Azim Premji University

A publication of Azim Premji University
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Rishi Valley

COMMUNITY MATH CENTRE
RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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Symmetry & Asymmetry

Symmetry (from Greek συμμετρία symmetria "agreement in dimensions, due proportion, arrangement") in everyday language refers to a sense of harmonious and beautiful proportion and balance.

We can observe symmetry in mathematics – particularly in geometry- in architecture, in music and in many, many other spheres of life. It appeals to our aesthetic sensitivities, our sense of rightness, our balance. Wellness is often linked to balance - of yin and yang, of sukha and sthira, of proactivity and reactivity.

Can asymmetry be disturbing?

Can it be productive?

Will it strive to correct or be corrected?

From the Editor's Desk . . .

2016 as a number is very interesting mathematically [see page - 26] of this issue for a variety of ways you can arrive at it – in *At Right Angles*, our aim is to ensure that our issues for the year 2016 will have a potpourri of mathematical articles that spark your interest and bring greater splashes of insight into your study of the subject.

The first issue of the year begins with Geetha Venkataraman's article on Symmetry which will have you looking at everything around you with new eyes and bring mathematics into every heritage site you visit. J. Shashidhar takes you on a magical tour of Infinity. This is followed by a variety of articles on triangles; would you have guessed that this simple polygon could open your eyes to such richness of mathematical concepts, conceptual structures and validation procedures? Look at it through the lenses provided by D. Maneesha, K.D. Joshi, A.Ramachandran, Swati Sircar and Sneha Titus among others. Very appropriately for the first issue of this year, 2016 is a triangular number. In fact, as pointed out in one of the fillers, 2016 is the only triangular year number that we're likely to come across in our lifetime!

In Features, Shailesh Shirali continues his series on elegant proofs; if seeing is believing, then proving is evangelizing! Peeking into ClassRoom, we have Bharat Karmarkar giving meaning and depth to a mundane procedure by building patterns into the procedure of rationalization of surds. Sangeeta Gulati takes us through some Desmos Activities that bring fun, investigation and the power of technology into the mathematics classroom. We have re-introduced the Number Crossword from this issue onwards. In Review, Tanuj Shah has looked at Ian Stewart's book 'Taming the Infinite' and it certainly seems worth reading. And Padmapriya Shirali wraps up the issue with Word Problems – historically an area of difficulty for school students.

AtRiA has some changes in the Editorial Committee this year. We are grateful to Gautam Dayal and Srirangavalli Kona for their contributions to the magazine, and we welcome onboard A. Ramachandran, Sangeeta Gulati and Swati Sircar. AtRiA is now present in four spaces, and if your curiosity is sparked by this statement, do look at the back cover for more details. We have also reworked some of the design elements; let us know if you like the new look of the magazine. As always, your comments and inputs are welcome at AtRiA.editor@apu.edu.in

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Reviews

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Features

Through the Symmetry Lens

Part I - Planar Symmetry and Frieze Patterns

Geetha Venkataraman

Symmetry seems to be very much a part of our genetic make-up. Even a young child, unschooled in matters, is able to differentiate between symmetrical or regular objects as compared with those that are irregular. Our hearing is tuned to recognise symmetry in rhythm, music and beats. We see beauty in symmetry of monuments, designs, decorations and art.

The aim of this article is two-fold. The first is to introduce the reader to the intuitive as well as the mathematical concept of symmetry. The other is to use the knowledge of symmetry to see the world around us. In a sense the aim here is to provide spectacles capable of discerning symmetry in our daily life and to use such a device to appreciate the many examples that prevail and that are so filled with symmetry.

This article has been written in two parts. Part-I covers an intuitive and mathematical approach to symmetry and discusses symmetries of two-dimensional objects or shapes that can be drawn on a sheet of paper as well as symmetries of certain two-dimensional infinite patterns known as frieze patterns or strip patterns. Part-II of the article focuses on wallpaper patterns and their symmetries.

Keywords: *symmetry, regular, plane, frieze, pattern, rotation, reflection, translation, glide, art, architecture*

What is Symmetry?

A basic introduction to symmetry of finite objects was given in the Review 'Of Monsters and Moonshine: A review of symmetry' by Marcus Du Sautoy, published in *At Right Angles*, Vol. 3, No. 1, March 2014. We shall augment the same here.

Intuitively, symmetry can be thought of as an action performed on an object, which leaves the object looking exactly the same and occupying the exact same place as before. To illustrate, think of a two-person game with one person as the 'doer' who performs the action on the object and the second person as the 'viewer' who can see the action performed by the doer. Imagine however that the viewer closes her eyes while the action is performed. When she opens her eyes, if it seems as though nothing has happened to the object—i.e., it is in exactly the same state and position as it was at the start—then the doer's action is said to be a *symmetry* of the object.

If on the other hand the viewer spots a change in the state of the object—say, that it has moved or has been broken—then the action is not a symmetry. A word of caution here: this game is not based on the viewer's perception; indeed, we assume for the purposes of the game that the viewer will be able to spot any change if it has occurred.

To illustrate further, let us take a simple geometric object like a square. The reader can make a cutout of a square and label the vertices A, B, C, D in the anti-clockwise direction. The labelling is simply a device used to track a symmetry. For, if the viewer has declared that nothing has changed in the object, then how do we even know that a symmetry has occurred? The markings are not considered part of the square.

The reader should also mark the back of the square with the corresponding vertices on the front and back coinciding. She should then place the cutout on a blank sheet of paper and mark its outline. The vertex labels on the outline square should correspond to those of the cutout.

In the initial state the cutout is placed within the outline so that the vertex markings coincide. The figure below shows a light-blue cutout placed in an outlined square so that the corresponding vertices match.

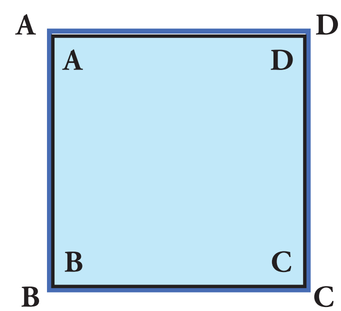

Figure 1

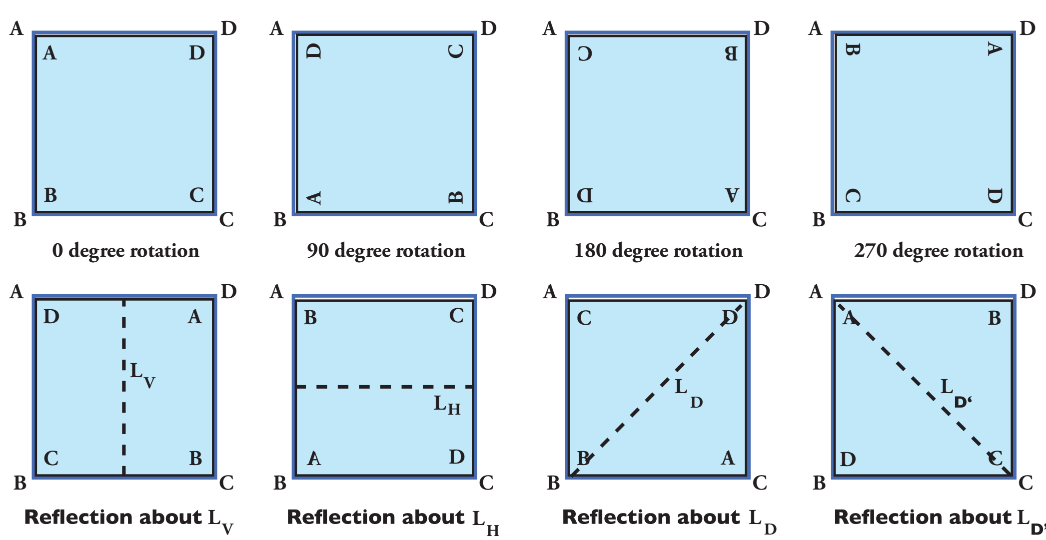

Figure 2

A symmetry of the square is an action that we can perform on the cutout such that it stays fully within the outlined square even after the action is performed. But now since the vertices have been labelled, we can track the symmetry by comparing the labels on the cutout and on the outline.

If the reader experiments with the cutout and the outline, she will be able to discover for herself that a square has 8 symmetries, as illustrated in the pictures below.

The reader should convince herself that a square has these 8 symmetries and no other. This can be seen by pinning the square cutout at the centre and rotating in the counter-clockwise direction. Only when 0°, 90°, 180° and 270° rotations are performed will the cutout fit into the outline. At other angles, the cutout will not fit into the outline.

Similarly folding the cutout along lines other than L_V, L_H, L_D, $L_{D'}$ will not see the two parts on the either side of the line overlapping exactly. So these are the only four lines about which reflection can take place. A reflection symmetry can also be seen by flipping the cutout along these lines.

From Figure 2 we note that the vertices in the cutout change position (or not) according to the symmetry. The relative positions of the vertices in the cutout with respect to those of the outline help us describe the symmetry. Figure 3 shows how the 8 symmetries of the square are tracked.

What we have described above is an intuitive definition of symmetry and we have also described a method for keeping track of the symmetry. While we can give a very general mathematical definition, we will confine our discussion to objects in a plane or to 3-dimensional space.

Let X denote an object in either a plane or space. We will think of X as a collection of coordinates (either two-tuples for a planar object or three-tuples for a 3-dimensional object). **A symmetry of an object X is a bijective function from X to itself, which preserves the distance between any two points of X.** In other words, a symmetry f of X is defined as follows:

1. **Function**: f is a rule which assigns to each point x in X, a unique point in X itself, called the *image* of x and denoted by $f(x)$.
2. **Injective**: Distinct points of X get mapped to distinct points in X. That is, if a and b are two different points in X, then $f(a)$ and $f(b)$ will be different points of X.
3. **Surjective**: Every point d in X is the image of some point c of X. That is, given d in X, there exists c in X such that $f(c) = d$.
4. **Distance preserving**: For u and v in X, let $d(u, v)$ denote the distance between the two points u and v. Then f is 'distance preserving' if for all points u and v in X, it happens that $d(u, v) = d(f(u), f(v))$. That is, the distance

0 degree rotation	90 degree rotation	180 degree rotation	270 degree rotation
A ⟶ A	A ⟶ B	A ⟶ C	A ⟶ D
B ⟶ B	B ⟶ C	B ⟶ D	B ⟶ A
C ⟶ C	C ⟶ D	C ⟶ A	C ⟶ B
D ⟶ D	D ⟶ A	D ⟶ B	D ⟶ C
Reflection about L_V	**Reflection about L_H**	**Reflection about L_D**	**Reflection about $L_{D'}$**
A ⟶ D	A ⟶ B	A ⟶ C	A ⟶ A
B ⟶ C	B ⟶ A	B ⟶ B	B ⟶ D
C ⟶ B	C ⟶ D	C ⟶ A	C ⟶ C
D ⟶ A	D ⟶ C	D ⟶ D	D ⟶ B

Figure 3

between any two points u and v is the same as the distance between their images $f(u)$ and $f(v)$.

We will denote by **Sym (X)** the set or collection of all symmetries of an object X.

There are some interesting observations we can make regarding symmetries. These can be seen intuitively using the working definition of symmetry or via the mathematical definition given above.

Symmetry Groups

Let f, g, h be symmetries of an object X. (The reader might find it helpful to think of X as a square.) We note that if we apply f first and then g then the result is again a symmetry of X which, we denote by $g * f$. In mathematical terms, $*$ represents the **composition** of the functions, and the above statement says that $*$ is **closed** on Sym (X). It can also be shown that $*$ is **associative**, that is, $(f * g) * h = f * (g * h)$. Further if we consider the 'do nothing' function defined as $I_X(a) = a$ for all a in X, then $f * I_X = f = I_X * f$. We say that I_X is an **identity** with respect to $*$. We can also show that for each symmetry f of X there exists a symmetry denoted as f^{-1} such that $f * f^{-1} = I_X = f^{-1} * f$. The symmetry f^{-1} is called the **inverse** of f. It simply reverses the action of f.

Thus $*$ on Sym (X) is a good way to combine symmetries, and $*$ is closed and associative; and identity and inverses exist with respect to $*$. We say that Sym (X) is a **group**[1] with respect to $*$ and we call Sym (X) the **group of symmetries** of X. If $f * g = g * f$ for all f and g in Sym (X), we say that Sym (X) is an **abelian** group.

The group of symmetries of an object helps in measuring symmetry, or the degree of regularity of the object. The larger the group of symmetries of an object, the more regular the object would be, and vice-versa. For example, if we were to consider Sym (X) where X is a quadrilateral, then Sym (X) would have the largest size when X is a square. Note that a circle has infinitely many symmetries, because rotation by any angle about the centre is a symmetry as is reflection about any diameter.

If X is a regular n-sided polygon, its group of symmetries is called the **dihedral** group of degree n. It is denoted as D_n. This group has $2n$ symmetries: n rotations and n reflections. The group of 8 symmetries of a square is denoted by D_4.

A useful learning device for studying a finite group is a *Cayley Table*. To make a Cayley Table for D_4 we create a 9 by 9 grid where the top row and first column list the eight symmetries of a square in the same order. The left topmost cell is empty. The (i, j) entry in the grid will be the symmetry $g * f$ (the symmetry f followed by the symmetry g). Here g is the entry in the first column and i$^{\text{th}}$ row, and f is the entry in the first row and j$^{\text{th}}$ column.

Figure 4 shows the Cayley Table for D_4 with only (5, 5) and (5, 6) positions filled. The reader may wish to use the cutout to compute the other

Cayley Table for D_4

	R_0	R_{90}	R_{180}	R_{270}	L_V	L_H	L_D	$L_{D'}$
R_0								
R_{90}								
R_{180}								
R_{270}								
L_V					R_0	R_{180}		
L_H								
L_D								
$L_{D'}$								

Figure 4

[1]A non-empty set G with a binary operation $*$ is a group if $*$ is associative, and identity and inverses exist with respect to $*$. For example, the set of integers is a group under addition.

entries. Note that R_0 will be the identity for D_4. Since $L_V * L_V = R_0$, L_V is its own inverse. The Cayley Table can be used to find the inverses and also check if the group is abelian.

In the rest of this article we will concentrate only on planar objects X. A few points about rotations and reflections that we ought to consider are discussed next.

In order to describe a rotation, we must specify the point about which the rotation takes place. This point will be referred to as the **rotocentre**. Also the direction of rotation must be specified. For planar figures, the axis of rotation will pass through the rotocentre and will be perpendicular to the plane of the object.

Every object X possesses the **do-nothing** symmetry or the $0°$ rotation symmetry. This symmetry is nothing but the identity function I_X of X. If X is a scalene triangle, then $\text{Sym}(X) = \{I_X\}$, that is, the only symmetry it possesses is the do-nothing symmetry.

Apart from the $0°$ rotation, all other rotational symmetries R of a planar object have the property that just one point remains unchanged by R; namely, the rotocentre itself. R will map every other point to a point different from itself.

In the case of a reflection symmetry L of a planar object, there is a line about which the reflection takes place. In other words, if we imagine a mirror placed along the **line of reflection (lor)**, then L maps points to their mirror images. All points on the lor are fixed by L (mapped identically to themselves), whereas points not on the lor are mapped to their mirror images which are different from themselves.

It can be proved mathematically that *finite planar objects have only rotational and reflection symmetries*. Here, 'finite' means that a rectangle can be drawn such that the object lies entirely inside the rectangle. Another result that is very interesting says that *if a planar object has only finitely many symmetries, then it will either have only rotational symmetries or an equal number of rotational and reflection symmetries.*

In mathematical language, the result can be restated as follows. Let X be a planar object, and suppose that Sym (X) is a finite group. Then Sym (X) is either a cyclic[2] group with n elements which are only rotations, denoted as C_n, or it is the dihedral group D_n with $2n$ elements, namely, n rotations and n reflections.

Figure 5 shows an example of an object with symmetry group C_4. The object has no reflection symmetries and only four rotational symmetries of $0°, 90°, 180°, 270°$ about the point of intersection, in the anticlockwise direction. The reader should draw a Cayley Table for C_4.

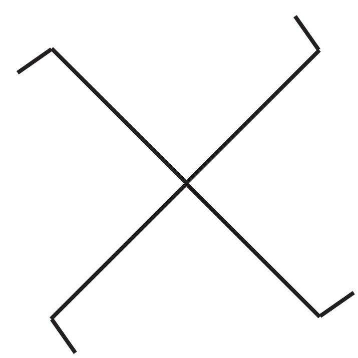

Figure 5

Strip Patterns or Frieze Patterns

We now turn our attention to infinite planar objects of certain types which will also help us analyse symmetry around us. The aim is to briefly introduce strip or frieze patterns.

A strip pattern is created by choosing a basic motif and repeating it at equal intervals to the left and right along a horizontal line. One can imagine the number line with the basic motif sitting at every integer place. A strip pattern is an infinite pattern that runs along a line in both directions. Any line would do but for ease we work with a horizontal line. Consider the example of a strip pattern (see Figure 6) made from repeating a square motif at

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[2]A group G is called *cyclic* if there is an element a in G such that every element of G is basically a composed with itself finitely many times or a^{-1} composed with itself finitely many times. In other words every element of G is of the form a^m for some integer m where $|m|$ represents the number of times either a or a^{-1} have been composed.

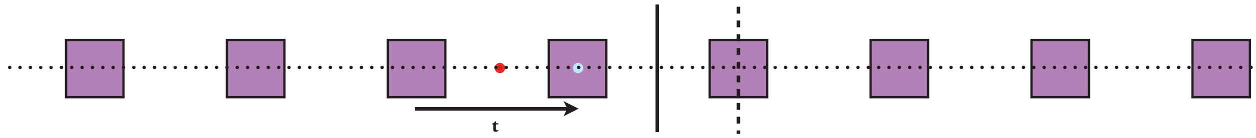

Figure 6

equal intervals. Remember that the pattern is infinite and continues indefinitely both to the left and to the right.

A strip pattern has a new type of symmetry not seen in the case of finite objects. In Figure 6 we have marked an arrow of a certain length **t** (the distance between two motifs), pointing to the right. If we move the entire strip by a distance **t** along the line to the right then we see that the strip will occupy the same position as it did originally. So this describes a symmetry called a **translation**. We denote it by T.

For any positive integer m, we denote by the symbol T^m the symmetry we get by moving the strip a distance $m\mathbf{t}$ to the right along the line, and by T^{-m} the symmetry we get by moving the strip a distance $m\mathbf{t}$ to the left along the line. The do-nothing symmetry will also be thought of as a translation symmetry in which we move by a distance of 0. This is also denoted as T^0.

Note that every strip pattern will have infinitely many translation symmetries. Indeed, formally, a finite object is one that does not possess a non-trivial translation symmetry (i.e., a translation symmetry which is not the do-nothing symmetry).

We say that two lines of reflection or two rotocentres of a figure are of the same type if there is a symmetry of the figure which takes one to the other. Otherwise they are said to be of different types.

In the above example, we see that there is only one horizontal reflection symmetry and vertical reflection symmetries along two different types of lines of reflection (denoted by solid and dotted lines). There are also 180° rotations in the anti-clockwise direction about two different types of rotocentres (denoted by red and pale blue circles).

In general for a strip pattern, the only rotational symmetries possible are of 0° and 180°. A 180° rotation symmetry may or may not exist. If a 180° rotational symmetry with a certain type of rotocentre exists, then there will be infinitely many rotational symmetries with that same type of rotocentre. (In our example, the centre of each square is a rotocentre of the red type.)

Similarly a reflection symmetry about a horizontal line or vertical line may or may not exist. If there is a reflection symmetry about a horizontal line, then it is unique (i.e., there can be only one such line). If there exists a reflection symmetry about a vertical line of a certain type, there will be infinitely many reflection symmetries about vertical lines of the same type. (In our example, there are infinitely many vertical lines of reflection passing through the centres of the squares.)

We had seen earlier that for any planar object X, the set Sym (X) of all symmetries of X is closed under composition of symmetries. In this context it might be useful to consider what symmetry we get if a reflection is followed by a translation or vice-versa. In general, this yields a new kind of symmetry that is neither a reflection, nor a rotation and not even a translation. This new symmetry is called a **glide reflection**. Thus if T is a translation symmetry and R is a reflection symmetry of an object X, then $R * T$ and $T * R$ are both symmetries knows as glide reflections. A glide reflection can also be defined independently as follows.

Consider Figure 7. The one on the left shows the effect of a reflection followed by a translation on the letter **R**. The one on the right shows the glide reflection, which is the composition of the reflection followed by the translation. (The shadow **R** in the figure on the right shows the intermediate position of **R** after undergoing a reflection about the dotted line.)

