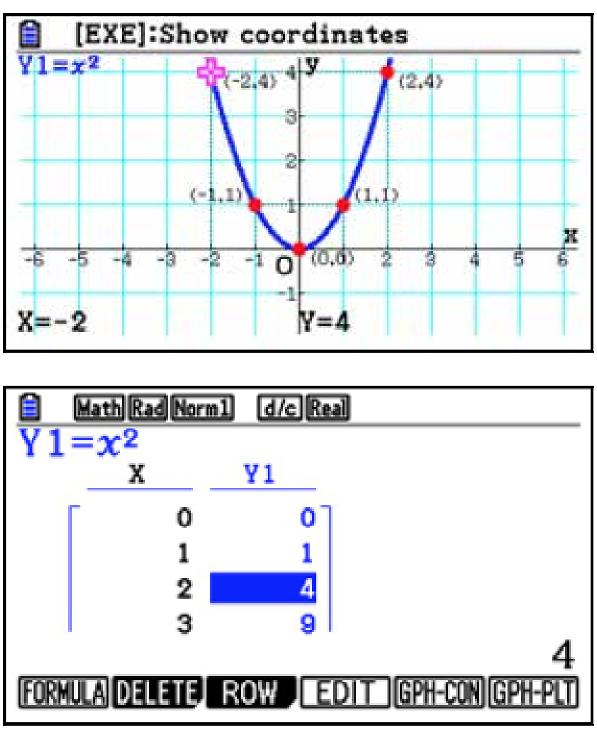


Figure 4

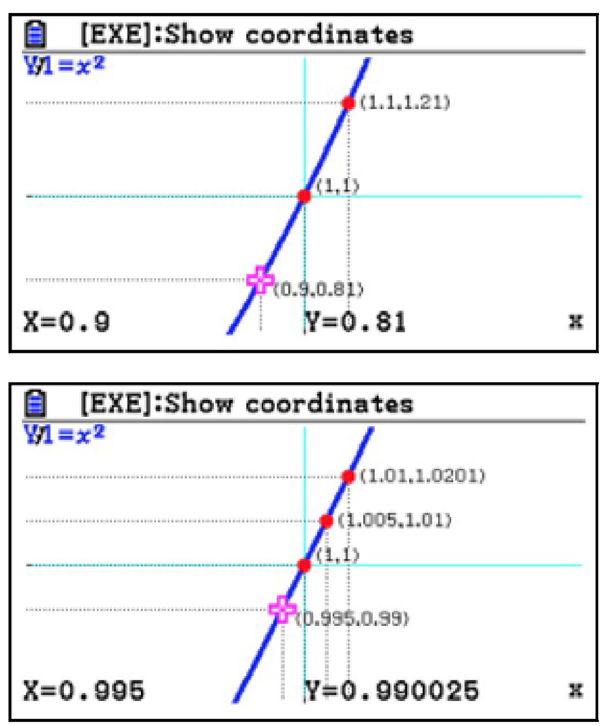


Figure 5

point (1,1) for the graph of $f(x) = x^2$. In each case, the graph that originally looked to be a curve looks (approximately) like a line:

In this case, the second graph has been zoomed in much more than the first graph, and the linearity is more evident. Of course, this process can be continued.

A table of values similarly can be zoomed by choosing a smaller step in the table, as shown in figure 6.

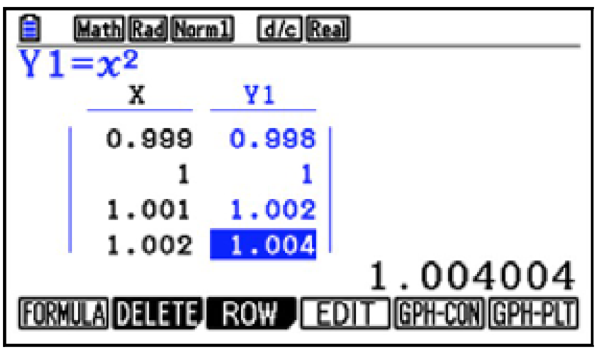


Figure 6

Such exploratory work serves two important purposes. Most importantly, it illustrates that, if examined on a small enough interval, even graphs that are clearly curved and not linear seem to be (almost) linear. Secondly, the idea of the slope of a graph can be seen to be still useful to describe change in a non-linear function, just as it was for linear functions. In this case, for example, it seems that, around the point where $x = 1$, the slope of the function is very close to 2. The table shows this clearly, as an increase of 0.001 in x seems to lead to an increase of twice as much, or about 0.002. In other words, the graph of $f(x) = x^2$ has an approximate slope of 2 when x is close to 1. Further zooming will increase the appearance of linearity and allow the slope of the function to be seen more precisely.

These explorations can of course be undertaken at different points (such as near $x = 2$) or with different functions (such as $f(x) = x^2 - x$.)

A derivative function

The idea of determining the slope of a function at different points as a means of describing its change is a powerful – and fundamental – one, so it is not surprising that the calculator has a command to do this automatically. A numerical derivative function, represented by the conventional symbol dy/dx allows students to determine the slope at any point of a graph when tracing. Once this is turned on by the calculator user, and a function explored either graphically or numerically, a value that represents the gradient of the function at each point graphed or tabulated is shown. There are examples of this shown for $f(x) = x^2$ in Figure 7 for both representations:

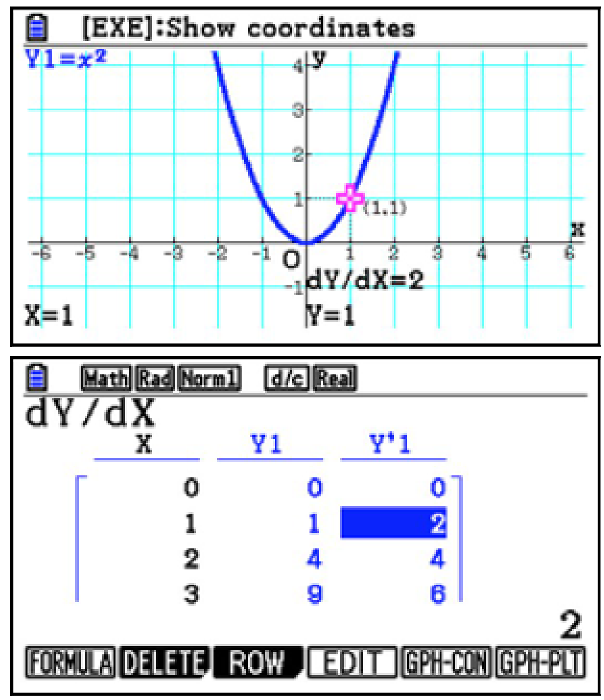


Figure 7

This facility offers strong opportunities for a student to explore what is happening. For example, if the graph is traced and the values of the numerical derivative observed, these are easily seem to be negative (but getting less so) as x increases from -2 to 0 and positive (and getting more so) as x increases from 0 to 2. The value of dy/dx at $x = 0$ is seen to be zero, as for a line of zero slope.

Similarly, the table shows such changes in slope, but also provides an opportunity for students to see that there is a strong relationship apparent between the value of the variable x and the value of the numerical derivative at that point: the derivative is twice the x value, which seems to be

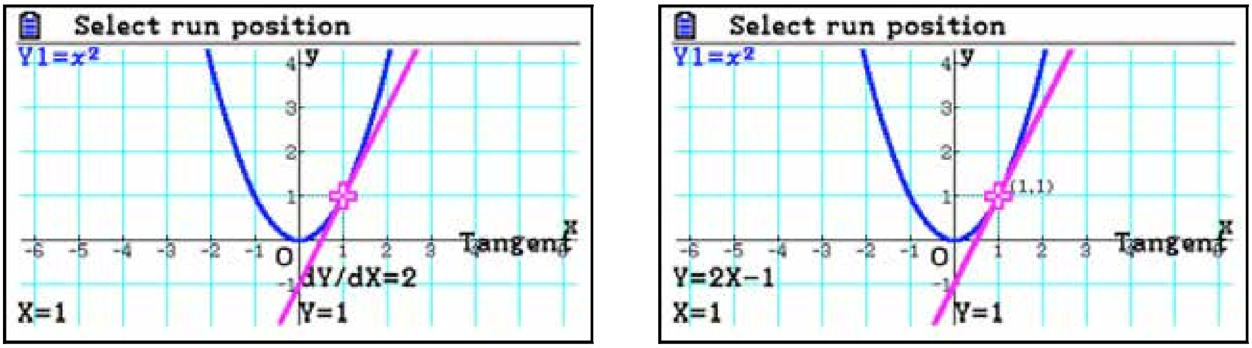


Figure 8

naturally represented as *dy*/*dx* = 2*x* for every value of *x*. This is the crucial idea of a derivative *function*, which describes the derivative at every point in a single relationship. The derivative of a linear function is easily described, since it is the same at every point: the example earlier has *dy*/*dx* = 1. The relationship for the derivative of a nonlinear function is more complicated, in the sense that it also involves the variable, as the gradient is different in different places.

This approach does not use formal limit proofs and abstract derivations, but is focused on providing a meaning for the idea of both the derivative at a point and a derivative function, which do not require excessive symbolism to make the ideas clear. (It is worth recalling that Newton and Leibniz, co-inventors of the calculus, and Archimedes before them, also did not use the formal mathematical ideas of limits to create meanings for basic calculus ideas.)

Tangents and derivatives

A common approach to introductory calculus involves the idea of a secant, becoming closer and closer to a tangent at a point on the curve. While this can also be revealing, it runs a significant risk of confusing the idea of the gradient of a function at a point with the *separate* idea of the gradient of a tangent to a curve at a point. So, it is not the author's preferred way of introducing the idea of a derivative; it seems more conceptually sound to describe the gradient of the function rather than the gradient of a tangent to the function. Nonetheless, a graphics calculator such as the CASIO fx-CG 20 allows for students and teachers interested in doing so to explore this idea by constructing tangents and examining their derivatives, as shown in Figure 8.

The first screen shows the value of the derivative at $x = 1$, while the second screen shows that this tangent at (1,1) can be represented by the line $y = 2x - 1$.

Students can explore the tangents at different points by tracing to different points, and even create an envelope of tangents by stopping at various points on the graph, as shown in Figure 9 for the function $f(x) = 3 - x$^2.

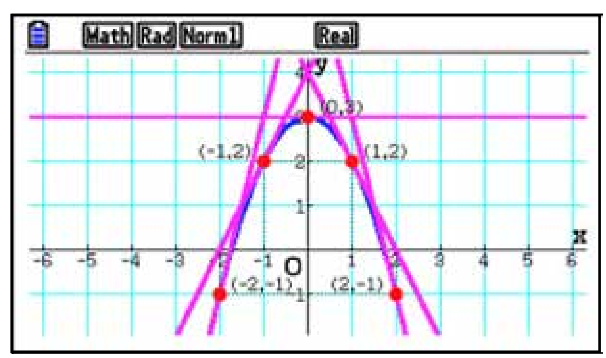


Figure 9

Students who have become familiar with the use of slopes to define functions as decreasing, constant or increasing are likely to find such explorations conceptually helpful.

Graphing derivative functions

An especially powerful exploration of the idea of a derivative function is achieved by graphing a function and its derivative function together on a single screen. A graphics calculator allows this to be undertaken by defining one function in terms of another, as shown in Figure 10 for the simple case of $f(x) = x^2$. The second function, Y2, is defined as the (numerical) derivative of the first function.

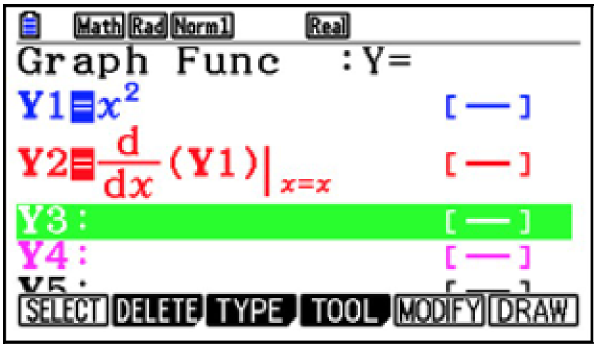


Figure 10

The resulting graph (Figure 11) allows students to see that the derivative function (graphed in red) can be used to describe how the original function (graphed in blue) is changing at various points. In this case, the derivative when $x < 0$ is negative and when $x > 0$ is positive, while there is a zero derivative at $x = 0$ where the red line crosses the x-axis.

The derivative is seen to be increasing steadily as x increases, a situation which can be explored more directly with a second derivative function as well, as shown in green in Figure 12. In this case, the second derivative is a constant (2).

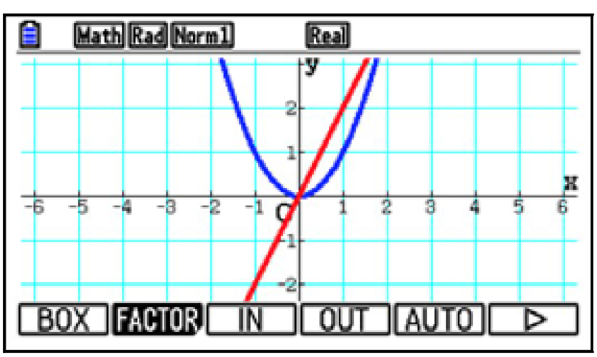


Figure 11

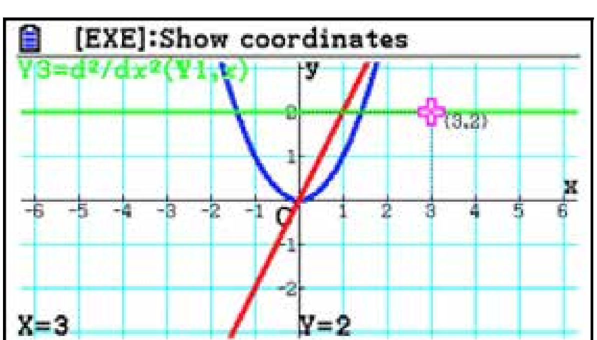


Figure 12

Examining derivatives of other functions is easily undertaken by changing (only) the Y1 function. If students explore quadratic functions in this way, they will see that the derivative functions are always linear, and describe the characteristics of the graph. They will also observe that many functions have the same derivative function – an important understanding needed later when differential equations are studied, helping students to realize that knowing a derivative function is not by itself sufficient to identify the associated function.

Of course, this idea allows a student to explore other kinds of functions than quadratic functions. The graph in Figure 13 of a cubic function $f(x) = x^3 - 3x + 1$ and its derivative shows that the derivative is quadratic and the turning points of the cubic occur where the derivative function crosses the x-axis at $x = -1$ and $x = 0$, while there is a point of inflection at the lowest point of the red derivative function ($x = 0$), where the derivative changes from a decreasing to an increasing function.

Explorations like these help to build meaning for the idea of the derivative. Indeed, many interesting class discussions can be undertaken by graphing only

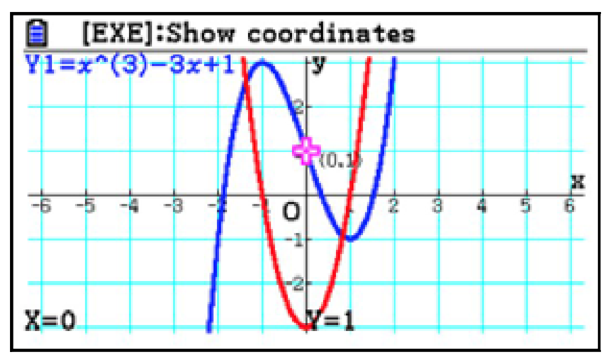


Figure 13

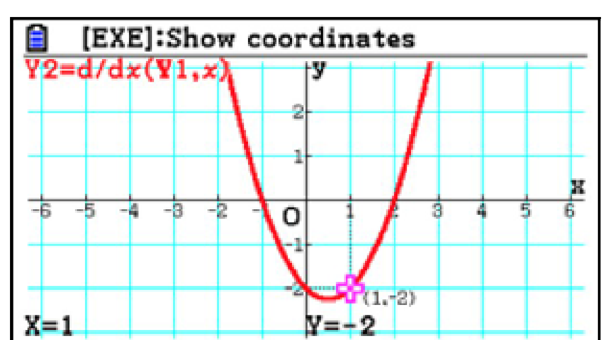


Figure 14

the derivative function and not the actual function, and asking students to describe the properties of the function that has a derivative of that shape. The example in Figure 14 is a case in point:

Students can use the graph of the derivative, which is describing how the function itself is changing, to describe the graph of the function, its two turning points, its point of inflection and its end behaviour for very large or very small values of x.

Derivatives from first principles

Differential calculus is often introduced by defining the derivative from first principles as the limit of the gradient of a secant as the secant gets closer and closer to being a tangent. The idea of 'closer and closer' is a difficult one for students to understand, but good approximations are available on a graphics calculator. We will use a more sophisticated example than previously to illustrate a possible form of exploration: the derivative of the sine function $f(x) = \sin x$, with radian measure being used. The theoretical formulation is usually represented as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In this case, we begin with defining a small value of h, say $h = 0.01$. Then the graphics calculator can be used to define the function as Y1 and the quotient as Y2 (Figure 15).

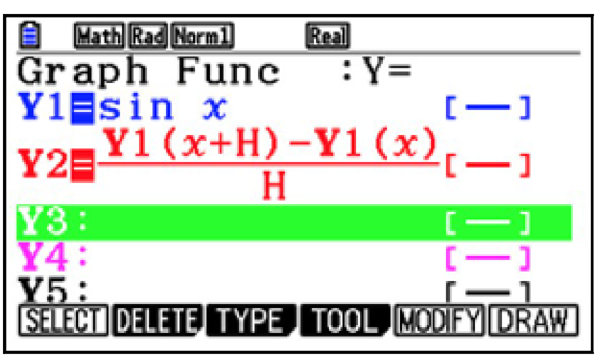


Figure 15

The resulting graph in Figure 16 (on a suitable scale with horizontal grid lines at every $\pi/2$) shows the sine function in blue and the approximate derivative in red. Students studying this situation will be expected to recognize that the red graph looks like that of the cosine function $f(x) = \cos x$, a quite unexpected result (for students, not their teachers).

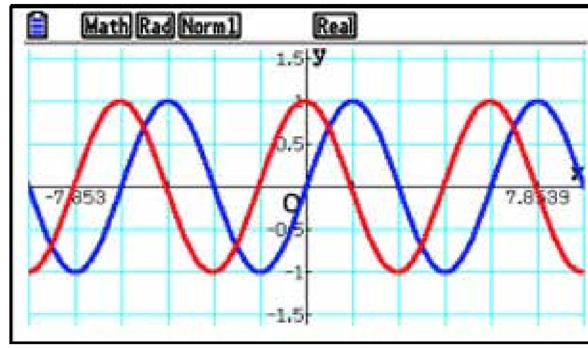


Figure 16

A good way to see whether this prediction is correct is to add a third graph (of the cosine function), as shown in the next screen (Figure 17). The new graph seems very close to the red graph, but they are not quite overlapping, suggesting that they are not quite the same:

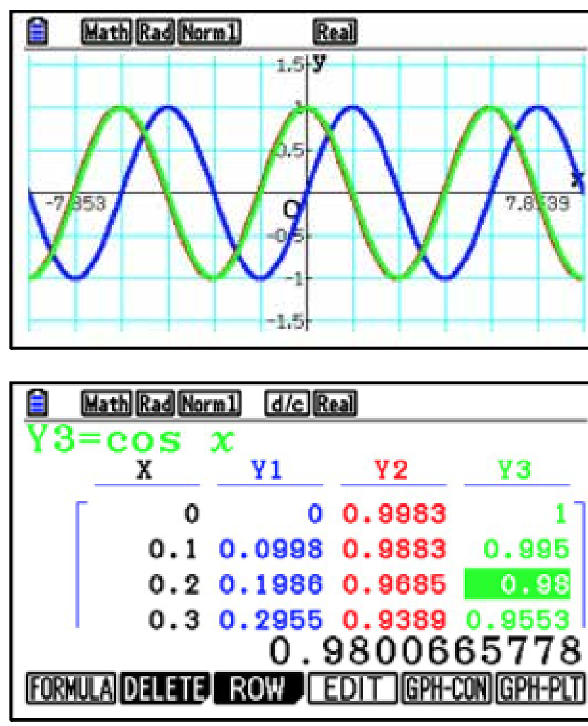


Figure 17

A table of values is also helpful here. Figure 17 shows that the second and third functions are similar, but not identical, which is a timely warning not to be deceived by the appearance of graphs.

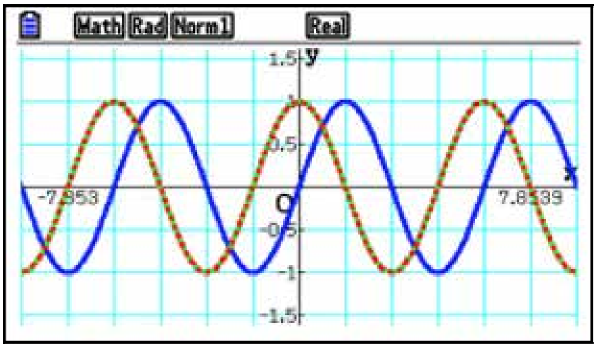


Figure 18

If the value of h is reduced, the idea of 'closer and closer' (and the formal idea of convergence and of a limit) can be further explored. For example, Figure 18 shows h = 0.001, for which the graphs of Y2 and Y3 are indistinguishable to the eye (with the third graph being drawn in a dotted style to show the overlap):

The tabled values are still slightly different, as seen in Figure 19 however, from around the third

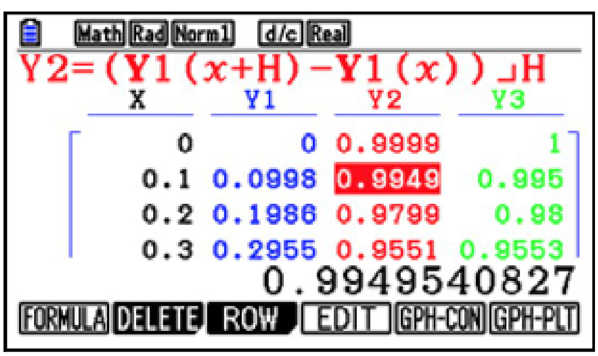


Figure 19

decimal place. Allowing h to be still closer to 0, but still positive will even give a table of values that gives an impression that the second and third functions are identical, as shown below, but a closer inspection in Figure 20 suggests that they are still slightly different.

Students can readily explore this situation and gain a good sense of the meaning of important concepts such as limit and convergence, as well as the derivative of the sine function. Without the use of technology, it is very hard for students to appreciate and understand the subtle ideas involved.

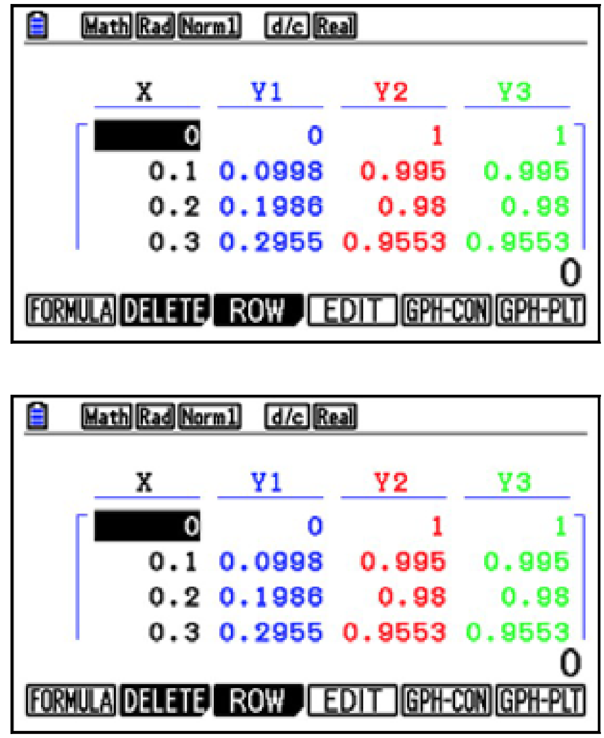


Figure 20

Of course, at some point, most mathematics curricula introduce formal mathematical proofs of results of these kinds, and handle the processes symbolically. However, the graphics calculator offers strong opportunities for this more formal work to be built upon practical experiences and intuitions; indeed, the work with the graphics calculator might even provide a motivation for a more formal and more theoretical development to be provided.

Conclusion

As claimed earlier, the graphics calculator is regularly misunderstood as a calculation device instead of its proper role as an educational device being recognized. Similarly, the focus on the graphing capabilities may often mask the importance of numerical approaches to understanding mathematical ideas, in addition to visual ones. The examples offered here show some of the ways in which the inbuilt capabilities of a modern graphics calculator can be used by students and exploited by teachers to help build a solid foundation for important mathematical ideas of differential calculus.

References

1. Kissane, B. & Kemp, M. (2008) Some calculus affordances of a graphics calculator. *Australian Senior Mathematics Journal*, 22(2), 15-27. (Available for download at http://researchrepository.murdoch.edu.au/6256/)

2. Kissane, B. & Kemp, M. (2014) A model for the educational role of calculators. In W.-C. Yang, M. Majewski, T. de Alwis & W. Wahyudi (eds.) *Proceedings of the 19th Asian Technology Conference on Mathematics (pp 211-220) Yogyakarta*: ATCM. (Available for download at http://researchrepository.murdoch.edu.au/24816/)

3. Kissane, B. & Kemp, M. (2014) *Learning mathematics with graphics calculators.* Tokyo: CASIO Computer Company. (Available for download at http://researchrepository.murdoch.edu.au/24814/)

BARRY KISSANE is an Emeritus Associate Professor at Murdoch University in Perth, Western Australia. He has worked with teachers and students to make effective use of calculators for school mathematics education in several countries. In a career spanning more than forty years, he has worked as a mathematics teacher and mathematics teacher educator, publishing several books and many papers concerned with the use of calculators. He has held various offices, such as President of the Australian Association of Mathematics Teachers, editor of *The Australian Mathematics Teacher* and Dean of the School of Education at Murdoch University. He may be contacted at b.kissane@murdoch.edu.au

SHORT PROOF OF AN INEQUALITY FOR THE AREA OF A QUADRILATERAL

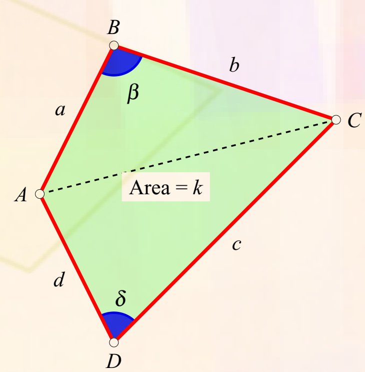
MOSHE STUPEL & DAVID BEN-CHAIM

"SHAANAN" – Academic Religious Teachers College, Haifa.

Here is an elegant and remarkably compact one-line proof for an inequality relating to the area of a quadrilateral:

Theorem. *The area of a quadrilateral is less than or equal to a quarter of the sum of the squares of the sides. Equality holds precisely when the quadrilateral is a square.*

Proof. With reference to the figure shown below, with sides, angles and area as indicated:



$$k = \frac{ab\sin\beta + cd\sin\delta}{2} \le \frac{ab+cd}{2} \le \frac{\frac{a^2+b^2}{2} + \frac{c^2+d^2}{2}}{2} = \frac{a^2+b^2+c^2+d^2}{4}.$$

Equality holds precisely when $\beta = 90^\circ = \delta$ and $a = b$, $c = d$. $\square$

Keywords: *Quadrilateral, area, inequality*

Problems for the Senior School

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem Corner

PROBLEMS FOR SOLUTION

Call a convex quadrilateral *tangential* if a circle can be drawn tangent to all four sides. In this edition of the problem set, all the problems that we have posed have to do with this notion.

Problem V-2-S.1

Let $ABCD$ be a tangential quadrilateral. Prove that $AD + BC = AB + CD$.

Problem V-2-S.2

Let $ABCD$ be a convex quadrilateral with $AD + BC = AB + CD$. Prove that $ABCD$ is tangential.

Problem V-2-S.3

Place four coins of different sizes on a flat table so that each coin is tangent to two other coins. Prove that the quadrilateral formed by joining the centres of the coins is tangential. Prove also that the convex quadrilateral whose vertices are the four points of contact is cyclic. Is the circle passing through the four points of contact tangent to the sides of the tangential quadrilateral formed by joining the four centres of the coins?

Problem V-2-S.4

Let $ABCD$ be a cyclic quadrilateral and let X be the intersection of diagonals AC and BD. Let P_1, P_2, P_3 and P_4 be the feet of the perpendiculars from X to BC, CD, DA and AB respectively. Prove that quadrilateral $P_1P_2P_3P_4$ is tangential.