Thus a glide reflection is defined as reflection followed by a translation in a direction parallel to

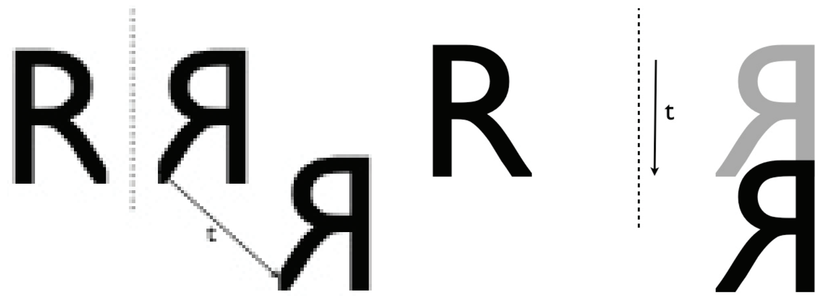

Figure 7

the line of reflection. The important fact for a glide reflection is that the translation has to occur in the same direction as that of the line of reflection. A strip pattern may or may not have a glide reflection.

It is possible to classify strip patterns on the basis of the combinations of possible symmetries of a strip pattern, namely, translation, horizontal reflection, vertical reflection, 180° rotation and glide reflection. All strip patterns are made up of just these basic elements. As may be expected, therefore, there are not too many such patterns possible. Analysis reveals that the number of different strip patterns is just seven.

Examples of the seven strip patterns are given in Figures 8 (a) and 8 (b).

The arrows marked in the patterns show the glide length, the distance a unit of the motif has to travel before the reflection takes place. It is possible for the glide length to be different from the basic translation length for a strip pattern.

Note that in Type II, the basic motif consists of two units, a standing P and an upside down P. Thus the basic translation will be the distance between two successive motifs and in this case we could take it to be the distance between the two successive standing P's. This is different from the glide length shown by the length of the arrow in the pattern.

Similarly in Type V, the basic motif consists of 4 units (P, reflected P, upside down P and its reflection). Here too the glide length and the basic translation length differ. However in Type VII, the glide length is also the translation length.

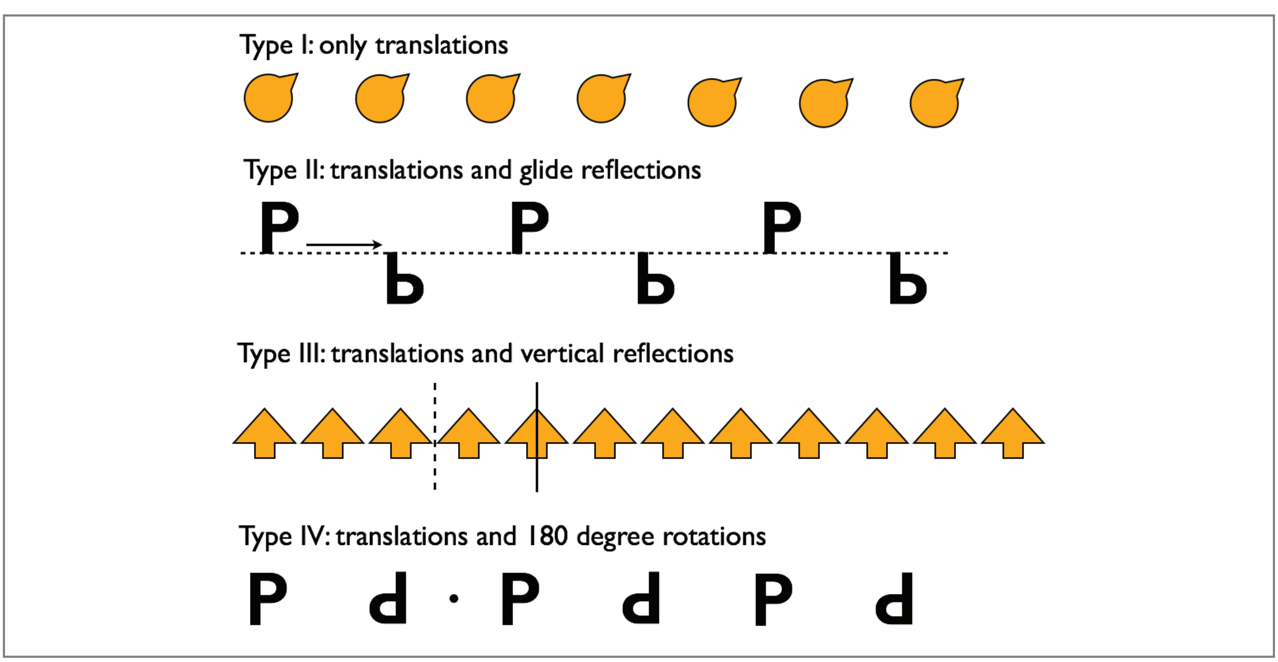

Figure 8 (a)

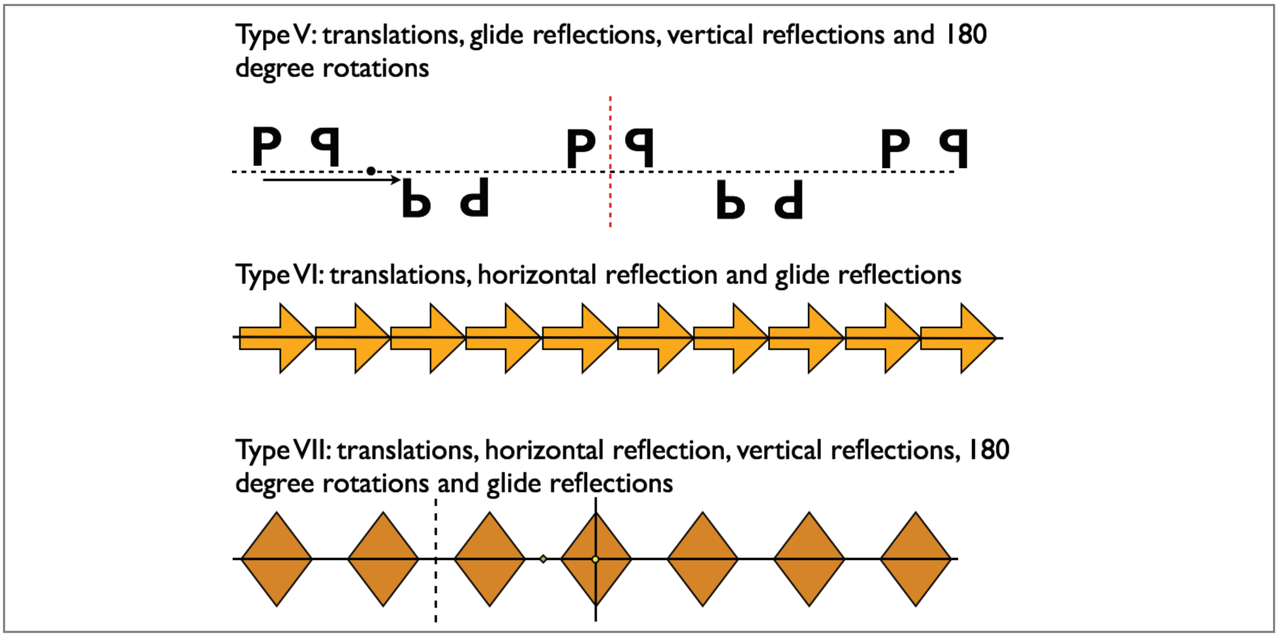

Figure 8 (b)

We conclude Part I of this article by giving examples of strip patterns that can be found in the clothes that we wear, in monuments and contemporary buildings, in fences and balcony railings among others. In a few cases we will classify these. The others are left as an exercise for the reader to undertake classification into one of the seven types.

Figure 9 shows a hand block-printing pattern used on cloth. Such strip patterns can be typically found on borders of saris, shirts or kurtas, bedsheets, curtains etc. This strip pattern has translations, horizontal reflection, vertical reflections, glide reflections and 180 degree rotations. So it is of Type VII.

The next strip pattern (Figure 10) is a decorative motif from the Humayun's Tomb, a 16th century monument built after the Mughal emperor Humayun's death in 1556. The tomb located in the Nizammuddin area in Delhi has been restored

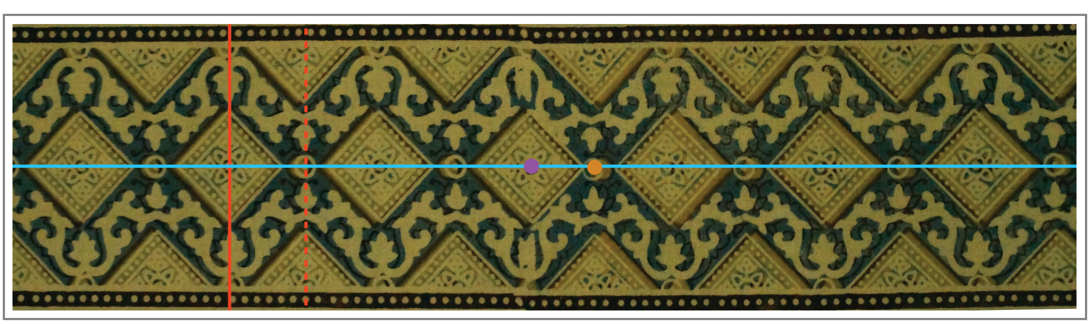

Figure 9: A strip pattern of Type VII

Figure 10: A strip pattern of Type III

Figure 11: A strip pattern of Type III

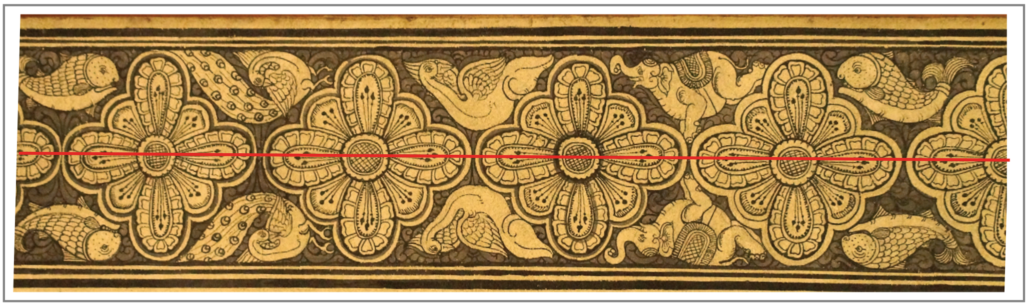

Figure 12: A strip pattern of Type VI

Figure 13

Figure 14

beautifully and is well worth a visit – not just because it has been restored beautifully, but also from the point of view of symmetry! As it has only translations and vertical reflections, this is a strip pattern of Type III.

The next two strip patterns (Figure 11, Figure 12) are decorative borders painted around a window in the Crafts Museum in Delhi. The style of the first painting is probably Madhubani from the state of Bihar, and the second one is probably done in a Patchitra style, which is a folk art form from Orissa. The strip pattern in Figure 11 is once again of Type III as it has only translations and vertical reflections.

On the other hand, the strip pattern in Figure 12 is of Type VI as it has translations, horizontal reflection and glide reflections.

Some more examples of strip patterns are given below, taken from the world around us. We invite the reader to classify them according to the symmetries present. The strip patterns in Figure 13 is an example of Kalamkari artwork from Andhra Pradesh.

The pattern in Figure 14 is again an example of hand-block printing on cloth.

The next two strip patterns (Figure 15, Figure 16) have been taken from a balcony railing and the side railing of a bridge.

The last two strip patterns (Figure 17, Figure 18) are borders or decorative motifs from a Buddhist temple in Seoul and the beautiful Gyeongbokgung Palace originally built in the 14th century and restored now, again in Seoul.

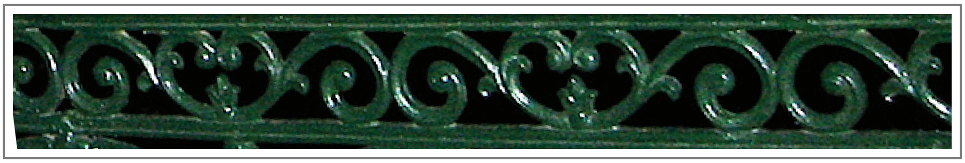

Figure 15

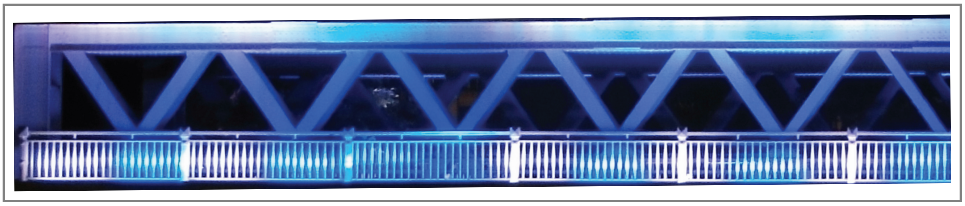

Figure 16

Figure 17

Figure 18

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The Magical World of Infinities

Part 1

To see a World in a Grain of Sand
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand
And Eternity in an hour.

From *Auguries of Innocence* by **William Blake**

Features

SHASHIDHAR JAGADEESHAN

Introduction

This article and its sequel hope to take the reader on a whirlwind tour of the infinities! For those who have never encountered these ideas, be ready for your world to be turned upside down and your intuition to be shot to pieces. Don't worry, you're not the only one who may react violently to the ideas presented below. Reputed mathematicians like Poincaré and Kronecker reacted with horror. Georg Cantor, who discovered these ideas, was literally driven to insanity because of the hostility he received, especially at the hands of Kronecker. But before we get caught up in this story, let us begin our journey into the world of Infinity.

Ready? Do you have your seat belts fastened? Then let us start with the so-called Hilbert hotel (an idea introduced by the famous German mathematician David Hilbert). The Hilbert hotel has an infinite number of rooms, and all the rooms in the hotel are full. Now if a new guest arrives, can you accommodate her? Some of you may have figured it out already! Yes, move each guest to the adjacent room, that is: ask the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, the guest in room 3 to move to room 4, and so on. As there are infinitely many rooms in

Keywords: *Hilbert hotel, Cantor, infinities, natural numbers, set of integers, set of rational numbers, one-to-one correspondence, interlacing, measure theory, cardinal number, cardinality*

the Hilbert hotel, you can accommodate everyone who has already checked in and you now have an empty room number 1, where you can accommodate the new guest!

Clever, you might say, but what if an infinite number of new guests arrive; can you accommodate all of them? The answer again is yes! This one needs a little more ingenuity. Here is what you do. Send the guest in room 1 to room 2, the guest in room 2 to room 4, the guest in room 3 to room 6, and so on. You have now managed to vacate room numbers 1, 3, 5 and so on. All the odd numbered rooms are vacant, and as there are infinitely many odd numbers, you can now accommodate the infinitely many new guests!

I am sure you think there is something fishy going on here. Now suppose an infinite number of new guests arrive in an infinite number of buses, can we accommodate all of them? The answer is still yes! It will be a while before we can show how to do it. In the meantime, let us put our common sense about infinite sets to some further testing!

1. *Which set has more elements, the set of all natural numbers or the set of even natural numbers?*
2. *Which set has more elements, the set of all natural numbers or the set of integers?*
3. *Which set has more elements, the set of all integers or the set of rational numbers?*
4. *In Figure 1, which circle has more points?*

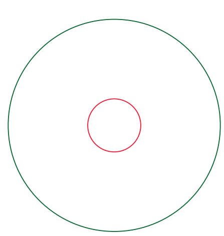

Figure 1

5. *In Figure 2, does the square have more points or does one of its edges?*

Figure 2

6. *In Figure 3, does the cube have more points or does one of its edges?*

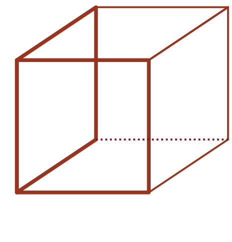

Figure 3

The answer to the first three questions is that in each case the two sets have the same number of elements, and the answer to the last three is that the two sets have the same number of points. Sounds crazy? Before we justify these answers, we need to lay some mathematical foundation to this madness!

One-to-one correspondence

You may have guessed that the basic issue here is: how do we compare infinite sets? How on earth can we say that one infinite set has more than, less than or equal number of elements as compared with another infinite set? A related question is, can we assign a 'value' to an infinite set?

Georg Cantor [1845–1918] was the first mathematician to dare to answer these questions. It is amazing that anyone could even conceive of the idea of counting infinite sets. What is even more startling is that the basic principle used is something even a young child can understand and uses all the time to count: the concept of **one-to-one correspondence**.

We say two sets A and B are in one-to-one correspondence (from now on, denoted by 1-1 correspondence) if there is a way of associating each element of A with a unique element of B, and similarly associating each element of B with a unique element of A. That is, every member of A has a unique partner in B, and every member of B has a unique partner in A. This principle is illustrated in Figure 4.

We see that there is a 1-1 correspondence between sets A and B. Moreover, we can also see that both sets A and B have the same number of elements.

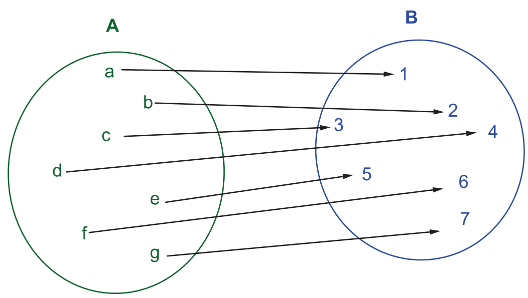

Figure 4

We often encounter this idea in daily life, for example if seats in a movie theatre are sold out, and if we know the theatre can house 472 seats, we can conclude (assuming there are no lap-tops!) that there are 472 people watching the movie.

Figure 5 shows an example of a function which is not a 1-1 correspondence. This is because more than one element in set C has the same partner in set D.

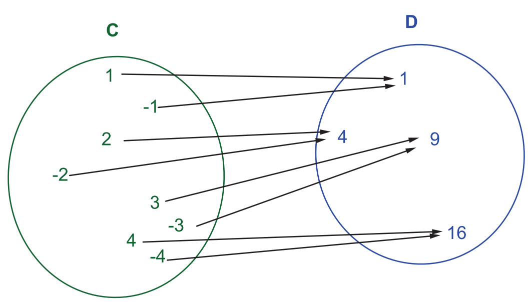

Figure 5

Cantor's brilliant insight was that the idea of 1-1 correspondence can be extended to infinite sets, and we can start comparing infinite sets and even count them. We say two sets A and B have the same number of elements if there is a 1-1 correspondence between them. In mathematical notation we write: $|A| = |B|$.

So in order to answer our six baffling questions, we need to create 1-1 correspondences between the pairs of sets in question. For example, if we want to compare the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

with the set of even natural numbers

$$E = \{2, 4, 6, \ldots\},$$

we need to find a 1-1 correspondence between them. This one is easy! We send each natural number to its double; in function notation this would be:

$$f : \mathbb{N} \to E, \quad f(n) = 2n, \quad \text{where } n \in \mathbb{N}.$$

It is easy to see that for every natural number, we have associated an even number, and similarly for every even number we have associated a natural number. This is exactly the principle used in accommodating the infinitely many passengers who arrived at the Hilbert hotel.

Now because we have established a 1-1 correspondence, we can say that there are as many natural numbers as even numbers. What about odd natural numbers? We leave it as an exercise to show that there are as many natural numbers as odd natural numbers. Here we see the first among many strange aspects of infinities: a set and its subset can have the same size! Our intuition that the whole is always greater than its part goes out of the window!

What about the set of integers? Surely, since they also have the negative numbers, there should be more integers than natural numbers? We use a technique called *interlacing* (frequently used by Cantor) to demonstrate pictorially a 1-1 correspondence (see Figure 6). This is of course not a proof; would you be able to construct one? In fact since many of the proofs will be rather technical, we are going to use pictures to indicate how the proofs work. Interested readers can look up the references below for formal proofs.

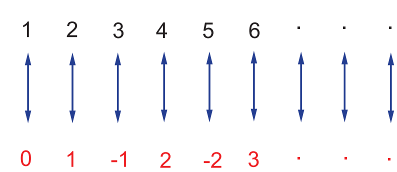

Figure 6

The question of whether the number of rational numbers equals the number of integers is a little more difficult, so we first tackle the question of the circles before we move to the other questions. Here is an easier question: In Figure 7, which line has more points?

Figure 7

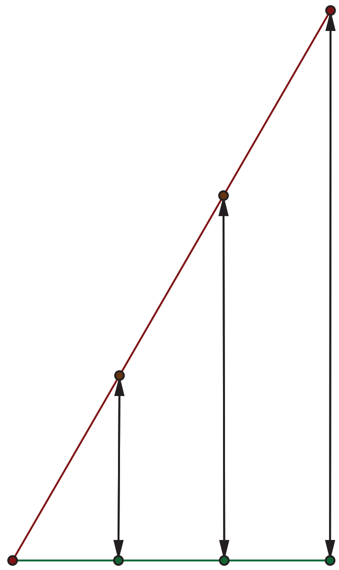

Figure 8

Consider the 1-1 correspondence shown in Figure 8. Readers will have to convince themselves that this can be done for any two pairs of lines with any length. The 1-1 correspondence clearly shows that the two lines have the same number of points. So, out goes the notion that longer lines have more points. (This raises an important question in a field of mathematics called Measure Theory: how does one define the length of a line?)