SOLUTIONS OF PROBLEMS IN ISSUE-V-1 (MARCH 2016)

Solution to problem V-1-S.1

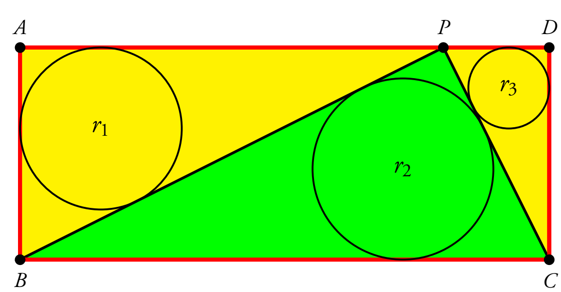
What is the greatest possible perimeter of a right-angled triangle with integer sides, if one of the sides has length 12?

We claim that it cannot be the hypotenuse whose length is 12. For if it were, let the other two sides be a, b; then $a^2 + b^2 = 12^2$. Since the RHS is an even number, a and b must be both odd or both even. The first possibility does not work out as it leads to $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$, implying that $a^2 + b^2 \equiv 2 \pmod 4$; but 12^2 is a multiple of 4. Hence both a and b are even numbers. Let $a = 2a_1$, $b = 2b_1$, where a_1, b_1 are integers. We now get $a_1^2 + b_1^2 = 36$. Repeating the same argument, we see that both a_1 and b_1 are even numbers; say $a_1 = 2a_2$, $b_1 = 2b_2$, where a_2, b_2 are integers. We now get $a_2^2 + b_2^2 = 9$. However, it is easy to check that 9 cannot be expressed as a sum of two perfect squares. Hence it cannot be the hypotenuse whose length is 12.

Let the hypotenuse be x units and the other side y units. Then we have: $x^2 - y^2 = 144$, hence $x + y$ and $x - y$ are factors of 144 and as both are of same parity, both must be even. The perimeter of the triangle is $x + y + 12$. Thus for maximum perimeter we must make $x + y$ as large as possible and hence $x - y$ as small as possible. The least possible value of $x - y$ is 2, so the greatest possible value of $x + y$ is 72; the corresponding perimeter is 84 units. (For the actual dimensions of the triangle: $x + y = 72$, $x - y = 2$, hence $x = 37$, $y = 35$; so the sides of the triangle are 37, 35, 12.)

Solution to problem V-1-S.2

Rectangle ABCD has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\measuredangle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?



Observe that

$$r_1 = \frac{AP + AB - BP}{2}, \quad r_2 = \frac{BP + PC - BC}{2}, \quad r_3 = \frac{DP + CD - PC}{2}.$$

Adding these we obtain

$$r_1 + r_2 + r_3 = \frac{AD + AB + CD - BC}{2} = AB = 8.$$

Solution to problem V-1-S.3

Let a, b, c be such that $a + b + c = 0$; find the value of P where

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}.$$

Since $a+b+c=0$ we have

$$2a^2+bc = a^2+a^2+bc = a^2-a(b+c)+bc = (a-b)(a-c).$$

So:

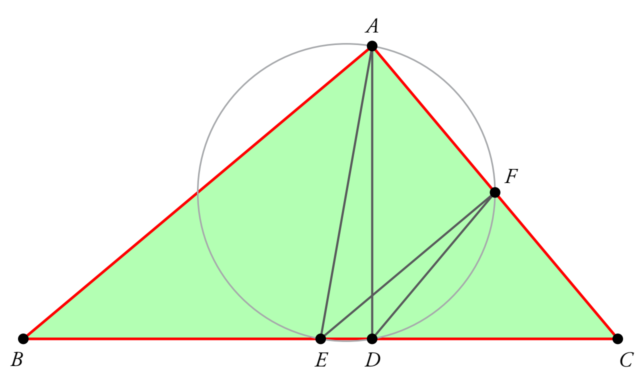
$$\frac{a^2}{2a^2+bc}=\frac{a^2}{(a-b)(a-c)},\quad \frac{b^2}{2b^2+ca}=\frac{b^2}{(b-c)(b-a)},\quad \frac{c^2}{2c^2+ab}=\frac{c^2}{(c-a)(c-b)}.$$

Hence:

$$\begin{aligned} P &= \frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)} \\ &= -\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)} \\ &= \frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = 1. \end{aligned}$$

Solution to problem V-1-S.4

In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\measuredangle BAE = 40^\circ$. If $\measuredangle DAE = \measuredangle DFE$, what is the magnitude of $\measuredangle ADF$ in degrees?



Since $EF \parallel AB$, $\measuredangle BAE = \measuredangle AEF = 40^\circ$. As $\measuredangle DAE = \measuredangle DFE$, points A, D, E and F are concyclic. Hence $\measuredangle ADF = \measuredangle AEF = 40^\circ$.

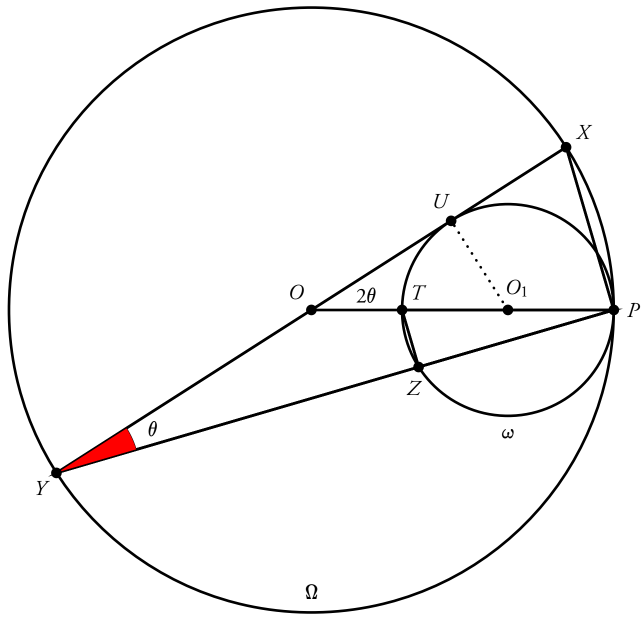
Solution to problem V-1-S.5

Circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume that $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\measuredangle PYX$ in degrees?

Let T be the point of intersection of OP and ω. Let $\measuredangle PYX = \theta$. Then $\measuredangle OPY = \theta$ and $\measuredangle XOP = 2\theta$. As ω is internally tangent to Ω, PT is a diameter of ω.

In $\triangle PZT$ and $\triangle YPX$, we have: $\measuredangle PZT = \measuredangle XPY = 90^\circ$ and $\measuredangle TPZ = \measuredangle PYX = \theta$. Hence $\triangle PZT \sim \triangle YPX$. This implies that

$$\frac{PT}{YX}=\frac{PZ}{YP}=\frac{1}{3}.$$



If r and R are the radii of ω and Ω, respectively, then we have

$$\frac{r}{R} = \frac{1}{3}.$$

If O_1 is the centre of ω, then we see that $OO_1 = R - r = 2r$. Let U be the foot of the perpendicular from O_1 to XY. Then

$$\sin 2\theta = \frac{UO_1}{OO_1} = \frac{r}{2r} = \frac{1}{2}.$$

Hence $\theta = 15°$.

Why these two jokes about a woman driver are actually reasons for her to be a good mathematician and a good teacher.

'Women drivers are like stars in the sky, you can see them but they can't see you.
http://creativefan.com/important/cf/2013/07/funny-quotes-about-women/see-you.jpg

That's because they're focusing on the driving, you dummy!

'If your wife, wants to learn to drive, don't stand in her way.'
http://quotesgram.com/funny-quotes-about-women-drivers/
Too right, nothing like a women possessed with the desire to learn!

In the words of G.K. Chesterton,
'A woman uses her intelligence to find reasons to support her intuition'
Could we make a more powerful argument??

Problems for the Middle School

Problem Editor: ATHMARAMAN R

Problem Corner

PROBLEMS FOR SOLUTION

In this edition of the problem section, all the problems are based on the fundamental concept of **divisibility** among numbers. The problems require for their solution only a basic understanding of this notion, and a basic knowledge of the rules of divisibility.

Problem V-2-M.1
What is the largest prime divisor of every three-digit number with three identical non-zero digits?

Problem V-2-M.2
Given any four distinct integers a, b, c, d, show that the product

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

Problem V-2-M.3
Let n be a natural number, and let $d(n)$ denote the sum of the digits of n. Show that if $d(n) = d(3n)$, then 9 divides n. Show that the converse statement is false.

Problem V-2-M.4
Let n be an arbitrary positive integer. Show that:

(a) $n^5 - n$ is divisible by 5;

(b) $n^7 - n$ is divisible by 7;

(c) $n^9 - n$ is not necessarily divisible by 9.

Problem V-2-M.5
Find all positive integers $n > 3$ such that $n^3 - 3$ is divisible by $n - 3$.

Problem V-2-M.6
Show that there cannot exist three positive integers a, b, c, all greater than 1, such that the following three conditions are simultaneously satisfied:

(a) $a^2 - 1$ is divisible by b and c;

(b) $b^2 - 1$ is divisible by c and a;

(c) $c^2 - 1$ is divisible by a and b.

Problem V-2-M.7
Using the nine nonzero digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, form a nine-digit number in which each digit occurs exactly once, such that when the digits are removed one at a time starting from the units end (i.e., from the "right side"), the resulting numbers are divisible respectively by $8, 7, 6, 5, 4, 3, 2, 1$. (So if the nine-digit number is $\overline{\mathsf{ABCDEFGHI}}$, then we must have:

$$8 \mid \overline{\mathsf{ABCDEFGH}}; \qquad 7 \mid \overline{\mathsf{ABCDEFG}}; \qquad 6 \mid \overline{\mathsf{ABCDEF}}; \qquad 5 \mid \overline{\mathsf{ABCDE}};$$

and so on. Here the notation $a \mid b$ means: "a divides b".) Is the answer unique?

SOLUTIONS OF PROBLEMS IN ISSUE-V-1 (MARCH 2016)

Solution to problem V-1-M.1
Find two non-zero numbers such that their sum, their product and the difference of their squares are all equal.

Let a, b be the numbers ($a \neq 0$, $b \neq 0$). Then we have: $a + b = ab = a^2 - b^2 = (a+b)(a-b)$. Since $ab \neq 0$, $a + b \neq 0$, hence $a - b = 1$, therefore $b = a - 1$. This leads to the equation $a(a-1) = 2a - 1$, or $a^2 - 3a + 1 = 0$. Solving this equation, we get two solution pairs:

$$a = \frac{3+\sqrt{5}}{2}, \quad b = \frac{1+\sqrt{5}}{2}; \qquad a = \frac{3-\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2}.$$

Solution to problem V-1-M.2
Prove that a six-digit number formed by placing two consecutive three-digit positive integers one after the other is not divisible by any of the following numbers: 7, 11, 13. (Adapted from the Mid-Michigan Olympiad in 2014 grades 7-9)

The key observation to be made here is that if $\overline{\mathsf{ABC}}$ is a three-digit number, then the six-digit number $\overline{\mathsf{ABCABC}}$ is necessarily a multiple of 1001. This should be clear since

$$\overline{\mathsf{ABCABC}} = \overline{\mathsf{ABC}000} + \overline{\mathsf{ABC}} = \overline{\mathsf{ABC}} \times 1000 + \overline{\mathsf{ABC}} = \overline{\mathsf{ABC}} \times 1001.$$

Since $1001 = 7 \times 11 \times 13$, the above equality implies that $\overline{\mathsf{ABCABC}}$ is divisible by each of the numbers $7, 11, 13$. This further implies that if $\overline{\mathsf{DEF}} = \overline{\mathsf{ABC}} + 1$, then the six-digit number $\overline{\mathsf{ABCDEF}}$ will leave remainder 1 when divided by any of the numbers $7, 11, 13$. This means in particular that $\overline{\mathsf{ABCDEF}}$ will not be divisible by any of the numbers $7, 11, 13$.

Solution to problem V-1-M.3

If n is a whole number, show that the last digit in $3^{2n+1} + 2^{2n+1}$ *is* 5.

Note that $3^{2n+1} + 2^{2n+1}$ is an odd number. Also note the general fact that if k is odd, then $a^k + b^k$ is divisible by $a + b$. Hence $3^{2n+1} + 2^{2n+1}$ is divisible by 5. It follows that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5.

Solution to problem V-1-M.4

(a) *Show that the sum of any m consecutive squares cannot be a square for* $m \in \{3, 4, 5, 6\}$.

Let S_m denote the sum of m consecutive squares:

$$S_m = n^2 + (n+1)^2 + \cdots + (n+m-1)^2.$$

We now consider each of the cases in turn.

- $S_3 = n^2 + (n+1)^2 + (n+2)^2 = 3n^2 + 6n + 5$, which is of the form $3k + 2$. However, no square is of this form. Hence S_3 cannot be a perfect square.
- In the same way, we get $S_4 = 4n^2 + 12n + 14$, which is of the form $4k + 2$. However, no square is of this form. Hence S_4 cannot be a perfect square.
- In the same way, we get $S_5 = 5n^2 + 20n + 30 = 5\left(n^2 + 4n + 6\right)$, which is a multiple of 5. If S_5 is a square, then it must be a multiple of 25, hence $n^2 + 4n + 6$ must be a multiple of 5. But $n^2 + 4n + 6 = (n+2)^2 + 2$, so for $n^2 + 4n + 6$ to be a multiple of 5, $(n+2)^2$ must be of the form $5k + 3$. However, no square is of this form. Hence S_5 cannot be a perfect square.
- In the same way, we get $S_6 = 6n^2 + 30n + 55 = 6n(n+5) + 55$, which is of the form $12k + 7$. However, no square is of this form. Hence S_6 cannot be a perfect square.

Observe that in each of the above cases, we followed a similar strategy. But in each case, the precise path taken was slightly different.

(b) *Can the sum of* 11 *consecutive square numbers be a square number?*

In the same way, we get $S_{11} = 11n^2 + 110n + 385 = 11\left(n^2 + 10n + 35\right)$. For this to be a square, $n^2 + 10n + 35$ must have the form $11m^2$. Since $n^2 + 10n + 35 = (n+5)^2 + 10$, this demand leads to the equation $x^2 - 11m^2 = -10$. This equation does indeed have solutions! For example, $x = 1$, $m = 1$ is a solution; but it yields $n = -4$, a negative value. The next solution after this one is $x = 23$, $m = 7$, which yields $n = 18$, $m = 7$. So: $18^2 + 19^2 + 20^2 + 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2 = 5929 = 77^2$.

Solution to problem V-1-M.5

(a) *Which positive integers have exactly two positive divisors? Which have three positive divisors?*

The positive integers which have exactly two divisors are clearly the prime numbers. (The prime number p has two divisors 1 and p itself.)

Now we consider the case when the positive integer n has exactly three divisors. From the above, we know that n is not a prime number. Suppose that n can be written in the form $n = ab$ where $1 < a < b < n$. In this case, n has at least the following four divisors: $1, a, b, n$. So if n is to have exactly three divisors, then it must be the case that n has a divisor a with $1 < a < n$; but it should also be the case that n cannot be written in the form $n = ab$ where $1 < a < b < n$.

The only option allowed by these restrictions is that n must be the square of a prime number. Indeed, if $n = p^2$ where p is a prime number, then n has just the following three divisors: $1, p, p^2$.

(b) *Among integers a, b, c, each exceeding 20, one has an odd number of divisors, and each of the other two has three divisors. If $a + b = c$, find the least value of c.*

The integers having an odd number of divisors are the perfect squares. Drawing from the result in part (a), we may suppose that the three numbers are n^2, p^2, q^2 where p, q are prime numbers. We are told that the sum of two of these squares equals the third square. The equation $p^2 + q^2 = n^2$ does not yield any solution; for if p, q are both odd, then we get $n^2 \equiv 2 \pmod 4$, which is not possible; and if $p = 2$, then we get $n^2 - q^2 = 4$; but this yields no solutions. So we consider instead the equation $p^2 + n^2 = q^2$. This turns out to have numerous solutions:

$$(p, n, q) = (3, 4, 5),\ (5, 12, 13),\ (11, 60, 61),\ (19, 180, 181),\ \dots,$$

i.e.,

$$(a, b, c) = (9, 16, 25),\ (25, 144, 169),\ (121, 3600, 3721),\ \dots$$

The first of these possibilities is ruled out, as it is given that a, b, c all exceed 20; so we must choose the second triple, which means that the least value of c is 169.

Solution to problem V-1-M.6

A group of 43 devotees consisting of ladies, men and children, went to a temple. After a ritual, the priest distributed 229 flowers to the visitors. Each lady got 10 flowers, each man got 5 flowers and each child got 2 flowers. If the number of men exceeded 10 but not 15, find the number of women, men and children in the group.

Let there be a ladies, b men and c children; then $a + b + c = 43$ and $10a + 5b + 2c = 229$; also $10 < b \leq 15$. The equations may be rewritten as: $a + b = 43 - c$, $10a + 5b = 229 - 2c$. Treating these as a pair of simultaneous equations in a, b and solving them in the usual manner, we get:

$$a = \frac{3c + 14}{5}, \quad b = \frac{201 - 8c}{5}.$$

Since $10 < b \leq 15$, we get

$$10 < \frac{201 - 8c}{5} \leq 15, \quad \therefore \quad \frac{63}{4} \leq c < \frac{151}{8},$$

i.e., $16 \leq c \leq 18$, since c is an integer. Also, since a is an integer, we have:

$$3c \equiv -14 \pmod 5 \equiv 6 \pmod 5, \qquad \therefore \quad c \equiv 2 \pmod 5.$$

The two conditions yield: $c = 17$, and therefore $a = 13$, $b = 13$. So there are 13 ladies, 13 men and 17 children.

Solution to problem V-1-M.7

There are two towns, A and B. Person P travels from A to B, covering half the distance at rate a, and the remaining half at rate b. Person Q travels from A to B (starting at the same time as P), traveling for half the time at rate a, and for half the time at rate b. Who reaches B earlier?

Let d be the distance between the two towns. The time T_P taken by P is given by

$$T_P = \frac{d/2}{a} + \frac{d/2}{b} = \frac{d}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{d(a+b)}{2ab} = \frac{d}{\text{harmonic mean of } a, b}.$$

Let T_Q be the time taken by Q; then we have:

$$d = \frac{aT_Q}{2} + \frac{bT_Q}{2} = \frac{T_Q}{2}(a + b), \therefore \quad T_Q = \frac{2d}{a + b} = \frac{d}{\text{arithmetic mean of } a, b}.$$

For arbitrary positive numbers a, b, it is true that the arithmetic mean of a, b is greater than or equal to the harmonic mean of a, b; hence $T_Q \leq T_P$.

A Problem Study

C⊗MαC

We present a close study of a problem posed in the March 2016 issue. In the process, we demonstrate that there may be more to a problem than just the solution of that problem.

Sums of consecutive squares

In the 'Middle Problems' section (March 2016 issue), Problem V-1-M.4 (b) (solved elsewhere in this very issue) asked:

> *Can the sum of* 11 *consecutive perfect squares be a perfect square?*

We shall show that this phenomenon occurs infinitely often. But first we ask a question that will almost certainly have occurred to you. Why 11? Why not 2? Why not 3? Why not 4? …So let us attempt to address this series of questions first.

Can the sum of two consecutive squares be a perfect square?

The answer is 'Yes'. Let the two consecutive squares be n^2 and $(n+1)^2$. Their sum is

$$n^2 + (n+1)^2 = 2n^2 + 2n + 1.$$

Suppose that this is a perfect square, x^2. Then we have:

$$x^2 = 2n^2 + 2n + 1,$$

$$\therefore \ 2x^2 = 4n^2 + 4n + 2 = (2n+1)^2 + 1.$$

Write $y = 2n + 1$; then we have $2x^2 = y^2 + 1$, i.e.,

$$y^2 - 2x^2 = -1, \tag{1}$$

and we have reached an equation which by now should be very familiar to you: the Brahmagupta-Fermat-Pell equation. Note that we want solutions in which x is odd. We will solve equation (1) in an ad hoc way for now. Computer-based trials yield the following x, y values which solve (1). The third row gives the corresponding values of n.
(Recall that $y = 2n + 1$.)

x	1	5	29	169	985	...
y	1	7	41	239	1393	...
n	0	3	20	119	696	...

These figures give rise to the following instances where the sum of the squares of two consecutive positive integers is itself a perfect square:

$$\begin{cases} 3^2 + 4^2 = 5^2, \\ 20^2 + 21^2 = 29^2, \\ 119^2 + 120^2 = 169^2, \\ 696^2 + 697^2 = 985^2, \quad \ldots. \end{cases}$$

As always in such settings, there are numerous patterns to be spotted in the sequences. Thus we have, for the sequence of y-values:

$$41 = 6 \times 7 - 1,$$
$$239 = 6 \times 41 - 7,$$
$$1393 = 6 \times 239 - 41,$$

and so on. That is, if the y-values are $y_1 = 1, y_2 = 7, y_3 = 41, \ldots$, then

$$y_{n+2} = 6y_{n+1} - y_n, \tag{2}$$

and exactly the same recurrence holds for the sequence of x-values:

$$x_{n+2} = 6x_{n+1} - x_n, \tag{3}$$

where $x_1 = 1, x_2 = 5, x_3 = 29, \ldots$.

Another striking recurrence relation which is also computationally very convenient is the following:

$$\begin{cases} x_{n+1} = 3x_n + 2y_n, \\ y_{n+1} = 4x_n + 3y_n. \end{cases} \tag{4}$$

All these relations can be proved using induction. It follows from them that there are infinitely many instances where the sum of the squares of two consecutive numbers is itself a perfect square.

Can the sum of three consecutive squares be a perfect square? Let the 3 consecutive squares be $n^2, (n+1)^2, (n+2)^2$. Their sum is

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 &= 3n^2 + 6n + 5 \\ &= 3(n+1)^2 + 2. \end{aligned} \tag{5}$$

We see that the sum is of the form $3k + 2$, i.e., it leaves remainder 2 under division by 3. However, no perfect square is of this form. (Under division by 3, all squares leave remainder 0 or 1.) Hence the sum of three consecutive squares can never be a perfect square.

Can the sum of four consecutive squares be a perfect square? Let the 4 consecutive squares be $n^2, (n+1)^2, (n+2)^2, (n+3)^2$. Their sum is

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 &= 4n^2 \\ + 12n + 14 = 4\left(n^2 + 3n + 3\right) + 2. \end{aligned} \tag{6}$$

We see that the sum is of the form $4k + 2$, i.e., it leaves remainder 2 under division by 4. However, no perfect square is of this form. (Under division by 4, all squares leave remainder 0 or 1.) Hence the sum of four consecutive squares can never be a perfect square.

Can the sum of five consecutive squares be a perfect square? Let the 5 consecutive squares be $n^2, (n+1)^2, \ldots, (n+4)^2$. Their sum is

$$\begin{aligned} \sum_{k=0}^{4} (n+k)^2 &= 5n^2 + 20n + 30 \\ &= 5(n+2)^2 + 10. \end{aligned} \tag{7}$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 5, hence a itself is a multiple of 5; say $a = 5b$ where b is an integer. This leads to:

$$\begin{aligned} 25b^2 &= 5(n+2)^2 + 10, \\ \therefore\ 5b^2 &= (n+2)^2 + 2. \end{aligned} \tag{8}$$

This equality implies that $(n+2)^2$ leaves remainder 3 under division by 5. However, no perfect square is of this form. (Under division by 5, all squares leave remainder 0, 1 or 4.) Hence the sum of five consecutive squares can never be a perfect square.

Can the sum of six consecutive squares be a perfect square? Let the 6 consecutive squares be $n^2, (n+1)^2, \ldots, (n+5)^2$. Their sum is

$$\sum_{k=0}^{5}(n+k)^2 = 6n^2 + 30n + 55. \tag{9}$$

Now observe that

$$6n^2 + 30n = 6n(n+5).$$

The quantity $n(n+5)$ is even, as it is the product of an even number, and an odd number. Hence $6n(n+5)$ is a multiple of 4. Since 55 leaves remainder 3 under division by 4, the quantity $6n^2 + 30n + 55$ leaves remainder 3 under division by 4. However, no perfect square is of this form. (Under division by 4, all squares leave remainder 0 or 1.) Hence the sum of six consecutive squares can never be a perfect square.

Can the sum of seven consecutive squares be a perfect square? Let the 7 consecutive squares be $n^2, (n+1)^2, \ldots, (n+6)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{6}(n+k)^2 &= 7n^2 + 42n + 91 \\ &= 7\left(n^2 + 6n + 13\right). \end{aligned} \tag{10}$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 7, hence a itself is a multiple of 7; say $a = 7b$ where b is an integer. This leads to:

$$49b^2 = 7\left(n^2 + 6n + 13\right),$$

$$\therefore\ 7b^2 = (n+3)^2 + 4. \tag{11}$$

This equality implies that $(n+3)^2$ leaves remainder 3 under division by 7. However, no perfect square is of this form. (Under division by 7, all squares leave remainder 0, 1, 2 or 4.) Hence the sum of seven consecutive squares can never be a perfect square.

Can the sum of eight consecutive squares be a perfect square? Let the 8 consecutive squares be $n^2, (n+1)^2, \ldots, (n+7)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{7}(n+k)^2 &= 8n^2 + 56n + 140 \\ &= 8n(n+7) + 140. \end{aligned} \tag{12}$$

The quantity $n(n+7)$ is even, as it is the product of an even number, and an odd number. Hence $8n(n+7)$ is a multiple of 16. Since 140 leaves remainder 12 under division by 16, the quantity $8n^2 + 56n + 140$ leaves remainder 12 under division by 16. However, no perfect square is of this form. (Under division by 16, all squares leave remainder 0, 1, 4 or 9.) Hence the sum of eight consecutive squares can never be a perfect square.

Can the sum of nine consecutive squares be a perfect square? Let the 9 consecutive squares be $n^2, (n+1)^2, \ldots, (n+8)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{8}(n+k)^2 &= 9n^2 + 72n + 204 \\ &= 9\left(n^2 + 8n + 22\right) + 6. \end{aligned} \tag{13}$$

Hence the quantity $9n^2 + 72n + 204$ leaves remainder 6 under division by 9. However, no perfect square is of this form. (Under division by 9, all squares leave remainder 0, 1, 4 or 7.) Hence the sum of nine consecutive squares can never be a perfect square.

Can the sum of ten consecutive squares be a perfect square? Let the 10 consecutive squares be $n^2, (n+1)^2, \ldots, (n+9)^2$. Their sum is

$$\sum_{k=0}^{9}(n+k)^2 = 10n^2 + 90n + 285. \tag{14}$$

Suppose that this quantity is a perfect square, say a^2. Then a^2 is a multiple of 5, hence a itself is a multiple of 5; say $a = 5b$ where b is an integer. This leads to:

$$25b^2 = 10n^2 + 90n + 285,$$

$$\therefore\ 10b^2 = 4n^2 + 36n + 114, \tag{15}$$

and so:

$$10b^2 = (2n+9)^2 + 33. \tag{16}$$

Since the quantity on the left side of this equality is a multiple of 10, it must be that $(2n+9)^2$ leaves remainder 7 under division by 10. However, no perfect square is of this form. (Under division by 10, all squares leave remainder 0, 1, 4, 5, 6 or 9.) Hence the sum of ten consecutive squares can never be a perfect square.

Remark. We have found something quite remarkable. A sum of 2 consecutive squares can be a perfect square, but not a sum of 3 or 4 or 5 or 6 or 7 or 8 or 9 or 10 consecutive squares! This phenomenon now prepares us to believe that a sum of 11 consecutive squares cannot be a perfect square either. But wait and see ….

Can the sum of eleven consecutive squares be a perfect square? Let the 11 consecutive squares be n^2, $(n+1)^2$, $(n+2)^2$, …, $(n+10)^2$. Their sum is

$$\begin{aligned}\sum_{k=0}^{10}(n+k)^2 &= 11n^2+110n+385\\ &= 11\left(n^2+10n+35\right). \qquad (17)\end{aligned}$$

Can this quantity be a perfect square? On this occasion, we shall not attempt a proof of impossibility—because a computer assisted search does uncover some solutions! We will not reveal this solution for now, but wait for the analysis to uncover it on its own.

For $\sum_{k=0}^{10}(n+k)^2$ to be a square, $n^2+10n+35$ must have the form $11m^2$ for some integer m. And since $n^2+10n+35=(n+5)^2+10$, this demand leads to the following equation (after the substitution $x=n+5$):

$$x^2-11m^2=-10. \qquad (18)$$

We shall solve this equation (and show the existence of infinitely many solutions in positive integers) in three stages:

Step 1: Find a single solution (x_0, m_0) to the equation, using trial and error. (Use computer experimentation if necessary.)

Step 2: Generate infinitely many solutions (in positive integers) to the 'auxiliary equation' $x^2-11m^2=1$. (There is a standard method for doing this, as described below.)

Step 3: Use the solution (x_0, m_0) and the infinite family of solutions to the auxiliary equation $x^2-11m^2=1$ to generate infinitely many solutions in positive integers to the original equation $x^2-11m^2=-10$. (We need to 'compose' the two solutions with one another; the method is described below.)