To answer the question about circles, once again we offer a pretty picture (Figure 9) and, a la Bhaskara, we say "behold"!

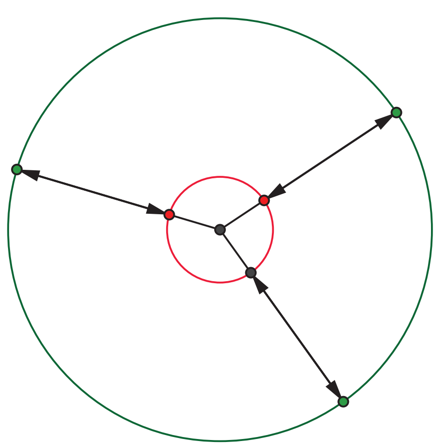

Figure 9

Before we go on to the question of the size of the set of rational numbers, let us look at one more comparison of sets of points. Consider all points in the interval $(-1, 1)$ on the real number line, and the real number line itself. Which has more points? By now, out of fatigue, you are probably saying 'the same'! Can we construct a 1-1 correspondence?

Here we can actually give the explicit 1-1 correspondence by the function

$$g : (-1, 1) :\to \mathbb{R}, \quad g(x) = \frac{2x}{1-x^2}.$$

Using algebra, you can establish that for every $t \in (-1, 1)$, $g(t)$ is a real number; for every real number s, $-1 < g(s) < 1$; and $g(t)$ does not repeat values, that is, if $g(a) = g(b)$ then $a = b$. This 1-1 correspondence shows that there are the same number of points in a line of finite length and in a line with infinite length. What is more, it shows that there are as many real numbers overall as there are between -1 and 1!

Let us look at the graph (Figure 10), where the 1-1 correspondence is self-evident (but not a proof!).

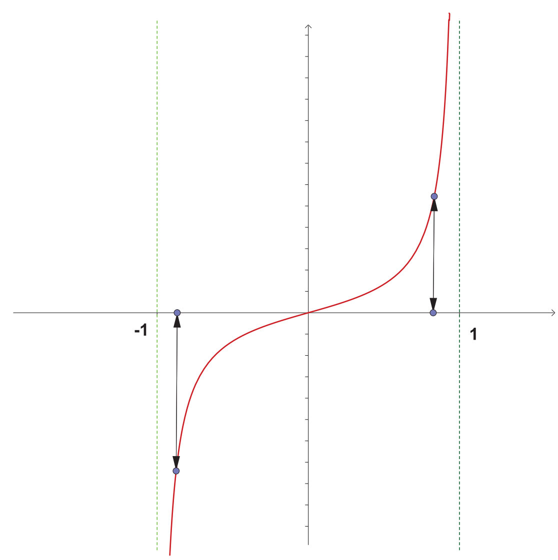

Figure 10

Counting the rational numbers and returning to the Hilbert hotel

Now we can take on the question about rational numbers. Here is where we encounter the genius of Cantor. He came up with the 1-1 correspondence shown in Figure 11 between the integers and the set of rational numbers. The explicit 1-1 correspondence is as shown in Figure 12.

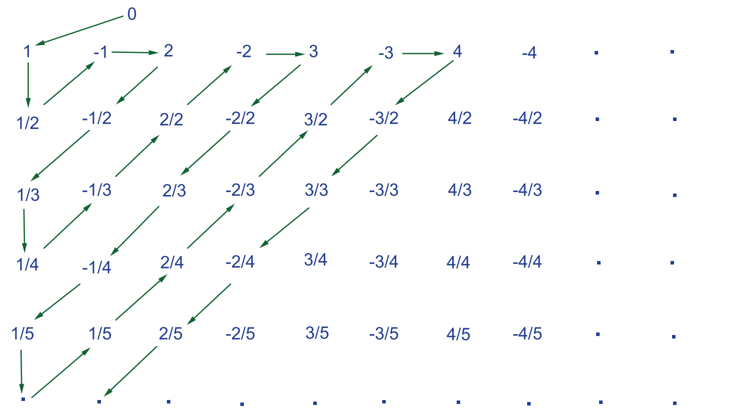

Figure 11

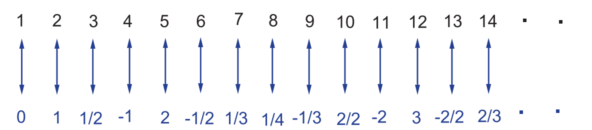

Figure 12

You might have noticed that the rational numbers repeat themselves, several times over. Does this cause a problem? Not really, because we have two options. One is to systematically skip any rational number that is repeated. Alternatively we use the following argument. Denote the collection of rational numbers with their repetitions by $\mathbb{Q}^*$. So we have shown that $|\mathbb{Q}^*| = |\mathbb{N}|$. Notice that $\mathbb{Q} \subset \mathbb{Q}^*$ and $\mathbb{N} \subset \mathbb{Q}$. Using a famous theorem (the Schroeder-Bernstein theorem), we can say $|\mathbb{Q}| \leq |\mathbb{N}| \leq |\mathbb{Q}|$ and hence $|\mathbb{Q}| = |\mathbb{N}|$.

Interestingly, the 1-1 correspondence shown above helps us solve the following question: if infinitely many buses arrive at the Hilbert hotel, each with infinitely many passengers, can we accommodate them? Of course we can. Let us look at Figure 13 which is a modification of Figure 11.

As you can see from Figure 13, we have arranged the infinitely many buses with their infinitely many passengers in rows and columns (so $B_2 4$ represents passenger 4 in bus 2). To accommodate all of them, once again shift each occupant as we did earlier to a room bearing double the number, and create infinitely many vacant rooms. We now use Cantor's method of criss-crossing to assign rooms to the passengers from the infinitely many buses, using the allocation shown in Figure 14.

How many infinities are there?
So far we seem to have encountered two kinds of sets. One is the set of rational numbers and its

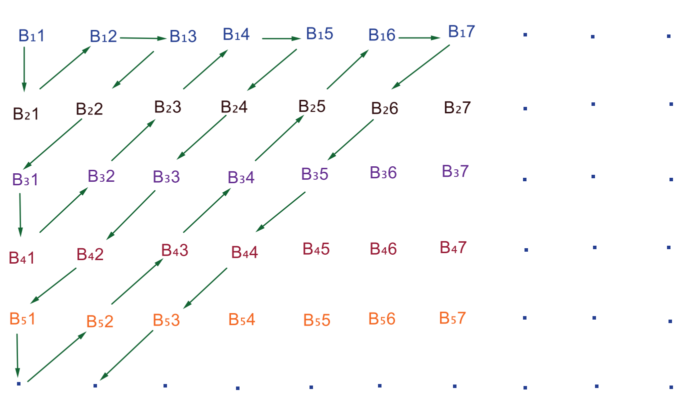

Figure 13

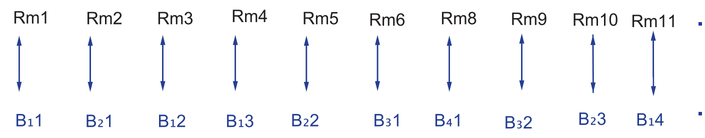

Figure 14

subsets, the integers, whole numbers and the natural numbers. We have seen that they all have the same number of elements in them. We also met points on curves and lines, and have been trying to compare them with each other and the real line. Here too we found that points on a line or a circle, no matter how long, have the same number of points. Now here's a question: how does the set of real numbers compare with the set of natural numbers? From our experience so far, we may be tempted to conclude that they are the same, especially when all our intuition about sizes of sets has been systematically destroyed! Once again Cantor pulls the rabbit out of the hat and shows that actually the set of real numbers has more elements than the set of natural numbers. In our language, he showed that no matter how clever you are, you cannot find a 1-1 correspondence between the set of natural numbers and the set of real numbers!

He went on to show much more. He denoted by the letter $\aleph_0$ the size (in mathematics: cardinality) of the set of natural numbers. The symbol $\aleph$ is the first letter in the Hebrew alphabet. He represented the cardinality of real numbers by **c**, the so-called 'continuum'. So, he showed

$$|\mathbb{N}| = \aleph_0 < \mathbf{c} = |\mathbb{R}|.$$

He assigned a symbol to represent the cardinality of any given infinite set (these symbols are called cardinal numbers), and showed that there exists a hierarchy among cardinal numbers, and in fact showed that $\aleph_0$ is the smallest cardinal number in this hierarchy.

In the second part of this article, we will discuss these ideas and return to the questions about the number of points in a square and one of its edges, and a cube and one of its edges!

Perhaps some of you will agree with Hilbert (talking about Cantor's work on infinities in 1926), when he says:

This appears to me to be the most admirable flower of the mathematical intellect and one of the highest achievements of purely rational human activity.

Maybe you will get hooked on to infinities the way I am!

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Triangle Equalizers

A Probabilistic Approach

Part I

Features

KAPIL D JOSHI

An equalizer of a triangle is a line which divides it into two regions having equal areas as well as equal perimeters. Every triangle has at least one and at most three equalizers. In [1], triangles of all three types are identified and it is remarked that triangles with two equalizers are quite rare. In [2], an example of such a triangle is given. In this two-part note, we simplify the results of [1]. In particular, we show that in a certain probabilistic sense, the probability that a triangle taken at random has two equalizers is 0. The method is based on the roots of quadratic equations, a technique already initiated in [2]. In Part II, we estimate the probabilities that a triangle taken at random has a given number of equalizers. In particular, we show that more than 85% of the triangles have only one equalizer. Using *Mathematica*, the figure is found to be close to 98%.

Introduction and Terminology

As usual, given a triangle ABC, we denote its sides opposite to A, B, C by a, b, c respectively, its semi-perimeter $\frac{1}{2}(a+b+c)$ by s and its incentre by I. Suppose a line ℓ is an equalizer of ABC. Then it is easy to show (see, e.g. [2]) that ℓ passes through I. Consequently, if ℓ passes through some vertex, say A, of ABC, then it must be the line AI whence the triangle must be isosceles with $AB = AC$. In all other cases, an equalizer must cut two of the three sides internally. Without loss of generality, we take these sides to be AB and AC and denote their points of intersection with ℓ by X and Y respectively. We say that this equalizer is *opposite* to the side a. (See Figure 1.)

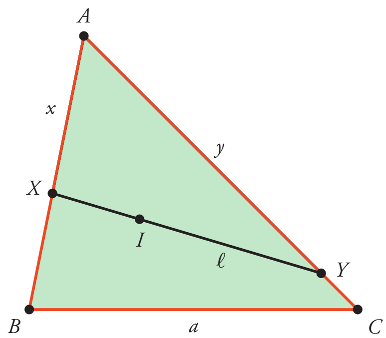

Figure 1. Equalizer opposite to a

Since a triangle can have at most three equalizers, the probability that a randomly taken line passing through the incentre of a triangle is an equalizer of it is 0. But the problem we address here is that of determining with what probabilities a randomly taken triangle has 1, 2 or 3 equalizers.

We first consider the problem of determining the conditions under which ABC has an equalizer opposite to a. For this we let $x = AX$ and $y = AY$. Then by definition of an equalizer, $x + y = s$ and $xy = \frac{1}{2}bc$. These equations together imply that x and y are the roots of the quadratic polynomial

$$f(p) = p^2 - sp + \frac{1}{2}bc. \tag{1}$$

A direct calculation gives

$$\begin{aligned} f(0) &= f(s) = \frac{bc}{2}, \\ f(a) &= \frac{(b-a)(c-a)}{2}, f(b) = \frac{b(b-a)}{2}, \\ f(c) &= \frac{c(c-a)}{2}. \end{aligned} \tag{2}$$

The existence of an equalizer opposite to a is equivalent to $f(p)$ having two (possibly equal) roots, one lying in the interval $(0, b)$ and the other in $(0, c)$. This observation, coupled with the equations above, enables us to determine the number of equalizers opposite to a depending upon how a compares with b and c. To avoid degeneracies, we first consider only scalene triangles, i.e., triangles ABC where a, b, c are distinct.

Numbers of Equalizers in Scalene Triangles

We assume that a, b, c are all distinct. We consider three cases depending upon where a is placed compared with b and c.

(i) Suppose first that a is the longest side. Then both $f(b)$ and $f(c)$ are negative. As the leading coefficient of $f(p)$ is positive, this means that b, c lie between the two roots of $f(p)$. Hence the larger of the two roots is bigger than both b, c, violating the requirement of an equalizer that one root must be less than b and the other one less than c. So in this case, though the quadratic (1) has real roots, there is no equalizer opposite to a.

(ii) Assume that a lies between b and c. Without loss of generality, assume that $b < a < c$. Then $f(0) > 0, f(b) < 0$ and $f(c) > 0$. So, by continuity of the quadratic function, f has at least one root in the interval $(0, b)$ and at least one root in the interval (b, c). As there are only two roots, there is exactly one root in $(0, b)$ and one in (b, c). So there is exactly one equalizer opposite to a, viz. the line XY with x representing the larger and y the smaller root. (Note that the roots cannot be interchanged because of the inequalities they must satisfy.)

(iii) Assume that a is the smallest side. This is the most interesting case. The quadratic (1) may not have any real roots in this case, as happens, for example, when $(a, b, c) = (3, 4, 5)$. But we claim that if at all (1) has a root, then every root gives rise to an equalizer opposite to a. (In [2], this is mentioned only as a possibility. We claim that it is a certainty.)

So assume that $s^2 \geq 2bc$, and let x, y be the roots of $f(p)$ with $x \leq y$. Without loss of generality, assume that $a < b < c$. Here $f(a), f(b), f(c)$ are all positive, so mere continuity of the quadratic is not of much help. Here we use the fact that the graph of $f(p)$ is a parabola with its lowest point at $p = s/2$ and symmetric about the line $p = s/2$. Figure 2 shows a sketch of the graph for

$0 \le p \le s$. The portion from 0 to $s/2$ is strictly decreasing, while that between $s/2$ and s is strictly increasing. Note further that x lies in the left half and y in the right half, i.e. $x \le s/2$ and $s/2 \le y$. (If $s^2 = 2bc$, then both the roots equal $s/2$, and the parabola touches the p-axis. This happens, for example, when $(a, b, c) = (7, 8, 9)$.)

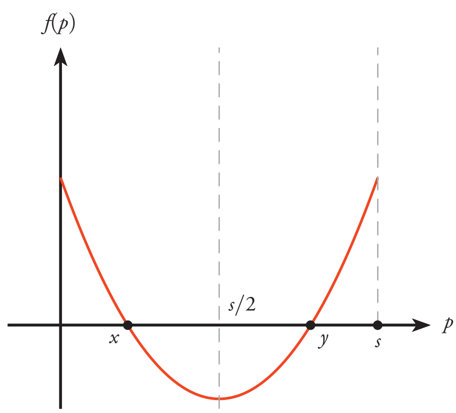

Figure 2. Graph of the quadratic $f(p)$

Since $f(b) > 0$, b cannot lie in the interval $[x, y]$. We claim that b lies to the right of this interval, i.e. that both x and y are less than b (and hence c too). For, if this is not so, then b and hence a too must lie to the left of x. But as f is strictly decreasing on $[0, x]$ this would mean $f(a) > f(b)$. But on the other hand $f(a) < f(b)$, a contradiction.

Thus we have shown that if at all (1) has a root, then both roots lie in the interval $(0, b)$ and hence in the interval $(0, c)$ too. If they are equal, there will be only one equalizer XY opposite to a. Moreover, it will be perpendicular to the angle bisector AI because $\triangle AXY$ is isosceles. But if the roots x and y are distinct, then XY will be an equalizer opposite to a and so will be $X'Y'$ where X', Y' are points on AB, AC at distances y and x respectively from A. (As already noted, a similar interchange was not possible in (ii) above.)

Summing up, we have shown that a scalene triangle has no equalizer opposite to its longest side, one equalizer opposite to its middle side, and either 0, 1 or 2 equalizers opposite to its shortest side. These three possibilities occur according as s^2 is less than, equal to or greater than, twice the product of the two longer sides. The three possibilities are illustrated by the triangles with sides $(13, 16, 19)$, $(14, 16, 18)$ and $(15, 16, 17)$ respectively, with s being 24 in each case.

The reasoning here can be modified to include cases of equality of some pairs of sides. Suppose, for example, that $\triangle ABC$ is isosceles with $AB = AC > BC$. Then a is the shortest side. Both b and c can be considered as the middle side if we drop the strictness of inequalities, because we have $a < b \le c$ and also $a < c \le b$. In this case, one root of the quadratic $p^2 - sp + \frac{1}{2}ab$ is b, and the other root is $\frac{1}{2}a$. So the equalizer opposite to b is simply the median through A. This is also the equalizer opposite to c.

Triangles with Two Equalizers

The above analysis provides a complete characterization of triangles having exactly two equalizers.

Theorem 1. *A triangle ABC has exactly two equalizers if and only if one of the following two possibilities holds:*

(i) *$\triangle ABC$ is scalene, and s^2 equals twice the product of the two longer sides.*

(ii) *$\triangle ABC$ is isosceles, and its unequal angle is equal to $2\sin^{-1}(\sqrt{2} - 1)$.*

Proof. The case of a scalene triangle was already considered above. An equilateral triangle has (at least) three equalizers; namely, its three medians. (It is easy to show that it cannot have any others. But that is not needed here.) So the only possibility left is one where $\triangle ABC$ is isosceles with $AB = AC \neq BC$ (say). Here we have $b = c$ and $a \neq b$. The median through A is an equalizer. It operates simultaneously as an equalizer opposite to both b and c. So, the second equalizer, if any, must be opposite to a. If $a > b$, then this is impossible; the reasoning is similar to case (i) of the scalene triangles, because in this case too, (1) cannot have any roots less than b (or c).

It remains to deal with the case where $b = c > a$. Here too, the argument in case (iii) for scalene triangles goes through and shows that there is a unique equalizer opposite to a if and only if $s^2 = 2bc = 2b^2$, i.e. if and only if $s = \sqrt{2}b$. But, on the other hand, $s = b + \frac{1}{2}a$. Hence:

$$\frac{a}{b} = 2(\sqrt{2} - 1). \tag{3}$$

The cosine formula (along with $b = c$) now gives

$$\cos A = \frac{2b^2 - a^2}{2b^2} = 1 - 2(\sqrt{2} - 1)^2$$
$$= 4\sqrt{2} - 5, \tag{4}$$

$$\therefore\ 2\sin^2 \frac{1}{2}A = 1 - \cos A = 2(\sqrt{2} - 1)^2, \tag{5}$$

which yields $A = 2\sin^{-1}(\sqrt{2} - 1)$. □

We remark that using the work done above, we can similarly give a complete classification of triangles with three equalizers in terms of the inequalities to be satisfied by its sides. This classification is simpler than that given in [1]. We omit it as our concern here is triangles which have exactly two equalizers. Later (Theorem 4, in Part II) we shall revisit the problem.

In [1], the angle $2\sin^{-1}(\sqrt{2} - 1) \approx 48°56'23'' \approx 49°$ is denoted by A_0, and the author regards it as full of surprise and drama. This angle also comes up as an upper bound on the smallest angle in any triangle which has exactly two equalizers as we now show. (Later we shall see that this angle also plays a crucial role in the calculation of the probabilities with which a triangle at random has a given number of equalizers.)

Theorem 2. *In a triangle with exactly two equalizers, the smallest angle can be at most equal to* $A_0 = 2\sin^{-1}(\sqrt{2} - 1)$.

Proof. The case of an isosceles triangle follows from (ii) of the last theorem. (In fact, here the smallest angle actually equals A_0.) Now suppose that (i) holds, i.e. that $\triangle ABC$ is scalene and has two equalizers. Without loss of generality, suppose $a < b < c$. Then we claim that $A \le A_0$ by showing that $\cos A \ge \cos A_0$. For such a triangle, we have $s^2 = 2bc$ and hence $(a + b + c)^2 = 8bc$, which implies $a = \sqrt{8bc} - b - c$. Putting this into the cosine formula and using the A.M.-G.M. inequality, we get:

$$\cos A = \frac{b^2 + c^2 - (\sqrt{8bc} - b - c)^2}{2bc}$$
$$= \frac{2\sqrt{8bc}(b + c) - 10bc}{2bc}$$
$$\ge \frac{4\sqrt{8}bc - 10bc}{2bc} = 4\sqrt{2} - 5 = \cos A_0,$$

which completes the proof. □

We now prove a result which is a sort of converse to this theorem.