Step 1 is trivial, because $x=1, m=1$ solves the equation; so we put $x_0=1$, $m_0=1$. (We can count ourselves lucky that this solution can be obtained by inspection, without any need to use computing machinery.)

For Step 2 we use a standard algorithm which works for all such equations. For now, we shall not discuss the theory behind this algorithm, but only remark that it works. We start by finding the simple continued fraction (SCF) for $\sqrt{11}$. Observe that:

$$\begin{aligned}\sqrt{11} &= 3+\left(\sqrt{11}-3\right)=3+\frac{2}{\sqrt{11}+3}\\ &= 3+\frac{1}{(\sqrt{11}+3)/2}\\ &= 3+\frac{1}{3+(\sqrt{11}-3)/2}\\ &= 3+\frac{1}{3+1/(\sqrt{11}+3)}\\ &= 3+\cfrac{1}{3+\cfrac{1}{6+(\sqrt{11}-3)}}.\end{aligned}$$

Noting that the form $\sqrt{11}-3$ has occurred again, we deduce that the SCF for $\sqrt{11}$ is periodic:

$$\sqrt{11}=3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\cdots}}}}. \qquad (19)$$

This is generally written in the following form:

$$\sqrt{11}=[3;3,6,3,6,3,6,3,6,\ldots], \qquad (20)$$

with the $3, 6$ repeating infinitely often. A convenient short form for this is to use a bar above the $3, 6$ (just as we do for recurring decimals):

$$\sqrt{11}=[3;\overline{3,6}], \qquad (21)$$

Now we compute the convergents to the continued fraction, i.e., the fractions obtained by truncating its tail:

$$[3] = 3,$$

$$[3;3] = 3 + \frac{1}{3} = \frac{10}{3},$$

$$[3;3,6] = 3 + \cfrac{1}{3 + \cfrac{1}{6}} = \frac{63}{19},$$

$$[3;3,6;3] = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3}}} = \frac{199}{60},$$

$$[3;3,6;3] = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6}}}} = \frac{1257}{379},$$

and so on. The solution pairs to $x^2 - 11m^2 = 1$ in positive integers are now:

$$(x, m) = (10, 3),\ (199, 60),\ \ldots. \tag{22}$$

All these solutions can also be generated from the very first one, by the following very simple procedure:

$$\left(10 + 3\sqrt{11}\right)^2 = 199 + 60\sqrt{11},$$

$$\left(10 + 3\sqrt{11}\right)^3 = 3970 + 1197\sqrt{11},$$

$$\left(10 + 3\sqrt{11}\right)^4 = 79201 + 23880\sqrt{11},$$

and so on. The solution pairs to $x^2 - 11m^2 = 1$ in positive integers are thus:

$$\begin{aligned}(x, m) = {} & (10, 3),\ (199, 60),\ (3970, 1197),\\ & (79201, 23880),\ \ldots.\end{aligned} \tag{23}$$

For Step 3 we compose the pair $(x_0, m_0) = (1, 1)$ with the above solution pairs via the process of multiplication by $10 + 3\sqrt{11}$:

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^0 = 1 + 1\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^1 = 43 + 13\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^2 = 859 + 259\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^3 = 17137 + 5167\sqrt{11},$$

$$\left(1 + \sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)^4 = 341881 + 103081\sqrt{11},$$

and so on; we thus get the following solution pairs to the equation $x^2 - 11m^2 = -10$:

$$\begin{aligned}(x, m) = {} & (1, 1),\ (43, 13),\\ & (859, 259),\ (17137, 5167),\\ & (341881, 103081),\ \ldots,\end{aligned} \tag{24}$$

and this list is infinite. Since

$$\begin{aligned}&\left(a + b\sqrt{11}\right) \cdot \left(10 + 3\sqrt{11}\right)\\ &\quad = (10a + 33b) + (3a + 10b)\sqrt{11},\end{aligned} \tag{25}$$

the pairs listed can also be generated by the following recursive procedure: $x_1 = 1$, $m_1 = 1$, and for $k \geq 1$:

$$\begin{aligned} x_{k+1} &= 10x_k + 33m_k,\\ m_{k+1} &= 3x_k + 10m_k.\end{aligned} \tag{26}$$

The procedures we have described clearly generate infinitely many solution pairs to the equation $x^2 - 11m^2 = -10$ and thus provide infinitely many instances to the question posed at the beginning: *How many instances are there when the sum of* 11 *consecutive perfect squares is itself a perfect square?* Thus we have:

$$38^2 + 39^2 + \cdots + 48^2 = 20449 = 143^2,$$

$$854^2 + 855^2 + \cdots + 864^2 = 8116801 = 2849^2,$$

$$\begin{aligned}&17132^2 + 17133^2 + \cdots + 17142^2\\ &\quad = 3230444569 = 56837^2,\end{aligned}$$

and so on.

A second family of solutions. Another infinite family of solutions gets generated if we start with a different solution in Step 1; namely: $x_0 = -1$, $m_0 = 1$. We get:

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^0 = -1+1\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^1 = 23+7\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^2 = 461+139\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^3 \\ = 9197+2773\sqrt{11},$$

$$\left(-1+\sqrt{11}\right)\cdot\left(10+3\sqrt{11}\right)^4 \\ = 183479+55321\sqrt{11},$$

and so on; we thus get the following solution pairs to the equation $x^2 - 11m^2 = -10$:

$$(x, m) = (23, 7),\ (461, 139),\ (9197, 2773),\\ (183479, 55321),\ \ldots, \tag{27}$$

and this list too is infinite. These pairs yield the following equalities:

$$18^2 + 19^2 + \cdots + 28^2 = 5929 = 77^2,$$

$$456^2 + 457^2 + \cdots + 466^2 = 2337841 = 1529^2,$$

$$9192^2 + 9193^2 + \cdots + 9202^2 \\ = 930433009 = 30503^2,$$

and so on.

Closing Remarks

A natural follow-up to the above is to ask whether the sum of 12 consecutive squares can be a perfect square; and likewise for 13, 14, 15, …consecutive squares. It would be quite difficult to anticipate the results in advance, i.e., in which cases we do obtain solutions, and which cases we do not. But we leave this analysis for you to take forward.

In closing, we add a few remarks amplifying on the comment made at the start: that there may be more to a problem than just the solution of the problem.

In his famous book *How To Solve It*, George Pólya suggests the following four key steps in problem solving: (i) understand the problem; (ii) devise a plan; (iii) carry out the plan; (iv) review/extend the solution and analysis. Our interest in this section is in suggestion (iv). There are two aspects to what Pólya has said: *reviewing* what one has done, which means taking the trouble to look back at one's effort and to identify the steps that worked and the steps that did not work; this little extra step can contribute significantly to one's 'problem-solving muscle.' The other aspect is: *extending one's work*. There is no set approach for doing this, but one can start by tweaking the problem, altering the numbers or the initial conditions and examining the consequences of these tweaks. Playing around in this manner with the problem, one can actually uncover fresh and new mathematics. In that sense, it can be a wonderfully enriching activity, one which every student and every mathematics teacher must experience. This is just what we have done in our article, and we hope that it has proved worthwhile for you to read it.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Adventures with Triples

Part I

Problem Editor: Ranjit Desai

Problem Corner

In this two-part note, we study the following two problems. Given three positive integers a, b, c, we say that the triple (a, b, c) has the **linear property** if the sum of some two of the three numbers equals the third number (i.e., either $a + b = c$ or $b + c = a$ or $c + a = b$); and we say that the triple has the **triangular property** if the sum of any two of the three numbers exceeds the third number (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$).

Fix any upper limit n, and let a, b, c take all possible positive integer values between 1 and n (i.e., $1 \le a, b, c \le n$). There are clearly n^3 such triples. How many of these triples possess the linear property? How many of these triples possess the triangular property? These are the questions that we plan to explore.

Notation

- $S(n)$ denotes the set of all triples (a, b, c) with $1 \le a, b, c \le n$. The number of such triples is clearly n^3.
- $L(n)$ denotes the number of triples in $S(n)$ which possess the linear property.
- $T(n)$ denotes the number of triples in $S(n)$ which possess the triangular property.

In Part I of the article, we shall focus on computing $L(n)$.

Keywords: *Integers, linear property, triangular property*

Counting the linear triples

To start with, let us enumerate by hand the values of $L(n)$ for a few small values of n.

n $= 1$: Since $S(1)$ has just the one triple $(1, 1, 1)$, and this does not have the linear property, $L(1) = 0$.

n $= 2$: The triples in $S(2)$ which have the linear property are clearly $(1, 1, 2)$ with all its permutations; hence $L(2) = 3$.

n $= 3$: The triples in $S(3)$ which have the linear property and have not been included in the previous list are $(1, 2, 3)$ with all its permutations; hence $L(3) = 3 + 6 = 9$.

n $= 4$: The triples in $S(4)$ which have the linear property and have not been included in the previous list are $(1, 3, 4)$ and $(2, 2, 4)$ with all their permutations; hence $L(4) = 9 + 6 + 3 = 18$.

n $= 5$: The triples in $S(5)$ which have the linear property and have not been included in the previous list are $(1, 4, 5)$ and $(2, 3, 5)$ with all their permutations; hence $L(5) = 18 + 6 + 6 = 30$.

n $= 6$: The triples in $S(6)$ which have the linear property and have not been included in the previous list are $(1, 5, 6)$, $(2, 4, 6)$ and $(3, 3, 6)$ with all their permutations; hence $L(6) = 30 + 6 + 6 + 3 = 45$.

Proceeding in this way, step by step, we construct by hand the following table of values of the L function:

n	1	2	3	4	5	6	7	8	9	10	11
$L(n)$	0	3	9	18	30	45	63	84	108	135	$\cdots$

Do you see any obvious pattern in the sequence of values of $L(n)$? A quick observation will reveal that all the L values are multiples of 3. This invites us to divide each number by 3; doing so, here is what we get:

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ \ldots.$$

Why, we have obtained the sequence of triangular numbers! What a nice surprise! We seem to have arrived at the following:

Conjecture 1. *$L(n)$ is equal to 3 times the $(n-1)^{\text{th}}$ triangular number, i.e.,*

$$L(n) = \frac{3n(n-1)}{2}.$$

Now we have:

$$\frac{3n(n-1)}{2} - \frac{3(n-1)(n-2)}{2} = 3(n-1).$$

Therefore, Conjecture 1 is equivalent to the following:

Conjecture 2. *The number of positive integer triples whose largest number is n and which possess the linear property is $3(n-1)$.*

For example, take $n = 4$. The positive integer triples with largest number 4 and which possess the linear property are $(1, 3, 4)$ and $(2, 2, 4)$ together with all their permutations; there are $6 + 3 = 9$ such triples. Note that $9 = 3 \times (4 - 1)$.

Or take $n = 5$. The positive integer triples with largest number 5 and which possess the linear property are $(1, 4, 5)$ and $(2, 3, 5)$ together with all their permutations; there are $6 + 6 = 12$ such triples. Note that $12 = 3 \times (5 - 1)$.

Proof of Conjecture 2. Suppose that n is even. The positive integer triples with largest number n and which possess the linear property are the following:

$$\begin{array}{ll} (1, n-1, n) & \text{together with its permutations,} \\ (2, n-2, n) & \text{together with its permutations,} \\ (3, n-3, n) & \text{together with its permutations,} \\ \dots & \dots \\ \left(\frac{1}{2}n-1, \frac{1}{2}n+1, n\right) & \text{together with its permutations,} \\ \left(\frac{1}{2}n, \frac{1}{2}n, n\right) & \text{together with its permutations.} \end{array}$$

Except for the very last triple, all these triples have distinct elements. Therefore, the total number of permutations of all these triples is

$$6\left(\frac{1}{2}n-1\right)+3=3n-6+3=3(n-1).$$

Next, suppose that n is odd. The positive integer triples with largest number n and which possess the linear property are the following:

$$\begin{array}{ll} (1, n-1, n) & \text{together with its permutations,} \\ (2, n-2, n) & \text{together with its permutations,} \\ (3, n-3, n) & \text{together with its permutations,} \\ \dots & \dots \\ \left(\frac{1}{2}(n-1), \frac{1}{2}(n+1), n\right) & \text{together with its permutations.} \end{array}$$

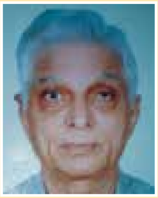
All these triples have distinct elements. Therefore, the total number of permutations of all these triples is

$$6\left(\frac{1}{2}(n-1)\right)=3(n-1).$$

The conjectured formula has been shown to hold in both the situations (when n is even and when n is odd). Hence it can be considered as proved. □

In Part II of the article, we shall study triangular triples.

Remark. The problem started here—that of counting positive integer triples possessing the linear property—seems ideal as exploration material for students of classes 8–10.



RANJIT DESAI worked as a high school teacher of mathematics from 1960 till 1998, when he retired as head and in-charge of the PG Centre in mathematics from B K M Science College, Valsad (Gujarat). He has been an extremely active member of Gujarat Ganit Mandal, which publishes the periodical Suganitam. All through his career, he has worked very hard to train students, teachers and parents alike. He has authored and published his own book titled "Interesting Mathematical Toys".

Review

Rich Maths for Everyone: A Review of

Jo Boaler's Mathematical Mindsets

PRABHAT KUMAR

Right at the start in the Introduction to her book *Mathematical Mindsets – Unleashing Students' Potential through Creative Math, Inspiring Messages and Innovative Teaching*, Jo Boaler describes her first meeting with Carol Dweck, soon after Dweck had joined Stanford University as a professor of mathematics education. By then Carol Dweck and her team had published enough research work about mindsets and their impact on learning. Dweck categorizes the beliefs people have about how they think about their learning abilities in two kinds of mindsets – a *growth mindset and a fixed mindset*. When people have a fixed mindset (yes, people can have different mindsets in different contexts), they think of their abilities as unchangeable. In contrast, when people have a growth mindset, they think they can learn by putting in more work and also most importantly their smartness can increase with hard work. Dweck's work has shown that these mindsets change our learning behaviour and hence our learning outcomes.

Keywords: *growth mindset, fixed mindset, plasticity of brain, celebrating mistakes, rich tasks, visualization, ego feedback, equity in classrooms, performance subject mathematics, algebra*

This work obviously has great significance in education since students' mindsets can affect their achievement levels. Equally important is what kind of mindset teachers have about their students, because that governs the choices teachers will make in the classrooms. While one can find people working from one of these mindset orientations in different fields of work, it perhaps would not be incorrect to assume that an unusually large proportion of students in many maths classrooms often are trapped in a fixed mindset about their abilities in the subject. So it was quite fortunate for Boaler and Dweck to have met and then to have decided to work together to find ways in which research about mindsets can enter the maths classrooms in order to change core beliefs that maths students and teachers generally hold about this subject and about their maths smartness.