Theorem 3. *Given any* α *with* $0 < \alpha \le A_0$, *there exists a triangle which has exactly two equalizers and whose smallest angle is* α. *Moreover such a triangle is unique up to similarity.*

Proof. The case where $\alpha = A_0$ is already settled by Part (ii) of Theorem 1. So suppose $0 < \alpha < A_0$. We construct a scalene $\triangle ABC$ in which $\measuredangle A = \alpha$, a is the shortest side and $s^2 = 2bc$. Indeed, we let s be arbitrary. In view of $s^2 = 2bc$, the requirement that $\measuredangle A = \alpha$ is equivalent to

$$\cos\alpha = \frac{b^2 + c^2 - (2s - b - c)^2}{s^2}$$
$$= \frac{4(b + c)s - 5s^2}{s^2},$$

which reduces to

$$b + c = \frac{(5 + \cos\alpha)}{4}s. \tag{6}$$

We solve this simultaneously with $s^2 = 2bc$ and find that b and c are the roots of the quadratic equation

$$q^2 - \frac{(5 + \cos\alpha)s}{4}q + \frac{s^2}{2} = 0. \tag{7}$$

By a direct calculation, the roots of this quadratic are

$$\frac{(5 + \cos\alpha) \pm \sqrt{(5 + \cos\alpha)^2 - 32}}{8}s. \tag{8}$$

For the roots to be real and distinct, we must ensure that $(5 + \cos\alpha)^2 > 32$, which reduces to $\cos\alpha > 4\sqrt{2} - 5 = \cos A_0$. Since we have assumed that $\alpha < A_0$, this requirement is satisfied. Thus, the quadratic (7) has two distinct real roots.

We let b be the smaller root and c the larger one. Finally, we let

$$a = 2s - (b+c) = \left(2 - \frac{5+\cos\alpha}{4}\right) s$$
$$= \frac{3-\cos\alpha}{4} s, \tag{9}$$

which is clearly positive. To ensure that a, b, c so defined form a scalene triangle with a as its shortest side, we must prove that $a < b$ and also that $a + b > c$. Recalling that b is given by the negative sign in (8), proving $a < b$ is equivalent to proving that

$$\frac{3-\cos\alpha}{4} + \frac{\sqrt{(5+\cos\alpha)^2 - 32}}{8} < \frac{5+\cos\alpha}{8},$$

which simplifies to showing that

$$\sqrt{(5+\cos\alpha)^2 - 32} < 3\ \cos\alpha - 1. \tag{10}$$

The RHS is positive because $\cos\alpha > 4\sqrt{2} - 5$. Therefore, (10) can be proved by comparing the squares of both the sides. Finally, showing that $a + b > c$ is equivalent to showing that $a > c - b$ and reduces to proving that

$$\frac{3-\cos\alpha}{4} > \frac{\sqrt{(5+\cos\alpha)^2 - 32}}{4},$$

which, again, is proved by comparing the squares of both the sides.

Thus we have found a scalene $\triangle ABC$ with a as its shortest side, $\measuredangle A = \alpha$ and $s^2 = 2bc$. The only other such triangle would be one where b and c are interchanged. But these two triangles are congruent to each other. Since the ratios of the sides to each other are uniquely determined by α, the triangle is unique up to similarity. $\square$

In the second part of this article, we shall show how to estimate the probability that a randomly chosen triangle has 1, 2 or 3 equalizers.

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USHERING IN 2016 . . .

$$2016 = 1 + 2 + 3 + 4 + \cdots + 61 + 62 + 63$$

$$2016 = 1^2 - 2^2 + 3^2 - + \cdots + 61^2 - 62^2 + 63^2$$

$$2016 = \sqrt{1^3 + 2^3 + 3^3 + \cdots + 61^3 + 62^3 + 63^3}$$

$$2016 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 = 2^{11} - 2^5$$

$$2016 = 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3$$

$$2016 = 666 + 666 + 666 + 6 + 6 + 6$$

$$2016 = 999 + 999 + 9 + 9$$

$$2016 = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 - 3 - 2 \times 1}$$

$$2016 = \frac{\int_0^1 \left(1 - x^{31}\right)^{64}}{\int_0^1 \left(1 - x^{31}\right)^{65}} \times 2015$$

$$2016^2 + 2016^3 = 8197604352$$

In the last relation, note that 8197604352 is a ten-digit number with all ten digits.

See: https://www.facebook.com/Mathematicx/photos/a.331529426887131. 81642.201776786529063/1116043688435697/?type=3. Adapted from the Facebook page "Mathematicx" and from other sources.

Comment. The first relation ($2016 = 1 + 2 + 3 + \cdots + 62 + 63$) tells us that 2016 is a triangular number. The triangular number previous to 2016 is 1953, and the next triangular number is 2080. This means that in our lifetime, 2016 will probably be the only year which happens to be a triangular number! (There may be a few readers of this magazine who were born before 1953, and there may well be a few readers who will still be around in 2080; but there couldn't be too many of them . . . !)

At Right Angles!

Integer-Sided Triangles with Perpendicular Medians

...and how we got there

Features

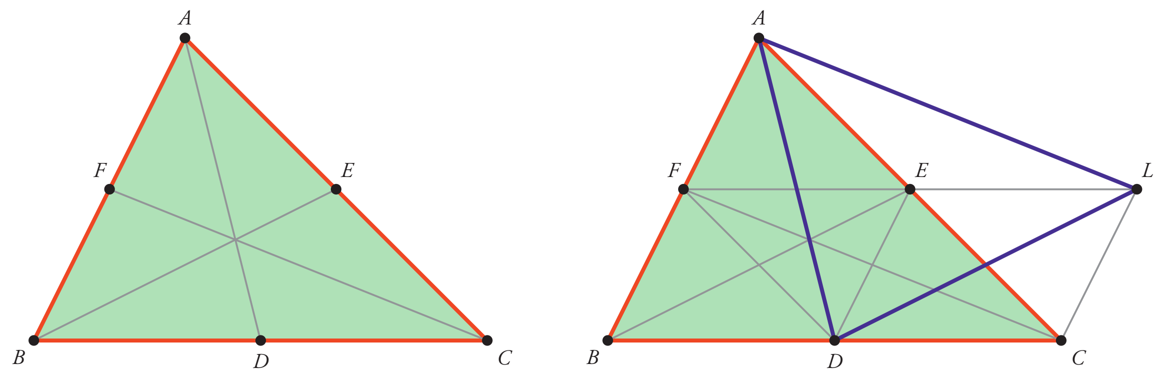
Ampagouni Divya Maneesha

It is well known (and easy to prove) that given any triangle ABC, there exists a triangle whose three sides are respectively congruent to the three medians of $\triangle ABC$. This triangle is sometimes called the ***median triangle*** of $\triangle ABC$. (Note that this is not the same as the *medial triangle*, which is the triangle whose vertices are the midpoints of the sides of the triangle. The two notions must not be confused with each other.) In this note, we ask for the condition that must be satisfied by the sides of $\triangle ABC$ in order that its median triangle be right-angled. After obtaining the condition, we consider the problem of generating integer triples (a, b, c) that satisfy this condition.

In this note we discuss the conditions that must be satisfied by the sides of an arbitrary integer-sided triangle if its medians can serve as the sides of a right-angled triangle.

Lemma 1. *Let $\triangle ABC$ is an arbitrary triangle with sides a, b, c. If m_a, m_b, m_c are the lengths of the medians drawn to the sides BC, CA, AB respectively, then there exists a triangle whose sides have lengths m_a, m_b, m_c. That is, if ABC is any triangle, another triangle can always be constructed whose sides are equal to the medians of $\triangle ABC$.*

Keywords: *Pythagoras theorem, Apollonius theorem, triangle, median, Diophantine equation*



(a) (b)

Figure 1. Construction of the median triangle ($\triangle ADL$) of $\triangle ABC$; note that its sides DL and LA are parallel to the medians BE and CF respectively

Proof. Our proof is constructive: we show how to actually construct the median triangle of ABC. Let AD, BE, CF be the medians of $\triangle ABC$; see Figure 1 (a). Through D draw DL equal and parallel to BE; see Figure 1 (b). Join AL; then we claim that $\triangle ADL$ is the required triangle.

For proof, we only have to establish that AL is equal and parallel to FC. We show this as follows. Since $EBDL$ is a parallelogram (by construction), EL is equal and parallel to BD, hence EL is equal and parallel to DC. This implies that $EDCL$ is a parallelogram; so LC is equal and parallel to ED. From this it follows that LC is equal and parallel to AF. Hence $AFCL$ is a parallelogram and AL is equal and parallel to FC, as required. Therefore the sides of $\triangle ADL$ are respectively equal and parallel to the medians AD, BE, CF. □

Lemma 2 (Apollonius)**.** *If m_a, m_b, m_c are the lengths of medians AD, BE, CF of $\triangle ABC$ drawn to the sides $BC = a, CA = b, AB = c$, then m_a, m_b, m_c are given by:*

$$4m_a^2 = 2b^2 + 2c^2 - a^2,$$
$$4m_b^2 = 2c^2 + 2a^2 - b^2,$$
$$4m_c^2 = 2a^2 + 2b^2 - c^2.$$

This is simply a statement of the theorem of Apollonius, and it follows from the application of the Pythagorean theorem to suitably constructed triangles.

A right-angled median triangle

We now ask the following question:

> *What conditions must a, b, c satisfy if the median triangle is to be right-angled?*

In view of the proof of Lemma 1, the above question is equivalent to asking the following question: *What conditions must a, b, c satisfy if some two medians of $\triangle ABC$ are to be perpendicular to each other?*

Referring to Figure 1 (b), let us require (without any loss of generality) that $\triangle ADL$ be right-angled at vertex D, i.e., that $\angle ADL$ is a right angle. (This is equivalent to requiring that the medians through A and B are perpendicular to each other.) In this case we must have $AD^2 + DL^2 = AL^2$, i.e.,

$$m_a^2 + m_b^2 = m_c^2. \tag{1}$$

Using Lemma 2, this may be rewritten as:

$$\begin{aligned}\left(2b^2 + 2c^2 - a^2\right) + \left(2c^2 + 2a^2 - b^2\right)\\ = 2a^2 + 2b^2 - c^2.\end{aligned} \tag{2}$$

This in turn simplifies to the following condition:

$$a^2 + b^2 = 5c^2. \tag{3}$$

It is easy to check that the converse holds as well, i.e., if (3) is true, then so is (1), implying that $\triangle ADL$ is right-angled at D. So we have obtained the required condition: given an arbitrary triangle ABC, the medians through A and B are perpendicular to each other if and only if $a^2 + b^2 = 5c^2$.

Finding integer solutions to the resulting condition

Having obtained the required condition on the triple (a, b, c), we now ask: *How do we generate the family of integer solutions to this equation?* Here is one approach which helps in obtaining this family. We write the equation in the following form:

$$a^2 + b^2 = 4c^2 + c^2. \tag{4}$$

We know that the following relation is an identity, true for all p, q, m, n:

$$\begin{aligned}(pm + qn)^2 &+ (pn - qm)^2 \\ &= (pm - qn)^2 + (pn + qm)^2.\end{aligned} \tag{5}$$

Looking closely at (4) and (5), let us put:

$$\begin{aligned} a &= pm + qn, \\ b &= pn - qm, \\ 2c &= pm - qn, \\ c &= pn + qm. \end{aligned}$$

From the last two equations we get $2(pn + qm) = pm - qn$, hence $p(m - 2n) = q(2m + n)$, i.e.,

$$\frac{p}{q} = \frac{2m + n}{m - 2n}.$$

Therefore let us write:

$$p = k(2m + n), \tag{6}$$

$$q = k(m - 2n), \tag{7}$$

where k is some rational number, suitably chosen. We now obtain, on substitution:

$$\begin{aligned} a &= pm + qn = k\left(2m^2 + 2mn - 2n^2\right), \\ b &= pn - qm = k\left(-m^2 + 4mn + n^2\right), \\ c &= pn + qm = k\left(m^2 + n^2\right). \end{aligned}$$

It follows that the triple

$$\begin{aligned}(a, b, c) = \Big(&k\left(2m^2 + 2mn - 2n^2\right), \\ &k\left(-m^2 + 4mn + n^2\right), k\left(m^2 + n^2\right)\Big)\end{aligned} \tag{8}$$

satisfies the condition $a^2 + b^2 = 5c^2$. We must choose k so that a, b, c turn out to be integers.

Some restrictions are needed on m, n in order for us to get a valid triangle: the sides of the triangle must be positive, and the three triangle inequalities must be satisfied (the sum of any two sides must exceed the third side). Hence we must have:

$$\begin{aligned} &a > 0, \ b > 0, \ c > 0, \ a + b > c, \\ &b + c > a, \ c + a > b. \end{aligned}$$

That is:

$$\begin{aligned} &m^2 + mn - n^2 > 0, \quad m^2 - 4mn - n^2 < 0, \\ &m^2 + n^2 > 0, \end{aligned} \tag{9}$$

and:

$$\begin{aligned} &3mn - n^2 > 0, \quad m^2 - mn - 2n^2 < 0, \\ &2m^2 - mn - n^2 > 0. \end{aligned} \tag{10}$$

To see what these inequalities lead to, it becomes simpler if we divide each one by n^2 and write $t = m/n$. Here is what we get (we have divided out by 2 in some cases):

Inequality	Solution set
$t^2 + t - 1 > 0$	$t < -\frac{1}{2}\left(\sqrt{5} + 1\right)$ or $t > \frac{1}{2}\left(\sqrt{5} - 1\right)$
$t^2 - 4t - 1 < 0$	$2 - \sqrt{5} < t < 2 + \sqrt{5}$
$t^2 + 1 > 0$	Always true
$3t - 1 > 0$	$t > \frac{1}{3}$
$t^2 - t - 2 < 0$	$-1 < t < 2$
$2t^2 - t - 1 > 0$	$t < -\frac{1}{2}$ or $t > 1$

The interval common to all the solution sets is: $1 < t < 2$. Hence the conditions on m, n are:

$$m > 0, \quad n > 0, \quad n < m < 2n. \tag{11}$$

With these conditions satisfied, any triple (a, b, c) computed using the following formula:

$$\begin{aligned}(a, b, c) = \Big(&k\left(2m^2 + 2mn - 2n^2\right), \\ &k\left(-m^2 + 4mn + n^2\right), k\left(m^2 + n^2\right)\Big),\end{aligned} \tag{12}$$

k being any rational number such that a, b, c are integers, can serve as the sides of an integer-sided triangle for which the medians through vertices A and B are perpendicular to each other. Note that k can be of the form $1/r$, where r is the gcd of $2m^2 + 2mn - 2n^2$, $-m^2 + 4mn + n^2$ and $m^2 + n^2$.

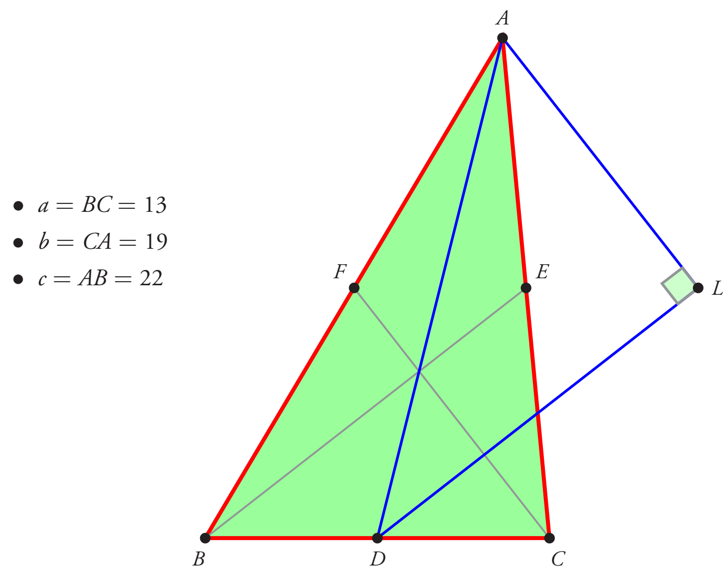


Figure 2. Triangle with sides $22, 19, 13$ and its median triangle; here $BE \perp CF$

Two examples are given below where k is not an integer.

Examples of some triples.

- Let $m = 3$, $n = 2$, $k = 1$; we get:
$$(a, b, c) = (22, 19, 13).$$
- Let $m = 4$, $n = 3$, $k = 1$; we get:
$$(a, b, c) = (38, 41, 25).$$
- Let $m = 5$, $n = 3$, $k = 1/2$; we get:
$$(a, b, c) = (31, 22, 17).$$
- Let $m = 12$, $n = 11$, $k = 1/5$; we get:
$$(a, b, c) = (62, 101, 53).$$

Figure 2 shows a triangle with sides 22, 19, 13 respectively, and its associated right-angled median triangle.

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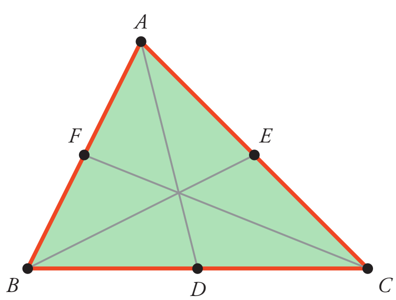
Addendum to

Integer-Sided Triangles with Perpendicular Medians

In the accompanying article, the author enquired into the conditions required on the sides a, b, c of $\triangle ABC$ in order to make its median triangle right-angled. We give an alternative treatment here.

Alternative proof of Lemma 1. The lemma states: *If $\triangle ABC$ is a triangle with medians AD, BE and CF, then there exists a triangle whose sides are respectively congruent to these three medians.*



Proof using vector algebra. Arbitrarily choose A to be the origin. Let the position vectors of B, C be $\vec{b}$, $\vec{c}$, respectively. Then the vectors representing medians AD, BE, CF are respectively:

$$\overrightarrow{AD} = \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{c}, \quad \overrightarrow{BE} = -\vec{b} + \tfrac{1}{2}\vec{c},$$
$$\overrightarrow{CE} = -\vec{c} + \tfrac{1}{2}\vec{b}. \tag{1}$$

Observe that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$ is the zero vector. Hence a triangle exists with sides that are congruent and parallel to AD, BE, CF respectively. □

Finding integer solutions to the equation $\mathbf{a}^2 + \mathbf{b}^2 = 5\mathbf{c}^2$: a different approach. Divide by c^2 and write $u = a/c$, $v = b/c$. Then u and v are rational numbers, and we have:

$$u^2 + v^2 = 5. \tag{2}$$

We must solve this equation over the rational numbers $\mathbb{Q}$.

Rewrite the equation as $u^2 - 4 = 1 - v^2$. Both sides yield to factorisation, and we get:

$$(u-2)(u+2) = (1-v)(1+v).$$

Hence:

$$\frac{u-2}{1+v} = \frac{1-v}{u+2} = t \text{ (say)}, \tag{3}$$

where t is some rational number. This yields:

$$u - 2 = t(1+v), \quad 1 - v = t(u+2),$$

i.e.,

$$u - tv = t + 2,$$
$$tu + v = 1 - 2t.$$

These two equations may be solved for u and v in terms of the parameter t; we get:

$$u = \frac{2\left(1+t-t^2\right)}{1+t^2}, \quad v = \frac{1-4t-t^2}{1+t^2}. \tag{4}$$

Now write $t = n/m$ where n, m are coprime integers. We get:

$$u = \frac{2\left(m^2+mn-n^2\right)}{m^2+n^2},$$
$$v = \frac{m^2-4mn-n^2}{m^2+n^2}. \tag{5}$$

Recalling the definitions of u, v as $u = a/c$ and $v = b/c$, we see that we may write:

$$a = 2k\left(m^2+mn-n^2\right),$$
$$b = k\left(m^2-4mn-n^2\right),$$
$$c = k\left(m^2+n^2\right),$$

for some rational number k. We have obtained a two-parameter solution which is identical to what had been obtained earlier.

Features

3, 4, 5 ...

And Other Memorable Triples

Part III

Shailesh Shirali

In Parts I and II of this article, we studied the triples $(3, 4, 5)$ and $(4, 5, 6)$; we noted some of their properties and some geometric configurations where they occur naturally. We started with $(3, 4, 5)$ and went on to study $(4, 5, 6)$, which we nicknamed as the 'elder sibling' of $(3, 4, 5)$. In this article, we wish to study the younger sibling: the triple $(2, 3, 4)$.

But before we do that, we spend a little more time with the triple $(4, 5, 6)$. We know from Part II of the article that the triangles with sides 4, 5, 6 has the feature that one of its angles has twice the measure of another of its angles. Also, we proved a general result:

In $\triangle ABC$ with sides a, b, c, the relation $\measuredangle A = 2\measuredangle B$ holds if and only if $a^2 = b(b + c)$.