This book is about developing such an open mindset about mathematics, through positive messages and creative teaching. Boaler sees the immense potential of the positive messages that teachers can convey to their students – about the brain and its learning potential as well as about maths. She also knows though that spoken words and written messages on posters on classroom walls have a limited impact if the teacher's approach and classroom practices continue to convey the opposite messages – that maths is a *performance subject,* that doing things faster is valued more than thinking deeply, that maths is all about rules and procedures, that there is little scope of discovering interconnections, that it is not possible to have a deep understanding of mathematical tasks, that boys are smarter than girls; and that once you have 'missed the bus', you cannot catch up. So, Boaler doesn't only talk about the many convincing brain studies and classroom studies which point to the need for a change in maths classrooms. She describes her own work with her team in classrooms and shares work from other teachers which shows how mindsets of students and therefore their engagement and performance in the subject can change with good teaching interventions at any stage of schooling. Though most of Boaler's work is based in the context of classrooms in the US and the UK, it is very relevant for Indian classrooms as well.

Brain plasticity

The book starts with the recent brain studies about the plasticity of the human brain and the implications of this for education. These studies have shown that changes in the brain can happen even after childhood and can happen within short spans of time that would have been considered unbelievable earlier. Based on such findings, Boaler asserts her belief that almost anybody can learn maths and can learn it well. She thinks that even if people are born with brains which function differently from the norm, the effect of those differences can easily be eclipsed with the right teaching inputs; and so ability in mathematics can be developed by everybody.

Celebrating mistakes

Another recent finding from brain studies that is mentioned is about the response of the brain to mistakes. For many, mistakes are discouraging events in a learning process or at best they are accepted as inevitable negative steps. Boaler mentions studies which show our brains 'lighting up' when mistakes are made; surprisingly they 'light up' even when we are not aware that we have made a mistake. So making mistakes is equivalent to making new connections in the brain. Further studies have linked mindsets and the response of the brain to mistakes – people with a growth mindset have their brains more active when a mistake is made, which means their brains grow more and there is a greater chance that one becomes aware of the mistake too.

The book has several examples of strategies teachers are using in their classrooms to change students' attitudes to mistakes. Some of these involve conveying growth mindset and positive brain messages; others involve welcoming mistakes and making use of them in the teaching-learning processes. Boaler's advice: "Value correct work less and mistakes more".

Mistakes are important for brain growth because they create opportunities for challenge. Boaler suggests that teachers give challenging work to students so that more such situations get created. She advocates promoting struggle and mistakes in the classroom.

Creative Mathematics

Doing a lot of routine, procedure-based problems for "practice" is however not a learning challenge. If we want to provide opportunities to our students to learn mathematics, we will have to take a hard look at the mathematical tasks we want the students to be engaged in. Everybody knows that mathematicians don't spend their time doing calculations, that mathematics is not about rules and number facts. Even employers are not looking for people who can calculate fast and accurately; machines are always going to be far ahead of us in that area. Boaler suggests that students get to do creative maths: asking questions, using maths for modelling, visualising problems, discovering connections between different representations and pathways, explaining and challenging each other.

Mathematical Mindsets

Which practices in maths classrooms reinforce fixed mindsets among students and convey a narrow view of mathematics to them? Boaler answers the age-old questions about the necessity of memorising maths facts, e.g. multiplication tables, importance of "practice", place of homework and timed tests.

She argues that developing *number sense* is more important than memorising number facts. Developing number sense involves visual and intuitive thinking, using multiple representations, using new information or concepts in different contexts, all of which makes learning more powerful. Her examples of strategies to make all this possible reminds me of Eleanor Duckworth (a former teacher and now-retired teacher educator) and her focus on providing opportunity to children to have their own ideas:

The having of wonderful ideas is what I consider the essence of intellectual development. And I consider it the essence of pedagogy to give Kevin the occasion to have his wonderful ideas and to let him feel good about himself for having them.

["The Having of Wonderful Ideas" by Eleanor Duckworth]

How often does the school curriculum give such occasions to children to feel good about having their own wonderful ideas?

Since maths so often is taught as a *performance subject*, curriculum tends to simplify the matter by dividing the content neatly into water-tight compartments, providing bland problems placed in pseudo-real contexts. It definitely helps some students to succeed, especially those who can memorise and repeat lots of procedures without seeking any connections among them. Others, however, find maths full of isolated procedures devoid of any meaning outside the "exercises" and the tests. Boaler sees this as one of the key reasons behind the narrow idea of maths that people generally have:

The oversimplification of mathematics and the practice of methods through isolated simplified procedures is part of the reason we have widespread failure in the United States and the United Kingdom. It is also part of the reason that students do not develop mathematical mindsets; they do not see their role as thinking and sense making; rather, they see it as taking methods and repeating them. Students are led to think there is no place for thinking in math class.

Equity in Maths Classrooms

There seems to be enough evidence to suggest that girls are more affected by this narrow treatment of maths than boys. It seems girls seek more connections in what they learn, and so maths 'ticks them off'. They find it difficult to move through the labyrinth of disconnected concepts and procedures and soon give up on maths.

There is another reason why girls don't do as well as boys in maths. Some recent studies have shown that the more people in any particular field think that ability in their field cannot be developed by hard work, the fewer women one finds in that field. In other words, the idea of 'effortless giftedness' in maths seems somehow to affect girls more than the boys.

Another way in which the idea of giftedness and hence the idea of a fixed mindset gets promoted is by having ability grouping in schools, called tracking in the US.

The strong messages associated with tracking are harmful to students whether they go into the lowest or highest groups. ... Students most negatively hit by the fixed messages they received when moving into tracks were those going into the top track.

These observations are the findings of several classroom-based research studies.

Seen closely, it shouldn't be surprising that students growing with fixed mindsets will gradually grow averse to taking risks, making mistakes and challenging themselves, because they begin to see any intellectual struggle as a threat to their smartness.

Boaler doesn't underestimate the challenges for a teacher responsible for teaching a heterogeneous group. After describing ways in which maths tasks can be adapted for a heterogeneous group, she shares in great detail a case study of an urban school in California, where teachers decided to stop tracking and instead started teaching maths using what is called *complex instruction*. Besides the examples of some actual maths work that the students in this school did as part of the complex instruction, the case study also describes in detail very efficient ways in which teachers in this school got the students to engage in *group work*, by becoming responsible for each other's learning. There is a wealth of very good ideas here.

Rich Maths
Hopefully, having convinced her readers about the need to revolutionise maths classrooms, Boaler spends five chapters and sixty pages of appendices to show how this can actually be done by teachers. She gives examples from her own classroom research work and she shares ideas from her classroom observations of other teachers at work.

The fifth chapter of the book titled *Rich Mathematical Tasks* is a collection of a few case studies of exciting maths experiences. One of them is from Boaler's interaction with the members of the Udacity team in Silicon Valley; the others are from the classrooms. My favourite one is the one where a "growing shapes" problem (seeing the pattern in a series of growing shapes and finding the nth shape) asks, after giving three figures numbered Fig. 2, Fig. 3 and Fig. 4:

What would Figure 100 look like?
And then: *Imagine you could continue your pattern backward. How many tiles would there be in Figure –1?* (That's figure negative one, whatever that means!)

Boaler, after describing these case studies, presents the questions teachers can ask themselves to make the maths tasks in their classrooms mathematically rich by design and sometimes by adaptation:

Can you open the task to encourage multiple methods, pathways, and representations?

Can you make it an inquiry task?

Can you ask the problem before teaching the method?

Can you add a visual component?

Can you make it low floor high ceiling?

Can you add the requirement to convince and reason?

Assessment for Growth Mindset
Timed tests continue to remain the primary tools of assessment of learning in most classrooms, even more so in maths classrooms. The problem is not just the public exams in India or standardised testing in the US as Boaler points out, "math teachers are led to believe they should use classroom tests that mimic low-quality standardized tests, even when they know the tests assess narrow mathematics."

Frequent testing in maths reinforces the fixed mindsets among students about their abilities in the subject. Grading following these tests creates further problems:

When students are given a percentage or grade, they can do little else besides compare it to others around them, with half or more deciding that they are not as good as others. This is known as "ego feedback," a form of feedback that has been found to damage learning. Sadly, when students are given frequent tests scores and grades, they start to see themselves as those scores and grades. They do not regard the scores as an *indicator of their learning or of what they need to do to achieve; they see them as indicators of who they are as people.*

Challenging the view that grades motivate students and help learning, Boaler mentions studies that have shown that grading reduces achievement of students. Removing grades from the assessment/feedback process has been shown to even improve the achievement gap between male and female students.

So, what is the alternative? The book lists nine different ways of assessing students' learning. Some of these involve students reflecting on their own learning and using a self-assessment rubric. Most of these are designed to also develop greater self-awareness and responsibility. Though these alternative assessment practices are general and can be used for any subject, Boaler has gone to the trouble of giving specific examples of how these strategies can be used by maths teachers.

Conclusion

Boaler has written this book in simple and accessible language. Though the book refers to several studies and research articles (the list of references run into eight pages!), one doesn't get a sense of reading technical writing. Many of the mathematical tasks and ideas for the classrooms have been put together separately in the form of a long appendix so that teachers can easily access them for use in their classrooms. In my view, the message of the book gains strength not so much from the mention of brain research and classroom studies but more from the detailed descriptions of actual work done by maths teachers in their classrooms.

It is a book that maths teachers should keep on top of their book pile, even after finishing reading it, to get back to for ideas and inspiration. It does a marvellous job of reinstating faith in all that is beautiful and possible in maths classrooms in the times when success has come to be equated with fast performance, when there is a relentless pressure on teachers to adopt dehumanizing practices of a system that is designed to sort, to eliminate and to select a few.

Jo Boaler starts the fifth chapter of this book with: I am passionate about equity. The chapter ends with:

I want to live in a world where everyone can learn and enjoy math, and where everyone receives encouragement regardless of the color of their skin, their gender, their income, their sexuality, or any other characteristic.

This book will, I hope, be a good step towards that world.

PRABHAT KUMAR graduated from IIT Kharagpur in 2003 and has been teaching Physics and Mathematics in Sahyadri School for the past 12 years. Among his many interests are the following: history of science and its relevance in the teaching of science; making classrooms safe learning spaces for all children; bird watching; and photography. He can be contacted at prabhat@sahyadrischool.org.

The Closing Bracket . . .

The talk about the need to learn mathematics, what it is and what role it plays in our lives has become intense in recent decades. The reflections in NCF 2000 and then subsequently in NCF 2005 along with the position paper on mathematics teaching that accompanied it, laid down some axiomatic beliefs that we would work with, while attempting to evolve programs and strategies for all learners. The commitment to having all learners provided with equivalent education has co-existed with the oppositely directed desire for differentiated learning for children with different dispositions and abilities. The latter has been a standard theme of all teacher education programs, now even more so, as an official effort to provide skill-based education to learners according to their disposition so as to make them more useful and provide them with a greater opportunity in society. There is an increasing effort to design tests that look for talent early and make greater resources available to those who are found to be more 'capable' and 'talented' by these tests. There is no difficulty in realising the socio-cultural, lingual and economic roots of this so called disposition and the implications on inclusion and universal opportunity with equity. In this context, the inclusive commitments of the NCF documents are to be reiterated and emphasised again and again.

In the development of the idea on what the focus of mathematics teaching should be, where the NCF 2005 breaks ground is in introducing the term *mathematisation*. It does not precisely define mathematisation or elaborate it but gives sufficient indication of what it may mean. The confusion however, remains as to whether this is more than the application of mathematics to daily life – and if so, how? If applying mathematics to daily life situations is the main purpose of doing mathematics, then how is this the higher purpose? For, after all, using mathematics in daily life would fall under the utilitarian perspective as well. Clearly therefore, examples of using surface area to estimate the expense of painting, using different interest schemes to choose the best investment, finding out the probability of winning before betting or doing other operational tasks that help in day to day activity may be interesting applications of mathematics but are no different from what a person who does not go to school to learn mathematics can do on his or her own. So what are we missing? The other question that arises from this conversation is; why should mathematics that is not immediately linked to life be done at all in the classroom?

Does *mathematisation* then mean the use of concrete models to explain abstract concepts? The idea of concrete materials and their relation to life has become a part of the general mathematics discourse in education. It is now stated in most workshops that mathematics is fun and can be learnt through activity, games and concrete models. The assumption in this is that concrete models constructed by the teacher educators or the teachers, when made available to learners, help them uncover the logic and the conceptual underpinnings. It sees concrete models as alternatives to the need for any abstract thinking that the child may have to do. The dual argument of relationship to life and known concrete models given by the teachers are both in the direction of looking at mathematics as something that can be explained and absorbed through explanations. For example, the balance is a concrete model of both sides of an equation, the corner of a room is the model of an angle, the area is the surface covered, negative numbers are going down the ladder in to the basement. And so on and so forth, for everything starting from numbers to more complex mathematical ideas. While we can filter out from this, those that are absolutely wrong and which can foster misconceptions, it is important to stress that this is **not** what NCF 2000 or NCF 2005 intends to communicate as its action plan. In fact, the latter makes this even more explicit as some amount of ambiguity may be detected in the former.

The question then is, if mathematisation is not merely about doing and using mathematics in real life and is not about modelling then what is it? This is a question that needs deeper thinking and needs to be considered in conjunction with the remaining ideas in NCF 2005. The fact is, that mathematics at the primary and elementary level is different from the mathematics at the secondary level. Mathematics is learnt best by doing mathematics in new situations and on new problems. Application of concepts and developing new procedures helps build confidence and capability in mathematics and this should build into a spiral of learning in which the student uses mathematical ideas differently and in a more inter-related manner. We can start with the fact that mathematisation has to include the ability to look at life situations differently using mathematics. That is, not in the way that we are expected to do so, but in some way that we have discovered ourselves. It could be appreciating the placement of motifs differently or recognising patterns in which the wild foliage grows, it means looking at many critical events in our life as chance events not determined by anyone's pre-mediated intervention or by the vagaries of fate. In a sense, it is the ability to look at life situations with a different lens so to speak and to see what you would not if you lacked the additional knowledge of mathematics. It is not merely about application but also about understanding and building a relationship. Application may come much later, or may not at all. It may start with concrete things but it is also about visualisations taking a leap from what is around, in a way that the link of the leap is unconventional and unusual. Mathematisation thus becomes an aspirational aim that is not to be limited to the elements that may be tested. These, as we saw, are very limited. The exercises to build mathematisation, instead of a maths phobia that makes mathematics exclusively for school and for doing examinations need to work on everything around us. It should make use of mathematical ideas and ways natural to the observation and the view of the world, absorbing, appreciating and enjoying it, thinking about it and using it for individual and large purposes.