So we start by posing the following number-theoretic problem: *Find all triples (a, b, c) of coprime positive integers satisfying the property $a^2 = b(b + c)$.* What solutions does the equation have (in coprime positive integers) other than $(a, b, c) = (6, 4, 5)$?

Keywords: *Triangle, angle, triple, ratio, double angle, observation, proof*

Solving the equation $a^2 = b(b+c)$
Note the use of the word 'coprime'. The reason for this should be clear: if a triangle with sides a, b, c has the geometrical property "one of its angles equals twice another one," then so will the triangle with sides $2a, 2b, 2c$, or one with sides $3a, 3b, 3c$; for these different triangles are all similar to one another. This statement has an exact parallel when we shift our sight to the equation $a^2 = b(b+c)$, or $a^2 = b^2 + bc$. Note that the equation is *homogeneous*: each term has degree 2. This implies that if the triple a, b, c is a solution of the equation, then so will be the triples $2a, 2b, 2c$ and $3a, 3b, 3c$; or, more generally, the triple ka, kb, kc where k is any positive integer. Hence there is no need to carry along any common factor shared by a, b, c.

There are many different ways of solving the given equation. Here is one approach. Write the equation as $a^2 - b^2 - bc = 0$, and divide through by b^2. We get:

$$\left(\frac{a}{b}\right)^2 - 1 - \frac{c}{b} = 0. \tag{1}$$

Let $u = a/b$ and $v = c/b$. Then u and v are rational numbers. The equation connecting u, v is the following:

$$u^2 - 1 - v = 0. \tag{2}$$

We must find pairs (u, v) of rational numbers that solve equation (2). This is obviously easy to do: choose any rational number $u > 1$, and compute v using the equation $v = u^2 - 1$. Then, using the values of u and v, deduce the values of a, b, c (keeping in mind the fact that they are coprime). Here are some examples:

- Take $u = 3/2$. This yields:
$$v = \frac{9}{4} - 1 = \frac{5}{4}.$$
Hence $a : b = 3 : 2$ and $c : b = 5 : 4$, giving $a : b : c = 6 : 4 : 5$. We have recovered the triple $(6, 4, 5)$.
- Take $u = 4/3$. This yields:
$$v = \frac{16}{9} - 1 = \frac{7}{9}.$$
Hence $a : b = 4 : 3$ and $c : b = 7 : 9$, giving $a : b : c = 12 : 9 : 7$. We have obtained the triple $(12, 9, 7)$. It follows that a triangle with sides $12, 9, 7$ has the property in question: one of its angles equals twice another one. (To be more specific: the angle opposite the side with length 12 is twice the angle opposite the side with length 9.)
- Take $u = 5/3$. This yields:
$$v = \frac{25}{9} - 1 = \frac{16}{9}.$$
Hence $a : b = 5 : 3$ and $c : b = 16 : 9$, giving $a : b : c = 15 : 9 : 16$. We have obtained the triple $(15, 9, 16)$. It follows that a triangle with sides $15, 9, 16$ has the property in question (the angle opposite the side with length 15 is twice the angle opposite the side with length 9).
- Take $u = 6/5$. This yields:
$$v = \frac{36}{25} - 1 = \frac{11}{25}.$$
Hence $a : b = 6 : 5$ and $c : b = 11 : 25$, giving $a : b : c = 30 : 25 : 11$. We have obtained the triple $(30, 25, 11)$. It follows that a triangle with sides $30, 25, 11$ has the property in question (the angle opposite the side with length 30 is twice the angle opposite the side with length 25).

Caution. But we obviously need to be careful when we choose a value for u. For example, suppose we choose $u = 5/2$. This yields:

$$v = \frac{25}{4} - 1 = \frac{21}{4}.$$

Hence $a : b = 5 : 2$ and $c : b = 21 : 4$, giving $a : b : c = 10 : 4 : 21$. But there clearly cannot be a triangle with sides $10, 4, 21$, because $10 + 4$ is less than 21, which means that the triangle inequality has been violated ("any two sides of a triangle are together greater than the third one").

Here is how we can resolve this problem. The triangle inequalities tell us that $a + b > c$, $b + c > a$, $c + a > b$. Also, $u = a/b$ and $v = c/b$. So in terms of u and v, we must have the following: $u + 1 > v$, $u + v > 1$, $v + 1 > u$. Or, since $v = u^2 - 1$:

$$u + 1 > u^2 - 1, \quad u + u^2 - 1 > 1, \quad u^2 - 1 + 1 > u.$$

The third condition simply tells us that $u > 1$. The second condition ($u^2 + u > 2$) is trivially satisfied if $u > 1$. So only the first condition is of relevance. It may be rewritten as $u^2 - u - 2 < 0$, i.e., $(u + 1)(u - 2) < 0$. This is true provided $-1 < u < 2$. So the three conditions together imply that $1 < u < 2$. If this condition is satisfied, we will obtain a meaningful triangle. Conversely, if the condition is not satisfied, then we obtain an "impossible" triangle. (This happens, for example, when $u = 5/2$.)

Remark. From the boundaries derived for u, we anticipate that if we choose values for u which are close to 2, we will obtain triangles which are 'thin,' i.e., with a large obtuse angle. We illustrate this remark with a numerical example. Take $u = 39/20$. This yields $a : b : c = 780 : 400 : 1121$, and the angles of the triangle are: $\measuredangle A = 25.68°$, $\measuredangle B = 12.84°$ and $\measuredangle C = 141.48°$.

Figure 1 summarises the algorithm.

Procedure for generating all coprime, positive integer triples (a, b, c) which give the sides of a triangle in which one angle is twice another

To generate all integer triples of the stated kind, we follow these steps:

- Choose a rational number u between 1 and 2.
- Compute v using the relation $v = u^2 - 1$.
- Let $(u, v) = (a/b, c/b)$ where a, b, c are positive integers and $\gcd(a, b, c) = 1$.
- Then the triangle with sides a, b, c has the required property.

Values of u which are close to 2 give 'thin' triangles with large obtuse angles.

Figure 1

The triple 2, 3, 4

We turn now to the triple $(2, 3, 4)$, the 'younger sibling' of $(3, 4, 5)$. Does it too possess some geometrical features of interest, like $(3, 4, 5)$ and $(4, 5, 6)$? Figure 2 shows a sketch of such a triangle. It has been labelled so that $a = 4$, $b = 3$, $c = 2$. Using GeoGebra, we find its angles; as we may anticipate, the triangle is obtuse-angled

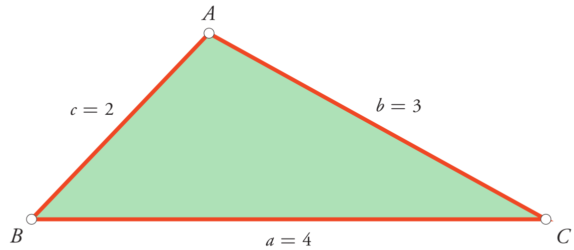


Figure 2. Triangle with sides $2, 3, 4$

(readers may recall that in Part I of this series of articles, we had proved that among all triangles whose sides are three consecutive integers, the 2-3-4 triangle is the only one which is obtuse-angled):

$$\measuredangle A = 104.48^\circ, \ \measuredangle B = 46.57^\circ, \ \measuredangle C = 28.96^\circ.$$

Examining these figures, a relationship connecting them does not immediately strike the eye. But we do find, after some searching, a curious relationship between them, prompted perhaps by the equality $96 = 2 \times 48$. We have:

$$\measuredangle A - 90^\circ = 14.48^\circ, \quad \measuredangle C = 28.96^\circ,$$

and $28.96 = 2 \times 14.48$, i.e., $\measuredangle C = 2\left(\measuredangle A - 90^\circ\right)$. Well, that is something! There is, after all, a relationship of note between the angles of the triangle.

Just as we did when we discovered a certain relationship between the angles of the triangle with sides $4, 5, 6$, we need to ensure that this observed relationship is exact and not approximate. (It could just be the case that equality holds till ten decimal places but not beyond that ....) Once again, we opt for a trigonometric proof of the equality.

The observed relationship may be rewritten as $\measuredangle C = 2\measuredangle A - 180^\circ$, which implies that $\cos C = -\cos 2A$. Does the reverse implication hold? That is, does the equality $\cos C = -\cos 2A$ imply that $\measuredangle C = 2\measuredangle A - 180^\circ$? We already know that $\measuredangle C$ is acute (because c is the smallest side), while $\measuredangle A$ is obtuse (because $a^2 > b^2 + c^2$). Now assume that $\cos C = -\cos 2A$. Then we can be sure that at least one of the following statements is true:

(1) $\measuredangle C + 2\measuredangle A$ is an odd multiple of 180°;

(2) $\measuredangle C - 2\measuredangle A$ is an odd multiple of 180°.

Of these, statement (a) is not possible; for if $0^\circ < \measuredangle C < 90^\circ$ and $90^\circ < \measuredangle A < 180^\circ$, then $180^\circ < \measuredangle C + 2\measuredangle A < 450^\circ$. Hence statement (b) must be true. But which odd multiple of 180° is $\measuredangle C - 2\measuredangle A$ equal to? Since $-360^\circ < \measuredangle C - 2\measuredangle A < -90^\circ$, it must be equal to -180°. So we only need to establish that $\cos C = -\cos 2A$. We have:

$$\cos C = \frac{3^2 + 4^2 - 2^2}{2 \times 3 \times 4} = \frac{7}{8},$$

$$\cos A = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4},$$

$$\cos 2A = 2\cos^2 A - 1 = \frac{2}{16} - 1 = -\frac{7}{8}.$$

We see that $\cos C = -\cos 2A$, and it follows that $\measuredangle C = 2\measuredangle A - 180^\circ$. Hence the observed relationship holds exactly.

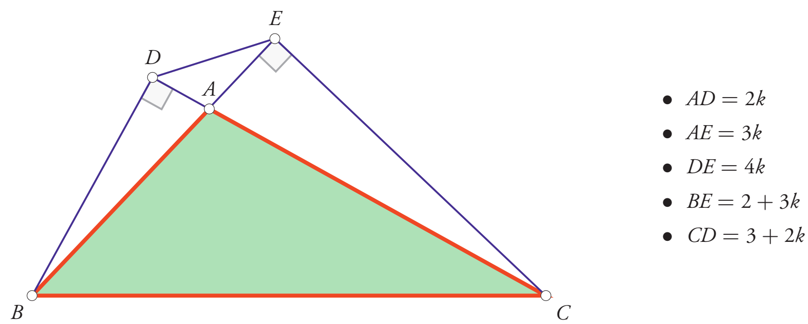


Figure 3. Demonstrating that $\measuredangle BED$ is twice $\measuredangle DBE$

A geometric interpretation

The angle relationship we have proved can also be illustrated and proved geometrically. Figure 3 shows the 2-3-4 triangle with sides BA and CA extended beyond vertex A, and perpendiculars BD and CE drawn from vertices B and C to the extended sides. Observe that quadrilateral $DBCE$ is cyclic (because $\measuredangle BDC$ and $\measuredangle BEC$ are both right angles; so BC is a diameter of the circumcircle of $DBCE$). In this figure we have $\measuredangle DBE = \measuredangle DCE$. The angle relationship $\measuredangle C = 2\measuredangle A - 180^\circ$ is equivalent to stating that each of $\measuredangle DBE$ and $\measuredangle DCE$ is equal to half of $\measuredangle BCD$ (for we have: $\measuredangle DBE = \measuredangle A - 90^\circ = \measuredangle DCE$). But we also have $\measuredangle BED = \measuredangle BCD$, by the property of a cyclic quadrilateral. Hence the stated property is equivalent to the following: *In $\triangle BED$, we have $\measuredangle BED = 2\measuredangle DBE$.*

The last statement should make us prick up our ears: it connects the property currently under study with what we studied in the previous part of this article (in the July 2015 issue of *At Right Angles*). We had earlier established the conditions under which one angle of a triangle is twice another angle of the same triangle. Invoking that result, we see that the desired angle relationship will be established if we show that $BD^2 = DE(DE + BE)$. This is what we now do.

As quadrilateral $DBCE$ is cyclic, $\triangle ABC$ is similar to $\triangle ADE$ (see Figure 3). Let the ratio of similarity be $1 : k$. Since the sides of $\triangle ABC$ are 2, 3, 4, the sides of $\triangle ADE$ will be $2k, 3k, 4k$. Since $\triangle ABD$ is right-angled at D, we get by Pythagoras's theorem:

$$BD^2 = 2^2 - (2k)^2 = 4\left(1 - k^2\right).$$

From $\triangle BDC$, which too is right-angled at D, we get $BD^2 + CD^2 = BC^2$, hence:

$$4\left(1 - k^2\right) + (3 + 2k)^2 = 4^2.$$

Solving this equation for k, we get $k = 1/4$. Hence:

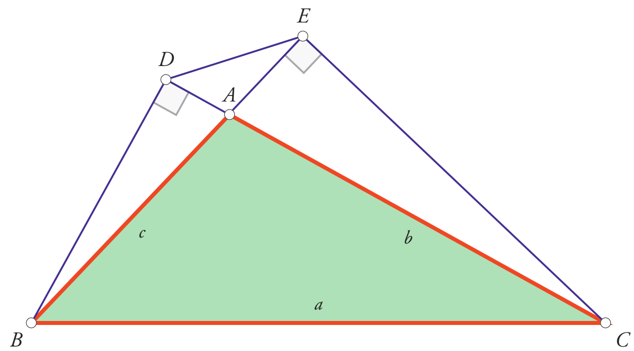
$$BD^2 = \frac{15}{4}, \quad DE = 1, \quad BE = 2 + \frac{3}{4} = \frac{11}{4},$$

$$DE(DE + BE) = \frac{15}{4},$$

and we see that $BD^2 = DE(DE + BE)$. It follows that $\measuredangle BED$ is twice $\measuredangle DBE$. The required property has thus been proved.

The general condition

Now we ask the following question: what condition must be placed on the sides a, b, c of $\triangle ABC$ so that it satisfies a property of the kind studied above? That is (see Figure 4), $\measuredangle BAC$ must be obtuse, and when perpendiculars BD and CE are drawn from vertices B and C to the extended sides CA and BA respectively, we must have: $\measuredangle ACB = 2\measuredangle DBA$.



- $AD = ck$
- $AE = bk$
- $DE = ak$
- $BE = c + bk$
- $CD = b + ck$

Figure 4. Under what conditions is it true that $\measuredangle BED$ is twice $\measuredangle DBE$?

We now ask: what conditions must a, b, c satisfy in order that $BD^2 = DE(DE + BE)$. We can opt either for a trigonometric approach now, or a pure geometry approach. We take the latter path.

As quadrilateral $DBCE$ is cyclic, $\triangle ABC \sim \triangle ADE$. Let the ratio of similarity be $1 : k$, as earlier. Since the sides of $\triangle ABC$ are a, b, c, the sides of $\triangle ADE$ will be ak, bk, ck. Since $\triangle ABD$ is right-angled at D, we get by Pythagoras's theorem:

$$BD^2 = c^2 - (ck)^2 = c^2\left(1 - k^2\right).$$

From $\triangle BDC$, which too is right-angled at D, we get $BD^2 + CD^2 = BC^2$, hence:

$$c^2\left(1 - k^2\right) + (b + ck)^2 = a^2.$$

Solving this equation for k, we get (algebraic details omitted):

$$k = \frac{a^2 - b^2 - c^2}{2bc}.$$

This yields (algebraic details omitted yet again; as the reader may have guessed, these algebraic computations have been done using a computer algebra system; I would not dare to go through this kind of algebra using hand calculation alone!):

$$DE = \frac{a\left(a^2 - b^2 - c^2\right)}{2bc},$$

$$BE = \frac{a^2 - b^2 + c^2}{2c},$$

and therefore:

$$DE + BE = \frac{a^3 + a^2 b - a\left(b^2 + c^2\right) - b\left(b^2 - c^2\right)}{2bc},$$

$$BE(DE + BE) = \frac{a\left(a^2 - b^2 - c^2\right)\left(a^3 + a^2 b - a(b^2 + c^2) - b(b^2 - c^2)\right)}{4b^2c^2},$$

$$BD^2 = \frac{(a + b + c)(a + b - c)(b + c - a)(c + a - b)}{4b^2}.$$

Note that the expression for BD can be obtained without needing to use the expression for k. We only need to see that $BD \times b/2$ is equal to the area of $\triangle ABC$, and an expression for the area is known from Heron's formula.

On equating the expressions for $BE(DE+BE)$ and BD^2 and going through the algebra, we obtain the following condition: $BE(DE+BE) = BD^2$ is true if and only if $a > b$ and

$$a^2 - ab - c^2 = 0.$$

We have obtained the conditions which will ensure the desired property. It is easily checked that $(a, b, c) = (4, 3, 2)$ satisfies the conditions.

Generating integer triples that satisfy the property

Just as we did earlier, we now ask for a way of generating more coprime integer triples for which the desired geometrical property holds. It turns out that the approach followed earlier works here as well.

We need to find integer triples (a, b, c) for which $a > b$ and $a^2 = ab + c^2$. Note that this relation automatically ensures that a is the largest side, and $\measuredangle A$ is obtuse. By dividing through by b^2, we may rewrite the relation as:

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b} + \left(\frac{c}{b}\right)^2.$$

Let $u = a/b$ and $v = c/b$. Naturally, u and v are positive rational numbers, with $u > 1$, $u > v$. The above relation takes the following form:

$$u^2 = u + v^2, \qquad \therefore\ u(u-1) = v^2.$$

To generate solutions to the equation $u(u-1) = v^2$, we adopt the following artifice. We write the above relation as:

$$\frac{u}{v} = \frac{v}{u-1},$$

and denote the value of u/v by t. We then have:

$$\begin{cases} u = tv, \\ v = t(u-1). \end{cases}$$

Treating t as a parameter (note that $t > 1$), we solve these two equations simultaneously for u and v. We get (algebraic details omitted):

$$u = \frac{t^2}{t^2-1}, \quad v = \frac{t}{t^2-1}.$$

Hence we have $u : 1 : v = t^2 : t^2 - 1 : t$, i.e.,

$$a : b : c = t^2 : t^2 - 1 : t.$$

This parametrisation allows us to generate infinitely many integer triples (a, b, c) which satisfy the desired property. For example:

- Take $t = 2$. We get $a : b : c = 4 : 3 : 2$. This yields the very triangle we have been studying.
- Take $t = 3$. We get $a : b : c = 9 : 8 : 3$. It may be checked that the triangle with sides 9, 8, 3 possesses the property in question: its angles are 99.59407°, 61.2178° and 19.18814°, and we have: $19.18814 = 2 \times (99.59407 - 90)$.
- Take $t = 4$. We get $a : b : c = 16 : 15 : 4$. It may be checked that the triangle with sides 16, 15, 4 possesses the property in question: its angles are 97.18076°, 68.45773° and 14.36151°, and we have: $14.36151 = 2 \times (97.18076 - 90)$.

Figure 5 summarises the algorithm.

Procedure for generating all coprime, positive integer triples (a, b, c) which give the sides of a triangle in which $\measuredangle C = 2(\measuredangle A - 90^\circ)$

To generate all integer triples of the stated kind, we follow these steps:

- Choose a rational number $t > 1$.
- Compute u and v using the relations
$$u = \frac{t^2}{t^2 - 1}, \quad v = \frac{t}{t^2 - 1}.$$
- Let $u : 1 : v = a : b : c$ where a, b, c are positive integers and the gcd of a, b, c is 1.
- Then the triangle with sides a, b, c has the required property.

Figure 5

What about the triangle inequality? We may wonder whether some restrictions have to be placed on t for the triangle inequality to be satisfied, i.e., for us to get a valid triangle. But in this case, unlike the previous one, the problem resolves itself on its own: any value of t which exceeds 1 will suffice. For if $t > 1$, we have $t^2 > t$; and $t^2 > t^2 - 1$ is always true; so the inequality to be checked is: $(t^2 - 1) + t > t^2$. But this is automatically satisfied, since $t > 1$.

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A limerick in disguise

Would you believe that the following innocent and unassuming numerical equality is actually a limerick in disguise?

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + (5 \times 11) = 9^2 + 0.$$

See page 87 for the solution.

(The equality is a correct one: both sides equal 81; please check!)

ClassRoom

Triangle Centres and Homogeneous Coordinates

Part I - Trilinear Coordinates

A Ramachandran

During a course in Euclidean geometry at high school level, a student encounters four classical triangle centres—the circumcentre, the incentre, the orthocentre and the centroid (introduced as the points of concurrence of the perpendicular bisectors of the sides, the bisectors of the angles of the triangle, the altitudes and the medians, respectively). We shall study two alternative ways of describing and characterising these four significant points. They are both known as homogeneous coordinate systems, but we explain the significance of this term later. In part I of the article, we consider the first of these: trilinear coordinates.

Trilinear coordinates

This approach was suggested by the German physicist-mathematician Julius Plücker in 1835 [1]. Here a triangle centre is characterised in terms of its perpendicular distances from the three sides of the triangle; or rather, the ratios of these distances. These ratios form the "trilinear coordinates" of the triangle centre. If $\triangle ABC$ is the triangle and P the point in question, then the perpendicular distances PD, PE, PF to the sides BC, CA, AB respectively are expressed as ratios involving the side lengths and/or trigonometric functions of the angles of the triangle.

Keywords: *Triangle centre, incentre, centroid, orthocentre, circumcentre, trilinear coordinates, homogeneous coordinates, trigonometric ratio*

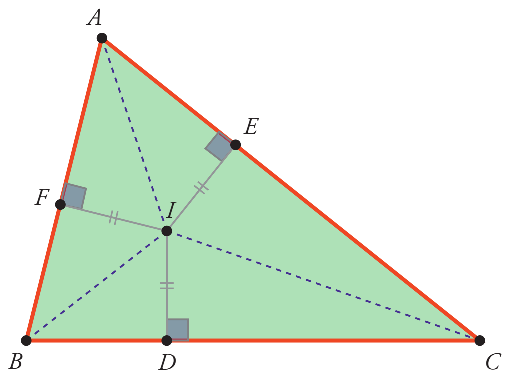

Figure 1. Incentre of a triangle; I is equidistant from the sides

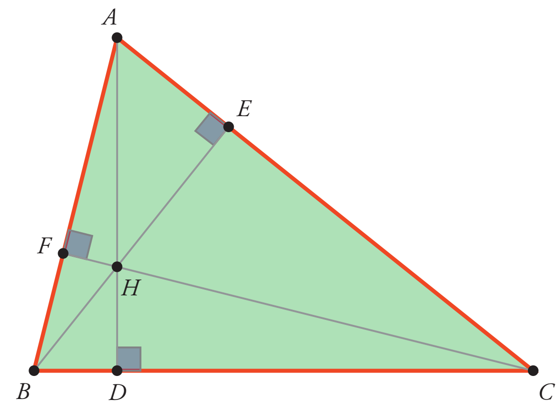

Figure 3. Orthocentre of an acute-angled triangle

Incentre. Since the incentre is equidistant from the sides, its trilinear coordinates are simply $1 : 1 : 1$ (see Figure 1).

Centroid. Let G be the centroid of $\triangle ABC$ (see Figure 2). It is a well-known result of Euclidean geometry that triangles GAB, GBC and GCA are equal in area. If GD, GE and GF are perpendiculars to the sides, then $GD \cdot a/2 = GE \cdot b/2 = GF \cdot c/2 = k$, say.

This yields: $GD = 2k/a$, $GE = 2k/b$, $GF = 2k/c$, hence $GD : GE : GF = 1/a : 1/b : 1/c$; these ratios form the trilinear coordinates of the centroid.

Alternatively, the coordinates could be given as $bc : ca : ab$ (multiplying through by abc), or as $\csc A : \csc B : \csc C$. The last relation arises from the fact that the sides of a triangle bear the same ratios to each other as the sines of the angles opposite them. So the reciprocals of the sides bear the same ratios to each other as the cosecant values.

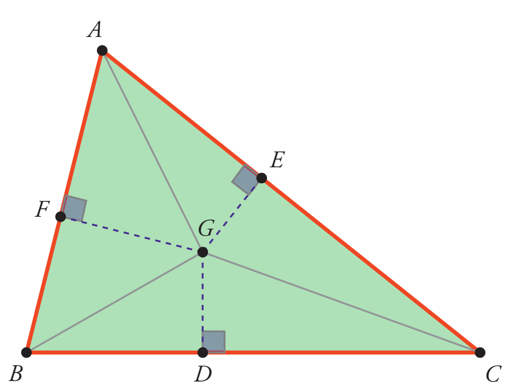

Figure 2. Centroid of an arbitrary triangle

Orthocentre. Now we turn our attention to the orthocentre. We first consider the case of an acute-angled triangle ABC (see Figure 3). Here, AD, BE, CF are perpendiculars from the vertices A, B, C to the sides BC, CA, AB, respectively; H is the orthocentre.

Since $\measuredangle HCD = 90^\circ - \measuredangle B$, we get $\measuredangle DHC = \measuredangle B$, and $\sec B = HC/HD$. Similarly, $\measuredangle HCE = 90^\circ - \measuredangle A$, so $\measuredangle EHC = \measuredangle A$, and $\sec A = HC/HE$. Hence:

$$\frac{HD}{HE} = \frac{HC}{HE} \Big/ \frac{HC}{HD} = \frac{\sec A}{\sec C}.$$

Similarly, $HE/HF = \sec B/\sec C$. Therefore the trilinear coordinates of the orthocentre are $\sec A : \sec B : \sec C$.

Let us see what happens as one of the angles (say $\measuredangle A$) approaches 90°. The other two angles also approach limiting values which we assume are distinct from 0° and 90°. Note that $HE/HD = \cos A/\cos B$ and $HF/HD = \cos A/\cos C$. As $\measuredangle A \to 90^\circ$, $\cos A \to 0$; so $HE \to 0$ and $HF \to 0$ (in the limit, A, H, E, F coincide; see Figure 4). Two of the three quantities HD, HE, HF are now zero, and it is customary to write the ratios as

$$HD : HE : HF = 1 : 0 : 0.$$

It follows that for a right-angled triangle ABC with $\measuredangle A = 90^\circ$, the trilinear coordinates of the orthocentre are $1 : 0 : 0$.

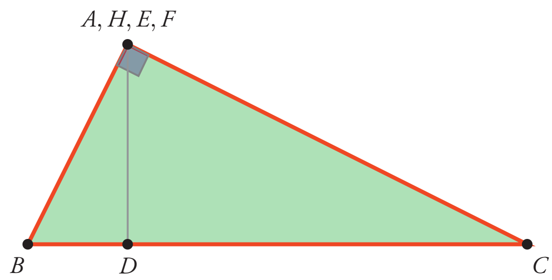

Figure 4. Orthocentre of a right-angled triangle

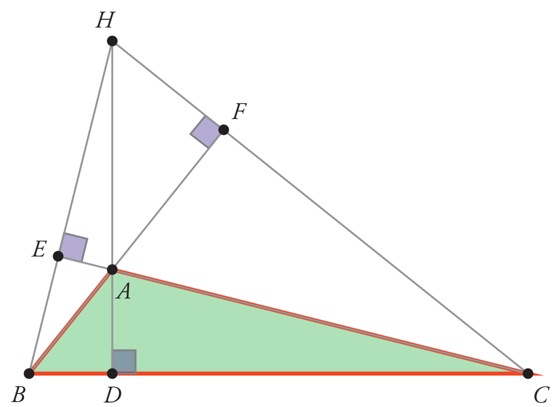

Figure 5. Orthocentre of an obtuse-angled triangle

Next, consider the case of an obtuse-angled triangle with say $\measuredangle A$ as the obtuse angle (see Figure 5); we will find that a negative sign appears in the trilinear relationship.

We have: $\measuredangle EHC = \measuredangle CAF = 180^\circ - \measuredangle A$. Then $HC/HE = \sec(180^\circ - A) = -\sec A$. Now, $HC/HD = \sec B$ (since $\measuredangle DHC = \measuredangle B$), so $HD/HE = -\sec A/\sec B$. However, $HE/HF = \sec B/\sec C$, while $HF/HD = -\sec C/\sec A$. So we get:

$$HD : HE : HF = -\sec A : \sec B : \sec C.$$

There is a way of looking at this relationship which restores the symmetry of signs. Note that if $\measuredangle A > 90^\circ$ (as in Figure 5), then H and A lie on the same side of BC, while H and B lie on opposite sides of CA, and similarly, H and C lie on opposite sides of AB. Recalling the sign convention for distances used in coordinate geometry, we see that it makes sense to regard HD as positive, and HE and HF as negative. Under this perspective, we have:

$$HD : HE : HF = \sec A : \sec B : \sec C,$$

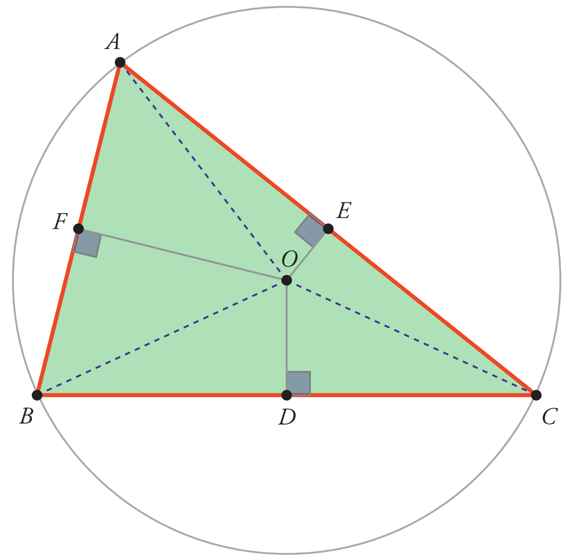

Figure 6. Circumcentre of an acute-angled triangle

and we find that the inherent symmetry of the formula has been restored.

Circumcentre. Next, we consider the circumcentre. We first look at an acute-angled triangle ABC (see Figure 6).

In the figure, O is the circumcentre of $\triangle ABC$, and OD is perpendicular to BC. We have: $\measuredangle BOC = 2\measuredangle A$, hence $\measuredangle BOD = \measuredangle A$ and $\cos A = OD/OB = OD/R$, where R is the circumradius. Thus $OD = R\cos A$. It follows that the distances from O to the sides of the triangle are proportional to the cosines of the angles opposite them. Hence the trilinear coordinates of the circumcentre are $\cos A : \cos B : \cos C$.

Just as we did last time, let us see what happens as one angle (say $\measuredangle A$) approaches 90°. The other two angles also approach limiting values which we assume are distinct from 0° and 90°.

The situation is depicted in Figure 7; D and O now coincide, and $\cos A = OD = 0$. Also

$$\frac{OE}{OF} = \frac{AB/2}{AC/2} = \frac{AB/BC}{AC/BC} = \frac{\cos B}{\cos C},$$

so $OD : OE : OF = 0 : \cos B : \cos C$. Hence the trilinear coordinates of the circumcentre are $0 : \cos B : \cos C$. This is consistent with the formula

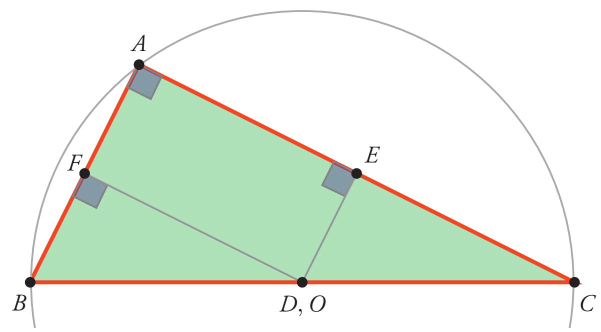

Figure 7. Circumcentre of a right-angled triangle

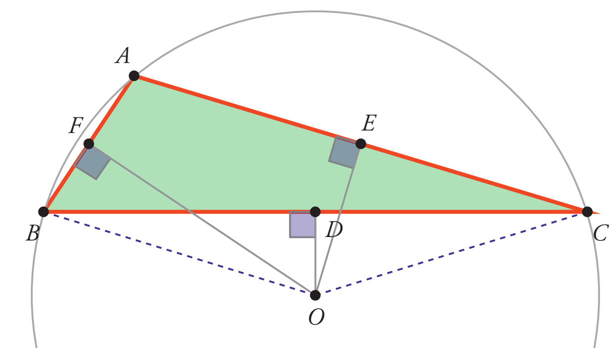

Figure 8. Circumcentre of an obtuse-angled triangle

obtained earlier; only, $\cos A$ has now assumed a zero value.

In the case of an obtuse-angled triangle, the circumcentre O lies outside the triangle (see Figure 8, where $\measuredangle A$ is obtuse).

Here we have $\measuredangle BOD = 180^\circ - A$, so $OD = -R\cos A$. The relations $OE = R\cos B$ and $OF = R\cos C$ remain unchanged. So we get $OD : OE : OF = -\cos A : \cos B : \cos C$. If we adopt the same sign convention as earlier, then $OD < 0$ since O and A lie on opposite sides of BC, while $OE > 0$ and $OF > 0$, since O and B lie on the same side of CA, and O and C lie on the same side of AB. With this understanding, the symmetry of the formula gets restored and we have: $OD : OE : OF = \cos A : \cos B : \cos C$. So the trilinear coordinates of the circumcentre are $\cos A : \cos B : \cos C$.

In Part II of the article, we shall describe another such coordinate system—barycentric coordinates.

Note from the editor: What is homogeneous about this system? The significance of the term *homogeneous* may not be immediately apparent. In 'ordinary' coordinate geometry, the equation of a line has the form $ax + by + c = 0$, where a, b, c are constants. Note that this equation is not homogeneous: two terms have degree 1, while one term has degree 0. Similarly, the question of a circle has the form $x^2 + y^2 + 2gx + 2fy + c = 0$; this too is not homogeneous. In some settings, it turns out to be advantageous to have equations which are homogeneous, in which all the terms have the same degree. The trilinear coordinates system described above has this feature, and so does the barycentric coordinates system to be discussed in part II. Here, the equation of a line has the form $lx + my + nz = 0$, where l, m, n are constants; note that this equation is homogeneous. In recent times it has been found that homogeneous coordinates are particularly convenient to use in computer graphics.

References

1. Julius Plücker. http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Plucker.html
2. Wikipedia, Homogeneous Coordinates. https://en.wikipedia.org/wiki/Homogeneous_coordinates

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

ClassRoom

The Rascal Triangle

C⊗MαC

This article is written with two purposes in mind, to illustrate two important themes. The first is to highlight the fact that even the most familiar settings can throw up surprises. The second is to point out that in a sufficiently nurturing classroom atmosphere, where the teacher and student are willing to listen to each other and travel together along uncharted territory, a great deal of learning can happen and students can go very far indeed in their explorations.

In a classroom of a school in America, a mathematics teacher had assigned the following work to his students. He had displayed the first four lines of the well-known Pascal triangle (namely, rows 0, 1, 2 and 3), and had asked them to guess or to deduce what could be the next few lines. What he had displayed was the array shown in Table 1 (see [2, 3]).

Presumably, he wanted them to spot the defining Pascal property and to therefore propose that the next row should be: $1, 4, 6, 4, 1$; the one after that: $1, 5, 10, 10, 5, 1$; and so on.

Instead, the students surprised him by proposing that the row after $1, 3, 3, 1$ should be $1, 4, 5, 4, 1$ and the one after that should be $1, 5, 7, 7, 5, 1$. This seemed wrong, and the teacher challenged them.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	?		?		?		?		?	
?		?		?		?		?		?

Table 1. The familiar Pascal array

Then he discovered that the students had used a totally different rule to generate the rows! In the Pascal triangle each new row is generated additively, using the numbers in the row above it (namely, by adding the two numbers closest to the entry to be filled). Thus if we have:

$$\begin{array}{ccc} a & & b \\ & x & \end{array}$$

then the new entry x is given by: $x = a + b$. But in the rule used by the students, the numbers in each new row are computed using the *two* rows preceding it. Thus if we have:

$$\begin{array}{ccc} & a & \\ b & & c \\ & x & \end{array}$$

then the new entry x is given by:

$$x = \frac{bc + 1}{a}.$$

It may be appropriate to call the generating rule for the Pascal array a "triangular rule" (based on the underlying shape), and the one used by the students a "diamond rule."

The diamond rule used by the students looks more complicated than the triangular Pascal rule: it involves both multiplication and division, whereas in the Pascal triangle we only do addition. In the Pascal triangle, all numbers will naturally be positive integers. But in the new array being considered, it is not at all obvious what kind of numbers are going to be produced. One wonders: do we get non-integral numbers? The surprising discovery we make is: despite the division, all the numbers do turn out to be positive integers. And though in one sense the new array is more complicated than the Pascal array, as it involves a more complicated recurrence rule, in another sense it turns out to be much simpler, as we shall see below.

The students who put forward this new rule and explored this new array whimsically named it the *Rascal triangle*. They also referred to the generating rules in a non-standard manner, using the cardinal directions. Thus, they referred to the rule that generates the Pascal array as the 'East-West rule':

$$\begin{array}{ccc} \text{WEST} & & \text{EAST} \\ & x & \end{array}$$

with $x = \text{WEST} + \text{EAST}$; and they described their own rule in the following way [1]. If the configuration is:

$$\begin{array}{ccc} & \text{NORTH} & \\ \text{WEST} & & \text{EAST} \\ & x & \end{array}$$

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		5		4		1					
				1		5		7		7		5		1				
			1		6		9		10		9		6		1			
		1		7		11		13		13		11		7		1		
	1		8		13		16		17		16		13		8		1	
1		9		15		19		21		21		19		15		9		1

Table 2. The first ten rows of the Rascal array

then x is given by:

$$x = \frac{\text{WEST} \times \text{EAST} + 1}{\text{NORTH}}.$$

The rest of this brief article will be devoted to finding a generating formula for the Rascal triangle.

Uncovering a formula for the array

Our first task will be to prove that all the numbers in the Rascal array are positive integers. To our surprise, we find that the simplest way to prove this is by establishing more: we discover and prove a generating formula for the array, and this formula then directly shows that all the entries are positive integers. This is an illustration of the maxim that in mathematics, “less can be more and more can be less” (meaning that it can be simpler to prove more than what is required). In this particular case, what we shall do is to use the diamond generating rule repeatedly to generate as much of the array as we can; then we shall study the array and guess its patterns and thus deduce its generating formula. Table 2 shows the first ten rows of the Rascal array.

Look at the diagonals of this array; what a delightful surprise! We see a collection of arithmetic progressions (see Table 3).

In indexing the diagonals, we intentionally start the numbering from 0 rather than 1. So Term(0) of each diagonal is 1. We now readily guess the generating formula for these diagonals: Term(k) of Diagonal(n) is

Terms	#0	#1	#2	#3	#4	#5	#6	...
Diagonal 0	1	1	1	1	1	1	1	...
Diagonal 1	1	2	3	4	5	6	7	...
Diagonal 2	1	3	5	7	9	11	13	...
Diagonal 3	1	4	7	10	13	16	19	...
Diagonal 4	1	5	9	13	17	21	25	...
Diagonal 5	1	6	11	16	21	26	31	...
Diagonal 6	1	7	13	19	25	31	37	...
Diagonal 7	1	8	15	22	29	36	43	...

Table 3. First few diagonals of the Rascal array

1									
1	1								
1	2	1							
1	3	3	1						
1	4	5	4	1					
1	5	7	7	5	1				
1	6	9	10	9	6	1			
1	7	11	13	13	11	7	1		
1	8	13	16	17	16	13	8	1	
1	9	15	19	21	21	19	15	9	1

Table 4

$kn+1$. *Note that this is purely an educated guess, based on the observed pattern.* We have not yet actually proved the formula.

We number the rows the same way, starting from 0. Now observe that as we traverse Row(n), Entry(0) lies on Diagonal(n) (remember that we are traversing the diagonals from top left to bottom right), the next entry lies on Diagonal$(n-1)$, the entry after that lies on Diagonal$(n-2)$, and so on. So Entry(k) of Row(k) is Entry(k) of Diagonal$(n-k)$. The array shown in Table 4, with Row(5) and Diagonal(2) highlighted, may make the meaning of this statement clearer: Entry(3) on the row (namely, 7) is also Entry(3) on the diagonal.

As per the conjectured formula, Entry(k) of Diagonal$(n-k)$ should be $(n-k)k+1$. Hence we have the following:

Conjecture. *Entry(k) in Row(n) of the Rascal array is $k(n-k)+1 = kn-(k^2-1)$.*

So according to this conjecture, the numbers in Row(n) are the following:

$$1,\ n,\ 2n-3,\ 3n-8,\ 4n-15,\ 5n-24,\ \ldots.$$

Having discovered a formula that seems to fit, it is an easy matter to verify that it is correct, using mathematical induction. First we verify that it is true for the first few rows of the Rascal triangle; this is easily done (please do so for yourself). Next, we observe that the diamond rule connects the following four elements:

	Entry $(n-1,k)$	
Entry (n,k)		Entry $(n,k+1)$
	Entry $(n+1,k+1)$	

As per the conjectured formula, where Entry $(n,k)=k(n-k)+1$, this translates to:

	$k(n-1)-k^2+1$	
$kn-k^2+1$		$k(n-2)-k^2+n$
	$k(n-1)-k^2+n+1$	

Hence we must verify whether the following is true:

$$\frac{(kn - k^2 + 1) \times [k(n-2) - k^2 + n] + 1}{k(n-1) - k^2 + 1} = k(n-1) - k^2 + n + 1.$$

We leave it to you to verify that equality does indeed hold here. Hence the conjectured formula is true.