Hriday Kant Dewan
June 2016

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.

2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

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Azim Premji University

PES Institute of Technology Campus, Pixel Park, B Block, Electronics City, Hosur Road (Beside NICE Road), Bangalore – 560100 India
Email: reachus@apu.edu.in

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TEACHING Area and Perimeter

PADMAPRIYA SHIRALI

A PRACTICAL APPROACH

At Right Angles

A Resource for School Mathematics

Area and Perimeter

Area and perimeter are forms of measurement that are used commonly in many day-to-day activities. In particular, area is used in an intuitive way on an everyday basis when we select a plate to cover a utensil, a table cloth for a specific table, a sheet of paper to cover a book, etc. Without really knowing the specific words, children also commonly make judgements which involve an intuitive understanding of area. A question that naturally arises is: When or why do we want to know the exact size of a space? Demonstration of this point needs to happen repeatedly through real world applications.

Area and perimeter are generally introduced together and dealt with as a single topic. However, perimeter is a linear measure, whereas area is a measure of a two dimensional space. One can also think of perimeter as the distance around the outside of a shape whereas area is the amount of space inside the shape.

Since they are introduced together, one frequently finds children mixing up the two concepts. Also formulae for arriving at these measures are brought in too quickly - well before the concepts are fully understood. One can avoid this problem by spacing out these two concepts. Area could be explored in the first stage as it occurs frequently in a child's everyday experiences. “Who has got a bigger portion of the chocolate?”

In this article, I have suggested plenty of activities which will slowly lead them into the topic and strengthen their conceptual understanding. It is best to let children work in groups of four so that enough discussion happens amongst them.

Keywords: *Shape, area, perimeter, mensuration, experience, formula, length, breadth, units, square units, geoboard*

ACTIVITIES FOR AREA

ACTIVITY ONE

Materials: Squares of different sizes (preferably wooden or plastic).

Objective: To compare sizes through stacking.

Let each group pick up 4 different squares and put them in order of size. Children may be able to visually compare and determine the smallest size, largest size etc. Ask them to show a way by which the order can be clearly seen. They may stack up the squares in order in different ways as shown in Figures 1 and 2.

Let children draw these in their books and describe the relationships.

The yellow square is smaller in size than the pink square.

Notice that the word 'Area' is not introduced immediately. It is important to focus on the concept of size and ensure that the sense of size is clearly understood. It is good to use familiar words like size when the concept is introduced. Once the concept is comprehended through commonly understood words one can introduce the terminology (area).

Repeat the same activity with circles.

Superimposing one shape over another shape is one way of comparing sizes which works for certain objects.

Figure 1

Figure 2

ACTIVITY *TWO*

Materials: Rectangles of different sizes (books can also be used).

Objective: To compare sizes through the usage of non-standard units.

Let each group pick up two rectangles and try to determine the smallest and the largest.

With some rectangles determining the largest size and the smallest size may be obvious.

Ask the children if it is possible to show the smallest by stacking one rectangle over the other.

This may be possible with some rectangles where both the length and breadth of the smaller sized rectangle are less than the other. But in the case of some rectangle pairs, the length of one might be shorter, but its breadth might be larger, as shown in Figure 3.

Figure 3

Ask the question "How can we show or how do we find out for certain which is smaller?" "What does *smaller* mean?" Help the children to express their understanding of what makes one shape bigger or smaller than another.

Most children are familiar with chocolate bars which have square partitions. Let them state how they would compare such chocolate bars. In a similar manner, lead children to the discovery that both the rectangles can be covered with smaller units like erasers, rectangular or square biscuits, post-it notes or any other appropriate objects available in the class to compare the sizes.

Let them experiment with different objects like triangle shapes, diamond shapes and circular objects like coins, as units of measure. Help them discover that only shapes that do not give rise to any gaps while covering areas, can be used as measuring devices. Discuss the reason for why circular objects would not give accurate answers.

Verify that the children understand the need for measuring both the rectangles with the same unit.

Observations are recorded using non-standard units as measure.

See Figures 4 and 5.

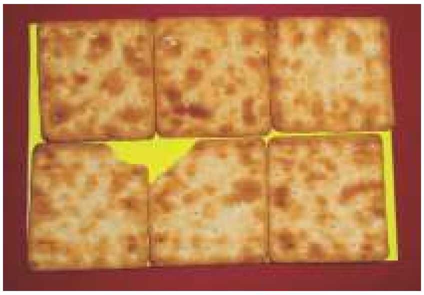

Figure 4

The yellow rectangle is 6 biscuits in size.

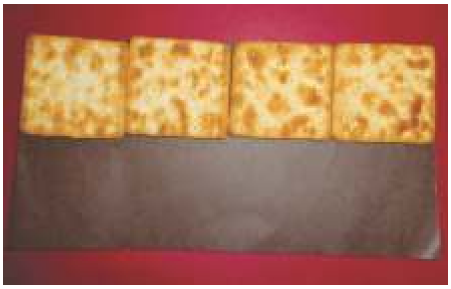

Figure 5

The brown rectangle is 8 biscuits in size.

Similarly, rectangles can also be measured with triangle units.

The brown rectangle in Figure 5 is 16 triangles in size. The yellow rectangle in Figure 4 is 12 triangles in size.

Repeat this activity with slightly larger sized objects. Size of the teachers table compared with the size of the students table, etc. Children can choose appropriate units of measure (compass boxes or small dictionaries).

While holding this activity, children will encounter situations where the unit of measure may not cover the space completely (e.g., while dealing with irregularly shaped objects). Discussion about this can lead to understanding the norm that more than half of a unit space is counted as one.

ACTIVITY ***THREE***

Materials: Regular and irregular shapes, cm squares, cm square grid paper.

Objective: To compare sizes through usage of cm squares

Tell them that the area of a shape refers to the space that it encloses or covers.

Pose the question: "How does one measure the space occupied (area) by a book? By a leaf? By a circle?" It is not necessary to use the word regular shape and irregular shape.

Children may suggest usage of small objects. They could do that. However by this stage they are already familiar with centimetre as a measure for lengths. They also use square ruled notebooks which have cm squares. They can stick some of these papers onto thick card sheets and cut cm squares to use as a measure for covering these shapes.

Tell the children that a square that measures one centimetre by one centimetre is a square centimetre.

See Figure 6.

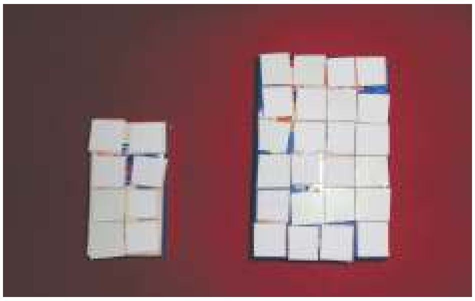

Figure 6

Pose the question: "Who has the biggest hand in the class?" Let children draw an outline of their hands and check the area by filling them in with cm squares. They could also measure the area of their foot. They could draw the outlines of these on square grid paper. It may also be interesting to find out how much variation can be there if the same hand is traced in different positions on the square grid paper.

See Figure 7.

Figure 7

My hand is ___ cm square in size (area).

See Figure 7a.

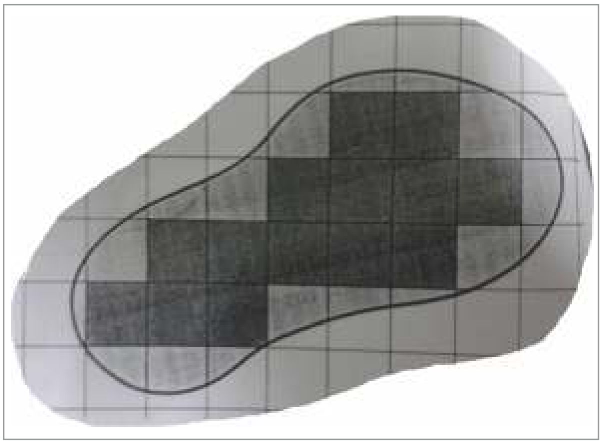

Figure 7a

As an extension they can do a craft activity of mosaic work with coloured cm squares and create beautiful motifs and write about them.

Does everything have an area?

At some point it is good to pause and ask: "Does everything have an area?" A discussion about this can reveal students' understanding and misconceptions about area. It can lead to questions about 3-D objects, curved spaces. Through discussion, difference between closed shape and open shape, area and capacity (volume) can be clearly brought out.

ACTIVITY ***FOUR***

Materials: Irregular shapes and shapes with curves, transparent grid or thread frame.

Objective: To compare sizes through usage of cm grid

A transparent grid can be prepared using hard transparent plastic sheet as shown in Figure 8 to create a cm square grid. A thread frame can also be prepared using a thick cardboard frame with threads running across in a grid form.

Objects can be placed underneath the grid and squares can be counted.

Figure 8

ACTIVITY ***FIVE***

Materials: Irregular shapes, square grid paper, triangular grid paper, hexagonal grid paper.

To compare sizes through usage of various types of unit measures.

In the earlier activity, the shapes to be measured were placed underneath square grids. In this activity, shapes are placed over grid paper and their outlines are drawn as shown in Figures 9 and 9a.

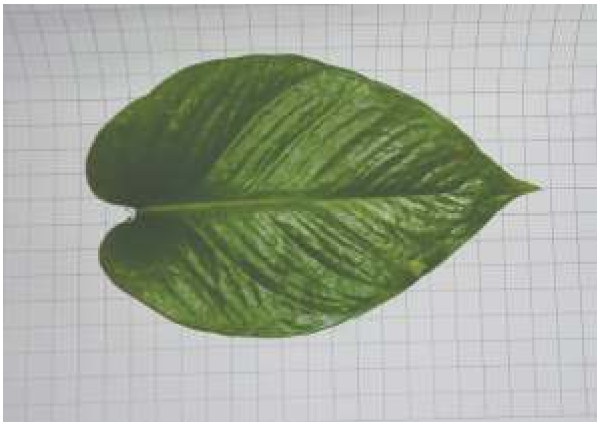

Figure 9

The area of the leaf is _______square centimetres.

The area of these shapes can also be found without drawing the outline; it can also be deduced from the grid around.

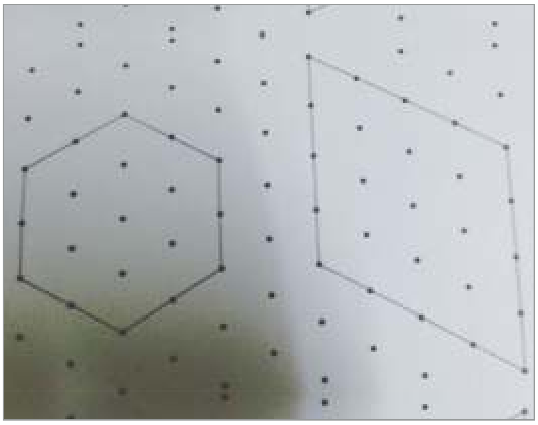

Figure 9a

ACTIVITY *SIX*

Materials: Geoboard and rubber bands.

Objective: To construct different polygons and count the squares to find area.

Initially children can construct different shapes (squares, rectangles, triangles, rhombi etc). They can record these on square dot papers and describe the shapes in terms of their measures.

In the next stage teacher specifies the measures (length or breadth) of different squares and rectangles. Children construct them on the geoboard and give the area in cm squares.

ACTIVITY *SEVEN*

Materials: Square grid paper.

Objective: To create composite shapes and sum the areas.

Children can create figures of robots or houses (as shown in Figure 10) to create composite shapes, and find the areas of what they have drawn.

My house has a door of 8 square cm. It has a window of 4 square cm.

Figure 10

See Figure 11.

Children can draw different shapes on square grid paper and find the total area by counting the full squares and half squares.

Figure 11

ACTIVITY ***EIGHT***

Materials: Square grid paper.

Objective: To discover that cutting and rearranging the pieces of any shape does not alter the area.

Children can draw the outline of any shape on square grid paper as shown in the picture. They record the area of the original figure. They can draw a couple of lines to cut along. Let them assemble the pieces together in some other shape and total the area.

ACTIVITY ***NINE***

Materials: Geoboard and rubber bands, square grid paper.

Objective: To discover the formula for the area of square and rectangle.

Let children construct various squares (1x1, 2x2, 3x3, 4x4) on the geoboard and draw the outlines on a square grid paper and write the area of each figure by counting the unit squares. Now pose the question: "Do you see any relationship between the length of the side and its area?" Lead them to discover that the area of a square is 'side times side'. Connect this to the multiplication model.

In a similar manner let children construct and draw outlines of various rectangles (2x3, 3x4, 2x4, and 3x5). Let them observe the patterns to realise that the area of the rectangle is length times breadth.

ACTIVITY ***TEN***

Materials: Self checking cards of specific measures.

Objective: To compute the missing measure and verify the result.

Make cards of different integral sizes (2x3, 2x4, 3x3, 3x4, 4x5 etc). Each card can pose a question related to the actual area of that card. Ex. 'My area is 12 cm square. My length is 4 cm. What is my breadth?' Children can work out the answer through calculation and verify by actually measuring the card. They could note down the related multiplication or division facts.

ACTIVITY *ELEVEN*

Materials: Square dot paper.

Objective: To discover that different shapes can have the same area.

Ask children to draw all possible shapes that can be made with three connected squares. (Squares are connected if they share a common side).

Let them then explore all possible shapes that can be made with four squares.

Specify an area, say 12 sq. cm and let children construct different shapes that have 12 sq. cm as their area.

Let them discover that different shapes can have the same area.

ACTIVITY *TWELVE*

Materials: Floor tiles or wall tiles.

Objective: To determine areas of class rooms and any other spaces which are already tiled.

Explain to the children that tiles can also be used as a unit of area.

Let children look around spaces around the school to compute the area of these spaces.

They can write the areas of these spaces in terms of the measuring unit, the tile and compare different areas in the school.

Area of the class room is (12 x 15) 180 tiles.

At this stage it may not be necessary to teach conversions of one unit to another. However if the question arises naturally, it would be interesting to see the strategies employed by children to calculate the area of a floor in other measures like square feet or square metres.

They could decide to calculate the area of each tile and multiply by the total number of tiles.

They may calculate the length of the room by multiplying the length of the tile by the number of tiles placed on the long side of the room. They may calculate the breadth in a similar manner, and then compute the area.

They may decide to ignore the tiles and measure the length and breadth with a tape and compute the area.