Having found the generating formula of the Rascal array, we see that this array is indeed simpler than the Pascal array, despite the fact that its generating rule looks more complicated. The patterns in the diagonals of the Rascal array are certainly far simpler than those in the Pascal array; they are arithmetic progressions (and APs are surely the simplest kind of progression). And the general term in the array is a two-variable quadratic expression. The corresponding formula for the Pascal array involves the factorial numbers, which are considerably more complicated than the square numbers.

Closing remarks. It is worth noting the conditions that enabled such a phenomenon to occur, namely: the discovery of a new array of numbers in a classroom setting. It needed a teacher who was not stuck to the 'book answer' and instead was alert to the possibility of a fresh discovery within the boundaries of a classroom; a teacher who was willing to travel with the students and go that extra mile to understand the thinking of his students. It also required that this entire event be recorded. Teachers must be encouraged to document and share accounts of such events.

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The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Accommodating Primes

Have a look at the prime numbers listed below. They are most accommodating! Starting with a prime number (7), we keep appending a digit to the left of the latest number in such a way that the new number is again prime. How long can we continue this accommodating sequence?

7
37
137
9137
29137
629137
7629137
67629137
567629137
6567629137
16567629137
216567629137
6216567629137
46216567629137
646216567629137
2646216567629137
12646216567629137
312646216567629137
6312646216567629137
86312646216567629137
686312646216567629137
7686312646216567629137
57686312646216567629137
357686312646216567629137

Note: Some of you may be skeptical about the above list. You may wonder:

"How do I know whether these numbers really are prime? What if the editors who assembled the page made some errors in listing the numbers? What if they made some typographical errors?"

For all such skeptics, here is a site which can actually check whether a number is prime or not: http://www.primber.com/prime-number-checker. html.

Please try it out! Note however that it does have one restriction: it can only accept numbers with 18 digits or less. So the largest number in the above list which it can check is 12646216567629137.

ClassRoom

An Astonishing Property of Third-Order Magic Squares

SHAILESH SHIRALI

In the previous issue of *At Right Angles*, it was shown that using the numbers from 1 to 9, there is essentially just one third-order magic square (here the phrase "essentially just one" actually means: "essentially just one, up to rotations and reflections"), namely:

8	1	6
3	5	7
4	9	2

In an article [1] written in the February 1999 issue of the *American Mathematical Monthly*, titled appropriately "Magic Squares Indeed!", the authors point out a truly remarkable property of this magic square; namely:

$$816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2,$$

$$834^2 + 159^2 + 672^2 = 438^2 + 951^2 + 276^2.$$

Please verify for yourself that these relations are true. In fact, there are even more such relations, but we will list them later. For the moment we ask: what makes these relations true? What we shall

Keywords: *Magic square, third order, quadratic, sum of squares, proof, generalisation*

show is still more surprising: these sum-of-squares equalities are true not just for this particular square but for any third-order magic square!

In their article, Benjamin and Yasuda use linear algebra to prove their results. We shall do so using only elementary algebra, accessible even at the ninth standard level.

Proof. We showed in the earlier article that if s is the magic sum of a third-order magic square, and m is the number in the central cell, then $s = 3m$. This simple relation allows us to find a generating formula for third-order magic squares. Let a, b be respectively the numbers in the two corner cells of the top row. Then we can express the numbers in all the other cells in terms of a, b, m, using the magic property of the square repeatedly (namely, that the sum of the numbers in each row, each column and each diagonal is $3m$). The result is shown below:

a	$3m-a-b$	b
$m-a+b$	m	$m+a-b$
$2m-b$	$a+b-m$	$2m-a$

It is easy to check that this is a magic square for any values of a, b, m. *Note that for the moment we drop the requirement that the entries of the square must be all different from one another and must all be positive integers.* We are treating the above array simply as an algebraic object with the property that its row sums, column sums and diagonal sums are all the same (equal to $3m$). For ease of notation, we denote the above magic square by $f(m, a, b)$. Here is an instance of a magic square generated by the above formula:

$$f(8,7,3) = \begin{array}{|c|c|c|}\hline 7 & 14 & 3 \\ \hline 4 & 8 & 12 \\ \hline 13 & 2 & 9 \\ \hline \end{array}$$

Here are two instances of magic squares generated by this formula, in which the requirement that the entries must be all different from one another and must all be positive integers does not hold:

$$f(2,3,4) = \begin{array}{|r|r|r|}\hline 3 & -1 & 4 \\ \hline 3 & 2 & 1 \\ \hline 0 & 5 & 1 \\ \hline \end{array} \qquad f(1,2,6) = \begin{array}{|r|r|r|}\hline 2 & -5 & 6 \\ \hline 5 & 1 & -3 \\ \hline -4 & 7 & 0 \\ \hline \end{array}$$

Associating two polynomials with each third-order magic square. Next, with the magic square $f(m, a, b)$ we associate two polynomials as follows.

- We first associate a quadratic polynomial with each row, using the entries in the cells as the coefficients. Thus, for the first row, we multiply the first number by x^2, the second number by x, and retain the third number as it is; then we add these three expressions together. We do exactly the same for the second row and for the third row. At the end, we have three quadratic expressions, one associated with each row. We square these and add the three resulting polynomials. The result (naturally) is a polynomial of degree 4.

  For example, for the magic square $f(8,7,3)$ shown above, the three rows yield the following three quadratics respectively:

  $$7x^2 + 14x + 3, \qquad 4x^2 + 8x + 12, \qquad 13x^2 + 2x + 9.$$

  On adding their squares, we get the following:

  $$\left(7x^2 + 14x + 3\right)^2 + \left(4x^2 + 8x + 12\right)^2 + \left(13x^2 + 2x + 9\right)^2,$$

which simplifies to the following fourth-degree expression:

$$234x^4 + 312x^3 + 636x^2 + 312x + 234.$$

- Now we do the same thing in reverse, starting from the third column and working our way towards the first column. Thus, for the first row, we multiply the third number by x^2, the second number by x, and retain the first number as it is; then we add these three expressions together. And we do exactly the same for the second row and for the third row. At the end, we have three more quadratic expressions, one for each row (but they are different from the ones obtained earlier). We square them and add the resulting polynomials. The result once again is a polynomial of degree 4.

  So, with the magic square $f(8, 7, 3)$, the three rows yield the following three quadratic polynomials respectively:

$$3x^2 + 14x + 7, \quad 12x^2 + 8x + 4, \quad 9x^2 + 2x + 13,$$

  and on adding their squares, we get the following:

$$\left(3x^2 + 14x + 7\right)^2 + \left(12x^2 + 8x + 4\right)^2 + \left(9x^2 + 2x + 3\right)^2,$$

  which when simplified yields the following fourth-degree expression:

$$234x^4 + 312x^3 + 636x^2 + 312x + 234.$$

  What do we notice? Why, it is the very same polynomial as the one obtained earlier! How very striking, how very odd, how very pleasing. But would this be true in general? Have we uncovered a new property about magic squares?

The answer: **Yes**, and it is easy to prove. For the magic square $f(m, a, b)$, the three rows yield the following three pairs of quadratic polynomials respectively:

	Polynomial I: Reading col 1 to col 3	Polynomial II: Reading col 3 to col 1
First row	$ax^2 + (3m - a - b)x + b$	$bx^2 + (3m - a - b)x + a$
Second row	$(m - a + b)x^2 + mx + (m + a - b)$	$(m + a - b)x^2 + mx + (m - a + b)$
Third row	$(2m - b)x^2 + (a + b - m)x + (2m - a)$	$(2m - a)x^2 + (a + b - m)x + (2m - b)$

For each column ('Polynomial I', 'Polynomial II'), we add the squares of the terms in the three rows. The result is a fourth-degree polynomial with coefficients as follows (we have chosen to display the polynomial in this form as it has too many terms to display in a single line):

Coefficient of x^4	$2a^2 - 2ab - 2am + 2b^2 - 2bm + 5m^2$
Coefficient of x^3	$-2\left(a^2 + 2ab - 4am + b^2 - 4bm + m^2\right)$
Coefficient of x^2	$3\left(4ab - 4am - 4bm + 7m^2\right)$
Coefficient of x^1	$-2\left(a^2 + 2ab - 4am + b^2 - 4bm + m^2\right)$
Coefficient of x^0	$2a^2 - 2ab - 2am + 2b^2 - 2bm + 5m^2$

The crucial finding is: *the two polynomials are identical*. (Note the palindromic patterns in the coefficients. Note also that each of the five coefficients is symmetric in a and b.)

With this result in our possession, it is easy to make sense of the finding reported by the two authors in the article mentioned at the start. For, the sums of the squares computed as described simply correspond to assigning the value $x = 10$ in the polynomials above.

Closing remark.

8	1	6
3	5	7
4	9	2

With reference to the 'standard' third-order magic square, the authors of [1] had drawn attention to still more such equalities, namely, the following:

$$\begin{aligned}
\text{Diagonals:}\quad & 456^2 + 231^2 + 978^2 = 654^2 + 132^2 + 879^2,\\
\text{Counter-diagonals:}\quad & 639^2 + 174^2 + 852^2 = 936^2 + 471^2 + 258^2,\\
\text{Diagonals:}\quad & 654^2 + 798^2 + 213^2 = 456^2 + 897^2 + 312^2,\\
\text{Counter-diagonals:}\quad & 693^2 + 714^2 + 258^2 = 396^2 + 417^2 + 852^2.
\end{aligned}$$

We invite the reader to show algebraically that any third-order magic square must possess the very same properties.

We also invite the reader to explore possible interpretations that can be given to our findings for particular values of x. For example, can any meaningful interpretation be given for the value $x = 1$? Or for the value $x = -1$?

Isn't it a truly wonderful exhibition of the power and reach of simple algebra that we have been able to demonstrate these properties of third-order magic squares?

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ClassRoom

How To Prove It

In a previous issue of AtRiA (as part of the "Low Floor High Ceiling" series), questions had been posed and studied about polyominoes. In this article, we consider and prove two specific results concerning these objects, and make a few remarks about an open problem.

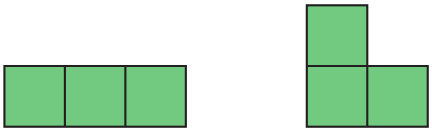
SHAILESH A SHIRALI

Polyomino as a natural generalisation of a domino
We are familiar with the notion of a **domino**, which is a shape produced by joining two unit squares edge-to-edge. If we divide this object into two equal parts, we get a **monomino**. See Figure 1.

Monomino Domino

Figure 1. Monomino and domino

A polyomino is a natural generalisation of this notion, where we allow the number of unit squares to vary. For example, using three unit squares, the shapes shown in Figure 2 are possible. They are called **trominoes**.



Straight tromino L-tromino

Figure 2. The two trominoes

Keywords: *Polyominoes, polygon, perimeter, parity, closed figure*

Similarly, by joining four unit squares edge-to-edge, the shapes shown in Figure 3 are possible. They are called **tetrominoes**. The names given to the individual shapes are shown alongside.

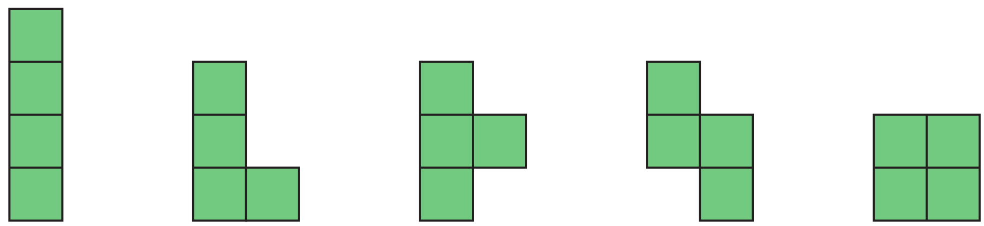


Figure 3. The five tetrominoes

For $n = 5$, the shapes are called **pentominoes**. It turns out that there are 12 possible pentominoes. We invite you to list them. For $n = 6$, the shapes are called **hexominoes**. As you may guess, there are a large number of these figures.

In general, we may define a polyomino as "a plane geometric figure formed by joining one or more equal squares edge-to-edge." (This is the definition given in [3]. See also [4]. For more about polyominoes, you may refer to the highly readable accounts in [1] and [2].)

As we increase the number of unit squares, more complex shapes become possible. For example, we may get shapes with 'holes' such as the one shown in Figure 4.

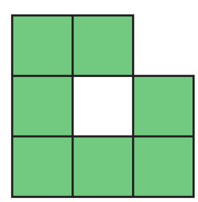


Figure 4. A 7-omino with a hole

Perimeter and number of sides of a polyomino

An n-omino is a plane geometric figure formed by joining n equal squares edge-to-edge. In this section we focus on two numbers associated with this object: its perimeter, and its number of sides. Note that by definition, the area of an n-omino is n square units. Table 1 lists the perimeter P and number of sides k for all polyominoes with area not exceeding 4.

Area, n	Shape	Perimeter, P	Number of sides, k
1	Monomino	4	4
2	Domino	6	4
3	Straight tromino	8	4
3	L-tromino	8	6
4	Straight tetromino	8	4
4	L-tetromino	10	6
4	T-tetromino	10	8
4	Z-tetromino	10	8
4	Square tetromino	8	4

Table 1

On examining the data, we see something noteworthy right away: *P and k are even numbers in every case*. Will this be the case for polyominoes with larger numbers of sides? We shall show that the answer is **Yes**. The proofs we offer are very instructive, as they use the notion of *parity*.

Proof that the perimeter P is even. The sides of the unit squares making up a polyomino give rise in a natural way to two mutually perpendicular directions which we may regard as a pair of coordinate axes. Let these axes be drawn; the outer boundary of a polyomino is now entirely composed of segments of unit length, each parallel to the x-axis or the y-axis.

Now let us take a walk around the outer boundary of the polyomino, advancing in steps of unit length and marking a dot at each lattice point along our path. (We must fix a direction for our tour; let us assume that we walk in the counterclockwise direction.) The number of steps we take clearly equals the perimeter P of the polyomino. With each step, our location changes in the following way: either the x-coordinate changes by 1 unit, or the y-coordinate changes by 1 unit; but not both at the same time. Hence if P_1 and P_2 are two successive lattice points along the path, then $P_1 - P_2$ is one of the following:

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

Let a, b, c, d be the respective number of steps of each of the above types, as we traverse the outer boundary. Then:

$$P = a + b + c + d.$$

After a full circuit, the total change in the x-coordinate must be 0; hence $a = b$. In the same way, we must have $c = d$, because the total change in the y-coordinate must be 0. Hence:

$$P = 2a + 2c = 2(a + c).$$

It follows that P is an even number. □

Figure 5 illustrates the meanings of the parameters a, b, c, d for the polyomino depicted in Figure 4, the path being described by the arrows.

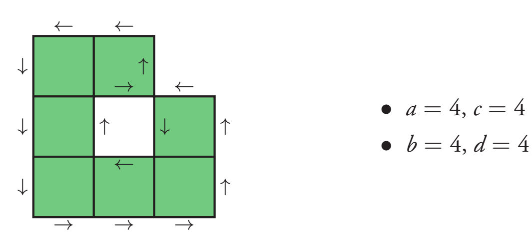


Figure 5. Traversing the outer boundary of the 7-omino shown earlier

Proof that the number of sides is even. Suppose that the polyomino is a k-sided polygon (or k-gon for short), with k vertices. (Note that the polygon need not be convex.) Let the vertices be $P_1, P_2, \ldots, P_k$. As we traverse the outer boundary in the counterclockwise direction, let the angle through which we need to turn at vertex P_i be θ_i (angles measured in degrees); then $\theta_i = \pm 90$ for each i. Let a be the number of vertices where we make a $+90^\circ$ turn, and let b be the number of vertices where we make a -90° turn; then $k = a + b$, and the total turning angle is $90(a - b)^\circ$.

Now, for any polygon, the total turning angle as we traverse the outer boundary is necessarily a multiple of 360°. In the case of a convex polygon, the total turning angle is exactly $\pm 360^\circ$, but for polygons with regions of non-convexity and/or holes, the total turning angle may be a higher multiple of 360°. Figure 6 shows examples: (a) where the total turning angle is 360°; (b) where the total turning angle is 720°.

(a) $(8 \times 90^\circ) - (4 \times 90^\circ) = 360^\circ$ (b) $8 \times 90^\circ = 720^\circ$

Figure 6. Total turning angle for a polyomino

From this reasoning, we deduce that $90(a-b)$ is a multiple of 360, and therefore that $a-b$ is a multiple of 4. Hence $a-b$ is an even number. Therefore, $a+b=(a-b)+2b$ is an even number as well. That is, k is even. So the number of sides of the polyomino is an even number. □

Unsolved problems

There is just 1 monomino, and just 1 domino; there are 2 trominoes, and 5 tetrominoes. These numbers give rise to an interesting but difficult problem. For any positive integer n, let $f(n)$ denote the number of different n-ominoes. Care is needed in interpreting the word 'different.' We regard two shapes as 'the same' if they are geometrically congruent to each other ('congruent' in the usual, Euclidean sense of that word); and two shapes are 'different' if they are not congruent to one another. With this understanding, we find the following values taken by the function f.

n	1	2	3	4	5	6	7	$\cdots$
$f(n)$	1	1	2	5	12	35	108	$\cdots$

The 12 possible pentominoes are depicted in Figure 7. We will leave it to you to sketch the 35 possible hexominoes. Or you may refer to [3] for some sketches.

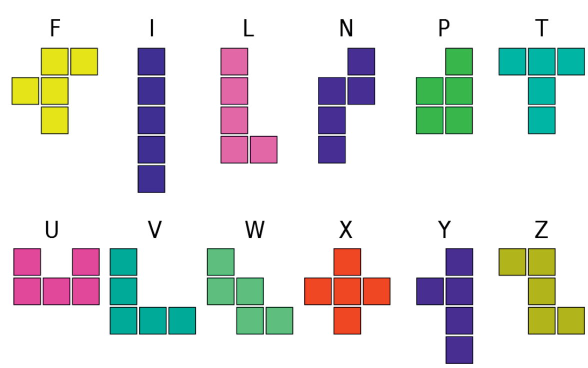


Figure 7. The 12 pentominoes
Source: https://simple.wikipedia.org/wiki/Pentomino#/media/File:Pentominos.svg.

Now for the problems that these numbers suggest: (a) Given n, is there a simple way of computing the value of $f(n)$? (b) Is there a simple formula for $f(n)$? As of now, the answers to both questions appear to be **No**, and the only known way to compute $f(n)$ for higher values of n is to use computer-assisted enumeration, based on clever algorithms. Using such means, the sequence of values of f has been obtained to many terms (the first two values are $f(1)=1$ and $f(2)=1$):

$$1,\ 1,\ 2,\ 5,\ 12,\ 35,\ 108,\ 369,\ 1285,\ 4655,\ 17073,\ 63600,\ 238591,\ 901971,\ 3426576,\ 13079255,\ 50107909,\ 192622052,\ 742624232,\ 2870671950,\ 11123060678,\ 43191857688,\ 168047007728,\ 654999700403,\ 2557227044764,\ 9999088822075,\ 39153010938487,\ 153511100594603,\ \ldots.$$

These two questions and others of their kind continue to remain unanswered at the present moment. However, it has been shown (see [1] and [4]) that $3.72^n < f(n) < 4.65^n$ for all positive integers n.

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Would you believe that . . .

69720375229712477164533808935312303556800

. . . is the smallest positive integer that is exactly divisible by each number from 1 to 100?
In how many ways can you arrive at this number?
Which is the smallest number which does *not* divide this number?
If any of your students come up with a well-thought-out response, do share it with us.

Our e-mail ID is atria.editor@apu.edu.in.

— Adapted from Moloy De's Facebook page "Math Believe It or Not".
See: https://www.facebook.com/photo.php?fbid=10206841050456928&set=gm.992345247471454&type=3&theater.

An Exploration with Surds

Bharat Karmarkar

In this short note we present a classroom vignette involving surds. Its origin lies in the following problem.

Problem. *Simplify the following expression:*

$$\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}. \tag{1}$$

We observe the following after the usual step of rationalising the denominators:

$$\begin{aligned}&\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}\\ &=\frac{4-2\sqrt{3}}{16-12}+\frac{2\sqrt{3}-2\sqrt{2}}{12-8}+\frac{2\sqrt{2}-2}{8-4}\\ &=\frac{4-2\sqrt{3}+2\sqrt{3}-2\sqrt{2}+2\sqrt{2}-2}{4}\\ &=\frac{4-2}{4}=\frac{1}{2}. \qquad \text{(Note the telescopic cancellation.)}\end{aligned}$$

It is nice to see an instance of irrational numbers adding up to a rational number!