ACTIVITY *THIRTEEN*

Materials: Bulletin boards, windows, floor mats.

Objective: To determine areas of larger spaces

Discuss with children the need for a larger unit to measure areas of bigger spaces. By this stage they may be already familiar with a foot as a measure. Tell the children that a square that measures one foot on each side is a square foot. They can measure the areas of larger spaces and record them as square feet or square metres etc. At this point it would be good to have a discussion on length unit getting converted to square length unit, e.g. cm → sq. cm, matchstick → sq. matchstick, inch → sq. inch etc.

Game: Area with dice

Materials: Square grid paper of A4 size, two dice

Objective: To develop a sense of area and minimise wastage of space between shapes.

Each child starts from one end of the sheet as shown in Figure 12. A line is drawn separating the sheet into two equal parts. Each child throws the two dice and draws a rectangle or square with the numbers that appear. They continue to take turns in throwing dice and building more and more rectangles attached to the previously drawn ones. Each child continues to throw the dice and build rectangles as long as there is space on their side even if the other has stopped. At the end they sum the areas of the gaps that arise. The one with the smallest gap area is the winner.

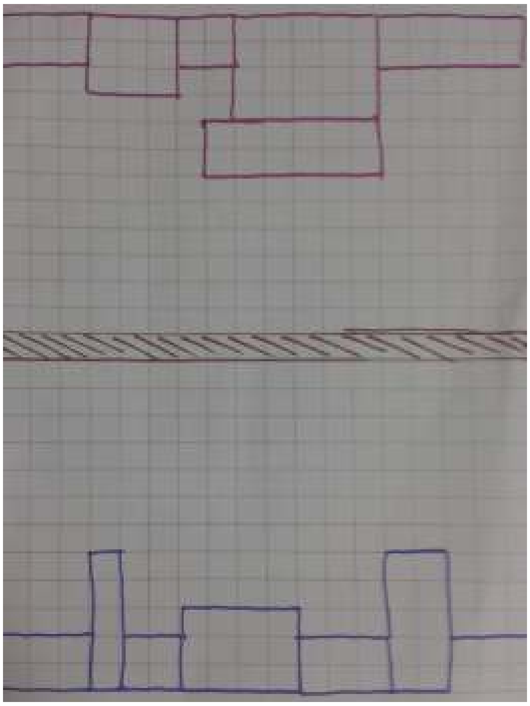

Figure 12

ACTIVITIES FOR PERIMETER

The word perimeter comes from the Greek word 'peri', meaning around, and 'metron', meaning measure. Help children to understand that perimeter means the distance around a figure.

Differentiate between perimeter and area using drawings as shown in Figure 13. Link it with their earlier understanding of the word metre as length.

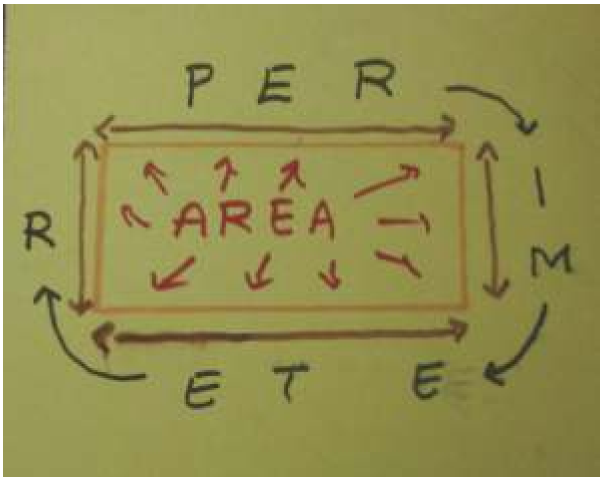

Figure 13

ACTIVITY ***FOURTEEN***

Materials: Classroom objects like children's tables, teachers table and bench.

Guess the shape with the largest Perimeter (distance around a figure)

Mention any three objects and pose the question 'Guess which of these objects has the longest length around?'

Let children place sketch pens or straws all around the edge of the top to find the object with the largest perimeter. Some rounding of numbers may be involved. Has their guess come right?

ACTIVITY ***FIFTEEN***

Materials: Different polygon cut outs and curved figures, Tape.

Objective: Find the shape with the largest Perimeter (distance around a figure)

Each group of children can be given four different polygonal shapes cut from chart paper. They can measure all the lengths in centimetres and total them up.

They could also measure the perimeter of curved figures with the help of a string or tape.

See Figure 13a.

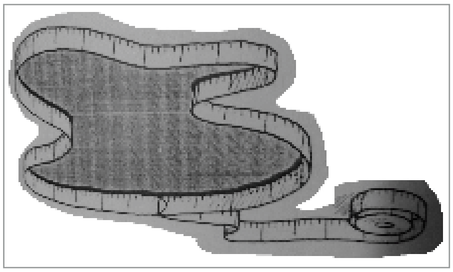

Figure 13a

ACTIVITY ***SIXTEEN***

Materials: Square tiles, square dot paper.

Objective: To discover that shapes with same area can have different perimeters.

Ask children to use 3 tiles or squares to make different shapes. Each tile must touch the other along one complete side.

How many different shapes can be made? What is the perimeter of each?

Repeat the same with 4 tiles and construct all possible shapes. Which shape gives the largest perimeter?

Repeat the same with 5 tiles and construct all possible shapes. Which shape gives the largest perimeter? The pentomino U stands out as the one with "more" perimeter. There can be a discussion on why this is the case.

ACTIVITY **SEVENTEEN**

Materials: Square dot paper.

Objective: To discover the formula for calculating perimeter of squares and rectangles.

Insist that they mark the measures on all sides with arrows as shown in Figure 14. They need to clearly understand that the number stands for the length of the side.

Let children draw squares with sides 1, 2, 3, 4 etc. in order, and let them calculate the perimeter of each one. They will easily see that it is 4 times the side.

Let children draw rectangles of different length and breadth combinations (1 x 2, 2 x 3, 2 x 4, 3 x 4 etc). Let them write the perimeter as a sum of all the four sides. As they begin to record these results, lead them to discover that the perimeter is twice the sum of the length and breadth.

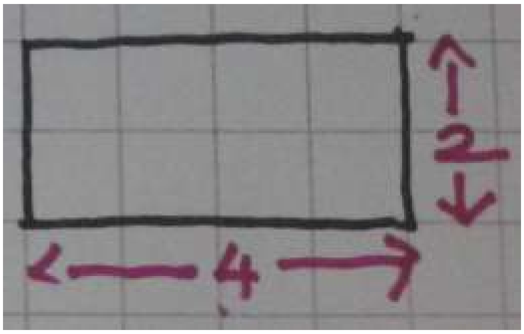

Figure 14

ACTIVITY **EIGHTEEN**

Materials: Sketch pens or toothpicks.

Objective: To create different shapes with the same perimeter

Construct rectangles and squares with a perimeter of 16 units. How many could you make?

This activity allows children to see that different shapes are possible with the same perimeter and they have different areas.

Repeat the activity with 24 units. What different sizes can be made? Which shape gives the largest area?

ACTIVITY **NINETEEN**

Materials: : Geo board, rubber bands.

Objective: To create different shapes of specified measures.

Pose some questions like the ones given below:

Can you construct the following shapes on a geoboard with one rubber band?

Square of length 2 units

Rectangle with an area of 3 square units

Pentagon with an area of 3 sq. units

Square with an area of 2 sq. units

ACTIVITY ***TWENTY***

Materials: Square dot paper.

Objective: To discover the effect of an increase in length or breadth on the perimeter and area.

Give children the measures of a rectangle, and let them find its perimeter and area. Now pose the questions:

"What happens to the perimeter of a shape when the length is increased by one unit?"

"What happens to the perimeter of a shape when the breadth is decreased by two units?"

"What happens to the area of a shape when the length is increased by one unit?"

"What happens to the area of a shape when the breadth is decreased by two units?"

One could also experiment with increasing both length and breadth by 1 unit, 2 units, 3 units etc to notice patterns and generalize. There should also be cases where length increases and breadth decreases and vice versa.

ACTIVITY ***TWENTY ONE***

Materials: : Tangram set.

Objective: To discover the relationship between the different pieces in terms of area.

Let children play around with the tangram pieces by creating different shapes. In the process of creating shapes they will begin to notice the size of different shapes and how they relate to one another in terms of area.

Identify a pair of shapes that have the same area. Can you find some more such pairs?

Do all these pairs look the same?

How can you show that they have the same area?

Can you find pairs where one shape is one half of the other? Which shapes are these?

How does the area of the square compare with the large triangle?

How does the small triangle compare with the medium triangle?

Note: This activity can come even at an earlier point when they are playing with non-standard units

ACTIVITY ***TWENTY TWO***

Materials: Square grid paper.

Objective: To discover the distributive law.

Use square grid paper as shown in Figure 15 to demonstrate distributive law.

What is the area of the rectangle on the left side? (3x4), that is 12 sq. units.

What is the area of the rectangle on the right side? (3x3), that is 9 sq. units.

What is the area of the whole rectangle? (3x7), that is 21 sq. units

Show that 3x4 + 3x3 = 3 x (4+3)

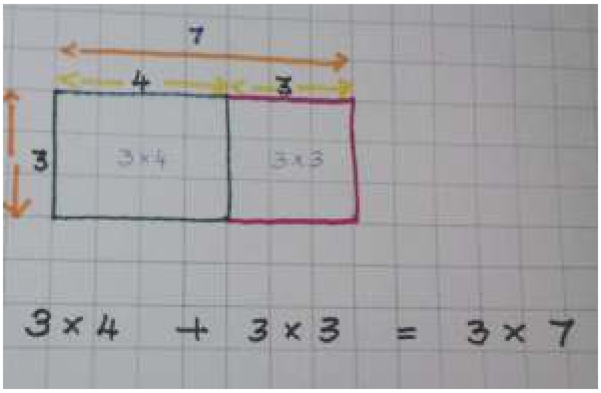

Figure 15

ACTIVITY ***TWENTY THREE***

Materials: Paper rectangles.

Objective: To discover the relationship between a triangle and its corresponding rectangle.

Let children fold some rectangles in half along the diagonal line and cut them along that line.

By placing one piece over the other they see that the two pieces are equal in size (area).

Help them to articulate their discovery that the area of a right-angle triangle is half of its length times its breadth.

It can also be demonstrated for any triangle as shown in Figure 15a.

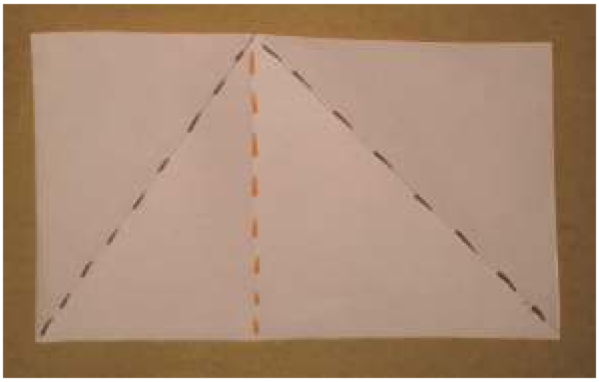

Figure 15a

ACTIVITY ***TWENTY FOUR***

Materials: To notice relationships in areas.

Objective: Ask children to take a square paper. Locate the midpoint of each side through folding. Draw connecting lines between adjacent midpoints as shown in Figure 16.

What is the relationship of the smaller square to the larger square? Can you show why?

Repeat the same process with the smaller square and draw another square inside it?

What is the relationship of this square with the bigger square? Can you show why?

The side of a large square is 20 cm. It is repeatedly folded three times by joining midpoints of the sides to produce smaller and smaller squares. What will be the area of the fourth square?

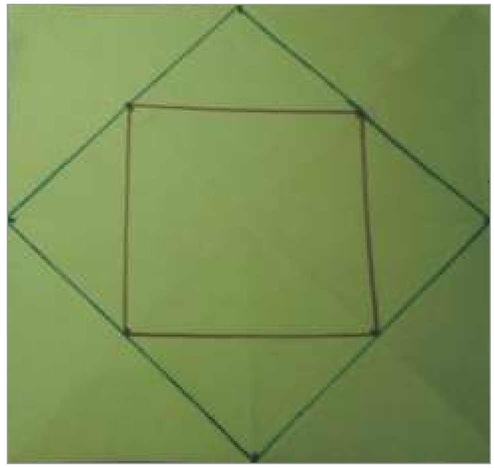

Figure 16

ACTIVITY ***TWENTY FIVE***

Materials: Square grid paper.

Objective: Spotting all possible squares, finding the shape with the largest perimeter

Outline a 6x6 square in a grid paper. How many squares of area 4 sq. units can be found here?

See Figure 17.

Here are some shapes that can be drawn in a 5 x 5 square. Draw some more shapes to find the shape that will give the largest perimeter.

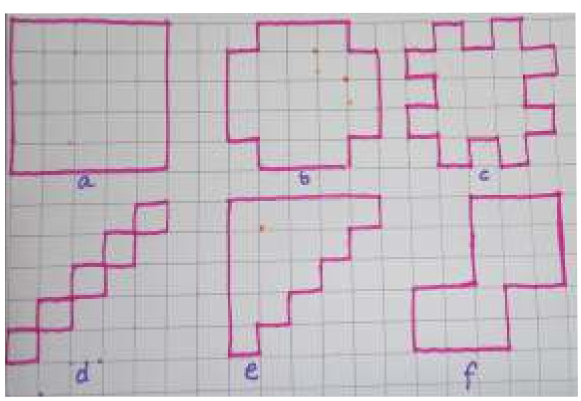

Figure 17

ACTIVITY *TWENTY SIX*

Materials: Square dot paper.

Objective: To reconstruct shapes based on data.

This is a class activity. Ask each group to draw some rectangles and squares on square dot paper. Without showing others the shapes they have drawn, each group shares the perimeter and area of these shapes with other groups.

'Perimeter of my figure is 12 units; area is 9 square units'

'Perimeter of my figure is 16 units; area is 12 square units'

'Perimeter of my figure is 14 units; area is 12 square units'

Each group has to try and create shapes which match the specifications of the other groups.

The same activity could be repeated by drawing triangles.

Triangle's area is 5 sq. units

Triangle's area is 8 sq. units

More challenging questions can be posed by the teacher:

'Find the side length of a square that has the same area as an 8 x 2 rectangle.'

'A square of side 3 cm is formed using a piece of wire. If the wire is straightened out and then bent to form a triangle with equal sides, what will be the length of each side of the triangle?'

'A rectangle of side 5 cm and 3cm is formed using a piece of wire. If the wire is straightened out and then bent to form a square, what will be the length of the square?'

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com