Keywords: *Surd, irrational, telescoping sum, arithmetic progression, exploration*

Uncovering the origin of the relation. Now the exploration commences. We first try to trace the origin of this problem. Note that

$$4^2 = 16, \quad 2^2 = 4, \quad 16 - 4 = 12, \quad \frac{12}{3} = 4. \tag{2}$$

Let us divide the interval from 4 to 16 into three equal parts; we get an AP (4, 8, 12, 16) with a common difference of 4. We write these numbers in decreasing order, from highest to lowest: 16, 12, 8, 4. Now we have (note the telescopic cancellation):

$$\left(\sqrt{16}-\sqrt{12}\right)+\left(\sqrt{12}-\sqrt{8}\right)+\left(\sqrt{8}-\sqrt{4}\right)=\sqrt{16}-\sqrt{4}=4-2=2. \tag{3}$$

From this relation we get by 'reverse-rationalisation':

$$\frac{4}{\sqrt{16}+\sqrt{12}}+\frac{4}{\sqrt{12}+\sqrt{8}}+\frac{4}{\sqrt{8}+\sqrt{4}}=2. \tag{4}$$

The numerators in these fractions are equal precisely because 16, 12, 8, 4 form an AP.

Next, the square roots of the numbers 16, 12, 8, 4 are, respectively:

$$4, \quad 2\sqrt{3}, \quad 2\sqrt{2}, \quad 2.$$

Hence we get:

$$\frac{4}{4+2\sqrt{3}}+\frac{4}{2\sqrt{3}+2\sqrt{2}}+\frac{4}{2\sqrt{2}+2}=2,$$

that is,

$$\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}=\frac{1}{2}. \tag{5}$$

We have recovered the relation with which we started the exploration.

Finding more such relations

Having uncovered the above, it becomes easy to generate more such relations. The same pair of numbers (16 and 4) can generate other APs and therefore more such relations. For example:

(1) **AP:** 16, 10, 4**.** The common difference is 6, the square roots of the numbers are, respectively: 4, $\sqrt{10}$, 2, and $(4-2) \div 6 = 1/3$; hence we get:

$$\frac{1}{4+\sqrt{10}}+\frac{1}{\sqrt{10}+2}=\frac{1}{3}. \tag{6}$$

(2) **AP:** 16, 12, 8, 4**.** This yields the relation we studied at the start:

$$\frac{1}{4+2\sqrt{3}}+\frac{1}{2\sqrt{3}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+2}=\frac{1}{2}.$$

(3) **AP:** 16, 13, 10, 7, 4**.** The common difference is 3, the square roots of the numbers are, respectively: 4, $\sqrt{13}$, $\sqrt{10}$, $\sqrt{7}$, 2, and $(4-2) \div 3 = 2/3$; hence we get:

$$\frac{1}{4+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{7}}+\frac{1}{\sqrt{7}+2}=\frac{2}{3}. \tag{7}$$

(4) **AP:** 16, 14, 12, 10, 8, 6, 4**.** The common difference is 2, and $(4-2) \div 2 = 1$; hence:

$$\frac{1}{4+\sqrt{14}}+\frac{1}{\sqrt{14}+2\sqrt{3}}+\frac{1}{2\sqrt{3}+\sqrt{10}}+\frac{1}{\sqrt{10}+2\sqrt{2}}+\frac{1}{2\sqrt{2}+\sqrt{6}}+\frac{1}{\sqrt{6}+2}=1. \tag{8}$$

(5) **AP:** 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4**.** This yields:

$$\frac{1}{4+\sqrt{15}}+\frac{1}{\sqrt{15}+\sqrt{14}}+\frac{1}{\sqrt{14}+\sqrt{13}}+\cdots+\frac{1}{\sqrt{7}+\sqrt{6}}+\frac{1}{\sqrt{6}+\sqrt{5}}+\frac{1}{\sqrt{5}+2}=2. \tag{9}$$

The numbers on the right-hand sides of the above equalities are the following fractions whose least common denominator is 6:

$$\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ 2, \quad \text{i.e.,} \quad \frac{2}{6},\ \frac{3}{6},\ \frac{4}{6},\ \frac{6}{6},\ \frac{12}{6}. \tag{10}$$

The numerators of these fractions are 2, 3, 4, 6, 12, which are the divisors of 12 arranged in increasing order.

For the sake of completeness, we can adjoin to the above list the fraction $1/6$ which we get as follows:

$$\frac{1}{\sqrt{16}+\sqrt{4}} = \frac{1}{6}. \tag{11}$$

We have obtained a recipé for generating such equalities!

Putting the recipé into practice

Take another pair of perfect squares, say $9 = 3^2$ and $25 = 5^2$. Since $25 - 9 = 16$, and the divisors of 16 are 1, 2, 4, 8 and 16, we obtain the following sequence of equalities:

$$\frac{1}{5+3} = \frac{1}{8},$$

$$\frac{1}{5+\sqrt{17}} + \frac{1}{\sqrt{17}+3} = \frac{2}{8},$$

$$\frac{1}{5+\sqrt{21}} + \frac{1}{\sqrt{21}+\sqrt{17}} + \frac{1}{\sqrt{17}+\sqrt{13}} + \frac{1}{\sqrt{13}+3} = \frac{4}{8},$$

$$\frac{1}{5+\sqrt{23}} + \frac{1}{\sqrt{23}+\sqrt{21}} + \frac{1}{\sqrt{21}+\sqrt{19}} + \cdots + \frac{1}{\sqrt{13}+\sqrt{11}} + \frac{1}{\sqrt{11}+3} = \frac{8}{8},$$

$$\frac{1}{5+\sqrt{24}} + \frac{1}{\sqrt{24}+\sqrt{23}} + \frac{1}{\sqrt{23}+\sqrt{22}} + \cdots + \frac{1}{\sqrt{12}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{10}} + \frac{1}{\sqrt{10}+3} = \frac{16}{8}.$$

Further extensions are surely possible. Please work them out on your own.

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PRIMES IN Arithmetic Progression

In the article *Prime Magic Squares* by Vinay Nair (*At Right Angles*, November 2015 issue), the following result was stated without proof: *If three prime numbers exceeding* 3 *form an arithmetic progression, then the common difference of the AP is a multiple of* 6. **Tejash Patel** of Patan, Gujarat has sent us a proof of this statement, using the approach of "proof by contradiction." He starts by noting that to prove that a number is a multiple of 6, it is sufficient if we prove that it is even and a multiple of 3.

Let the three prime numbers be p, $p+d$, $p+2d$ where the number d is the common difference of the AP; here $p>3$ and $d>0$. Since $p>3$, it follows that $p, p+d, p+2d$ are all odd; hence d is an even number. Next, since $p>3$, it must be true that p is not a multiple of 3; it is either of the form $3a+1$ or $3a+2$ where a is some positive integer. Now consider d; suppose that d is not a multiple of 3; then it is either of the form $3b+1$ or $3b+2$ where b is some nonnegative integer. There are $2\times 2=4$ possibilities for the pair (p,d). We consider the implications of each possibility for the triple $(p,p+d,p+2d)$. The possibilities are shown in the box below.

p	d	$(p,p+d,p+2d)$
$3a+1$	$3b+1$	$(3a+1, 3a+3b+2, \boxed{3a+6b+3})$
$3a+1$	$3b+2$	$(3a+1, \boxed{3a+3b+3}, 3a+6b+5)$
$3a+2$	$3b+1$	$(3a+2, \boxed{3a+3b+3}, 3a+6b+4)$
$3a+2$	$3b+2$	$(3a+2, 3a+3b+4, \boxed{3a+6b+6})$

We see that in each case, one of the numbers $p,p+d,p+2d$ is a multiple of 3 (the multiple of 3 has been highlighted) and therefore not a prime number (since it also exceeds 3). This contradicts the supposition that $p,p+d,p+2d$ are all prime numbers.

It follows that d is necessarily a multiple of 6.

— C⊗MαC

Keywords. *Primes, arithmetic progression*

Low Floor High Ceiling Tasks

Try-Angles!

How acute can you be?

In the November 2014 issue, we began a new series which was a compilation of 'Low Floor High Ceiling' activities. A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that each student is pushed to his or her maximum while attempting the work. There is enough work for all; but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task. With pentominoes, pentagons and tangrams out of the box, the first issue of this year sees us un-wrapping the eternal triangle.

ClassRoom

Swati Sircar & Sneha Titus

In this issue's task we work with right-angled triangles, isosceles as well as scalene. The activity has enormous scope for creativity, visualisation, investigation, pattern recognition, documentation and conjecture. Facilitators should encourage students to come up with proofs for conjectures that they make.

You need two sets of triangles to work with. Each set will need at least 20 triangles; they may be made of cardboard so that they do not fray too easily. The triangles need not be very big; in fact, our idea sprung from the waste paper that was being trimmed off the corner of some rectangular cards [Fig. 1].

Keywords: *Triangle, quadrilateral, square, rectangle, rhombus, parallelogram, trapezium, angle, congruence, similarity, collaborative, pattern, conjecture*

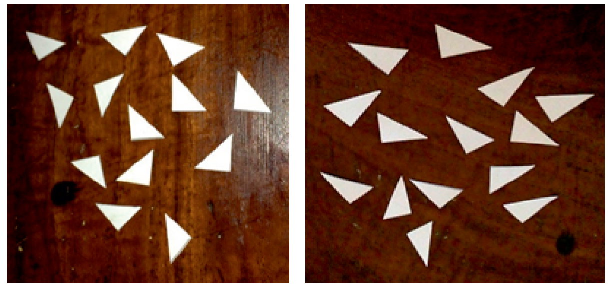

Figure 1

The first is a set of congruent right isosceles triangles (assume that the sides are 1-1-√2) [Fig. 2]. The second is a set of congruent scalene right-angled triangles (assume that the sides are a, b, c where $a < b < c$); see Fig. 3.

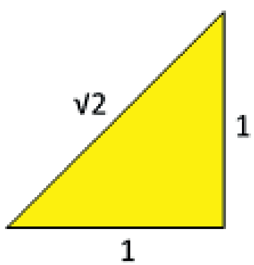

Figure 2

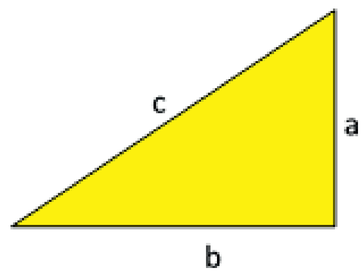

Figure 3

In all the described tasks, the aim is to use these triangles as building blocks to create specified geometric shapes. A necessary condition for tiling these triangles is that there are no gaps and no overlaps. These shapes can be created in a variety of ways and the aim is to encourage students to classify these ways and generalise from them.

This low floor high ceiling task would be appropriate for students of middle school. They need to know the properties of triangles, the sum of angles of a quadrilateral, the different types of quadrilaterals and their properties, and the Pythagoras theorem. An enterprising facilitator can push this activity to including crossed quadrilaterals too, but there is plenty of scope for mathematisation and skill development even within the ambit of triangles and convex quadrilaterals.

TASK 1: To make different Quadrilaterals from the given Triangles

1.1 Try to make all possible quadrilaterals with the right isosceles set and identify the resulting shapes. You may use as many as you wish but there should be no gaps between the pieces. Fill in the following table:

Name of quadrilateral	Number of right isosceles triangles used	Sketch/picture	Possibility of enlarging this quadrilateral using more triangles	
			Sketch	**Observation**

1.2 Which shape is missing?

1.3 Repeat the above with right scalene triangles. Document your results in a similar table.

1.4 Did you get any new quadrilateral with the right scalene triangles?

1.5 Is there any quadrilateral that cannot be made with the right scalene triangles?

Teacher's Note:

This task is a great way to get students to apply their understanding of the properties and differences between various classes of quadrilaterals. A stab at the high ceiling would be to ask the students to make a general quadrilateral and investigate its special properties. The facilitator should note that the same quadrilateral may be made with different numbers of triangles. Or, the same quadrilateral can be made with the same number of triangles but in a different orientation. The quadrilateral could be an enlarged version of the original, with the triangles in the same or different orientation. Interestingly, the dimensions of the quadrilaterals may or may not change and this aspect would be worth recording and studying for pattern development. Striking number patterns may emerge here. The facilitator should encourage students to document their findings systematically and help them to generalise from these. When the right isosceles triangles are used, a general rhombus (excluding the specific case of a square) and a kite cannot be made, and when general scalene triangles are used, a square cannot be made.

TASK 2: To make different Triangles from the given Triangles

2.1 Try to make as many triangles as possible with the right isosceles triangles. What kind(s) of triangles do you get? Tabulate your results as above.

2.2 Which triangles are possible? Explain why.

2.3 Repeat the above with right scalene triangles and tabulate your results.

2.4 What kind(s) of triangles did you get? How is this different from the previous case?

Teacher's Note:

With the right isosceles triangles, it is only possible to make right isosceles triangles; for hints for a proof, the reader can refer to the Low Floor High Ceiling article in *At Right Angles*, Volume 4, No. 3 (November 2015). With the scalene triangles, right scalene and acute and obtuse isosceles triangles are possible. Nice patterns emerge, such as the triangles made with 1, 4, 9, 16, … triangles and the connection between this and sums of consecutive odd numbers. The facilitator must encourage students to document their work systematically and explain their findings logically.

This task is especially exciting because theorems such as the mid-point theorem and Thales' basic proportionality theorem are clearly demonstrated in some configurations of the triangles. (See Fig. 4.) This may be a good opportunity to distinguish between demonstration and proof.

Note from Fig. 5 how this orientation of triangles clearly marks the circumcentre of the outer triangle for both right isosceles as well as for right scalene. More subtly, the figure demonstrates that in a right isosceles triangle, the perpendicular bisectors of all three sides are concurrent.

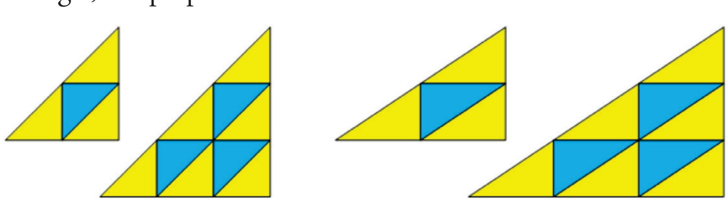

Figure 4

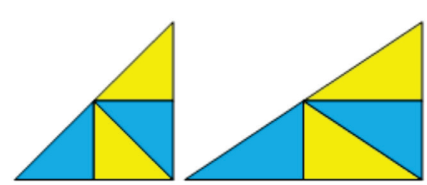

Figure 5

TASK 3: To extend the previous task to make Triangles from Acute/Obtuse-angled triangles

3.1 Which of the above arrangements to form triangles can be made with acute-angled triangles?

3.2 Which arrangements cannot be made?

3.3 Which of the above arrangements can be made with obtuse-angled triangles?

3.4 Which arrangements cannot be made?

Teacher's Note:

Arrangements like those in Fig. 4 are possible with acute- and obtuse-angled triangles, whereas arrangements like those in Fig. 5 are not possible with acute- or obtuse-angled triangles; they are only possible with right-angled triangles. This task moves from working with concrete materials to a more abstract style. If the student finds this difficult, the facilitator could suggest the use of pencil and paper or dynamic geometry software such as GeoGebra.

TASK 4: Squares (and Triangles) with the right Isosceles Triangles – to generalise the number of Triangles used

4.1 Make squares with the triangles and fill in the following table:

Number of right Isosceles Triangles used	Picture of Square (if it is possible to construct it)	Side of the Square
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		

Continue until a pattern emerges. (You could go up to 100 triangles.)

4.2 If we can make a square with m triangles, can we make a square with $2m$ triangles?

4.3 Give a general representation for the number of right isosceles triangles with which you can make a square?

4.4 Give a general representation for the number of right isosceles triangles with which you can make a right isosceles triangle?

Teacher's Note:

This task invokes visualisation and kinesthetic skills but students who are not very adept at these need not be daunted because soon a pattern emerges; once they realise that the sides of the possible squares can only be multiples of 1 or $\sqrt{2}$, they are able to speed ahead with the configurations possible for squares corresponding to these sides. Skillful facilitation can also elicit recognition of arithmetic progressions as the squares are enlarged. Students should be able to generalise their finding, that squares can only be made with $(2n)^2$ and $2n^2$ triangles.

The last question provides an interesting variation and students may be guided to recognise that these triangles may be made with n^2 or $2n^2$ right isosceles triangles – challenge them to prove that this can be generalised as n^2 triangles or $2n^2$ triangles.

TASK 5: More generalisations with right Isosceles Triangles – findings for a combination of Squares and Triangles

5.1 If we can make a square with m triangles, can we make a triangle with m triangles?

5.2 If we can make a triangle with m triangles, can we make a square with m triangles?

Teacher's Note:

Slicing the square along a diagonal and reorienting the triangles will always give a triangle, and this is a good activity to develop visualisation skills in students. So the answer to 5.1 is always 'Yes', but the same is true for 5.2 only when the number of triangles used is even.

Conclusion

Idle hands make for a math learning activity! The power of working with scrap material unfolded as we doodled with the triangles and we are quite sure that we haven't plumbed the depths or soared to the heights that more play can reveal. These tasks are sure to keep students learning experientially and we hope that teachers will document and post their classroom discoveries on AtRiUM (our FaceBook page)!

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ClassRoom

20-30-130, A Trigonometric Solution

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

The problem we study in this short note belongs to an extremely interesting class of geometrical problems. Typically, they deal with triangles with many lines drawn within them, intersecting at angles whose measures are an integer number of degrees. We are required to find the measure of some indicated angle. A common feature of these problems is that solutions using only pure geometry are difficult to find (though not impossible), and one is forced to fall back upon trigonometry. The present problem is of just this type.

Figure 1 shows $\triangle ABC$ with $\measuredangle A = 130^\circ$, $\measuredangle B = 30^\circ$ and $\measuredangle C = 20^\circ$. Point P is located within the triangle by drawing rays as shown from B and C, such that $\measuredangle PBC = 10^\circ$ and $\measuredangle PCB = 10^\circ$. Segment PA is then drawn. We are asked to find the measure of $\measuredangle PAC$.

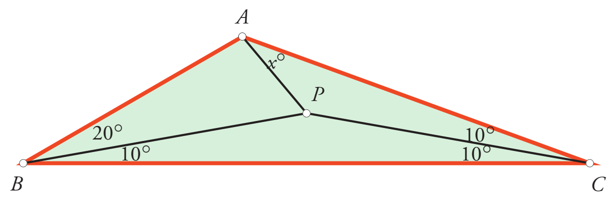


Figure 1

Keywords: *Integer degree, sine rule, complementary angle, supplementary angle, double angle identities, sine of the sum of two angles*

Solution. Let $\measuredangle PAC = x^\circ$; then $\measuredangle PAB = (130 - x)^\circ$. We now use the fact that $PB = PC$.

From $\triangle APB$ we have

$$\frac{AP}{PB} = \frac{\sin 20^\circ}{\sin(130 - x)^\circ}.$$

From $\triangle APC$ we have:

$$\frac{AP}{PC} = \frac{\sin 10^\circ}{\sin x^\circ}.$$

Hence:

$$\frac{\sin 20^\circ}{\sin(130 - x)^\circ} = \frac{\sin 10^\circ}{\sin x^\circ}.$$

This yields:

$$\begin{aligned} 2\cos 10^\circ &= \frac{\sin(130 - x)^\circ}{\sin x^\circ} \\ &= \sin 130^\circ \cot x^\circ - \cos 130^\circ \\ &= \sin 50^\circ \cot x^\circ + \cos 50^\circ. \end{aligned}$$

Hence:

$$\begin{aligned} \cot x^\circ &= \frac{2\cos 10^\circ - \cos 50^\circ}{\sin 50^\circ} \\ &= \frac{2\sin 80^\circ - \sin 40^\circ}{\cos 40^\circ} \\ &= 4\sin 40^\circ - \tan 40^\circ. \end{aligned}$$

Computation using a calculator shows that $4\sin 40^\circ - \tan 40^\circ \approx 1.732$. This *suggests* (but obviously does not prove) that $x = 30$.

Let us now formally prove that $x = 30$. For this, we must prove the following identity:

$$\tan 60^\circ = 4\sin 40^\circ - \tan 40^\circ,$$

i.e., $\tan 60^\circ + \tan 40^\circ = 4\sin 40^\circ$. Here, the LHS is equal to:

$$\begin{aligned} &\frac{\sin 60^\circ}{\cos 60^\circ} + \frac{\sin 40^\circ}{\cos 40^\circ} \\ &= \frac{\sin 60^\circ \cos 40^\circ + \sin 40^\circ \cos 60^\circ}{\cos 60^\circ \cos 40^\circ} \\ &= \frac{\sin 100^\circ}{\cos 60^\circ \cos 40^\circ} = \frac{2\sin 80^\circ}{\cos 40^\circ} \\ &= 4\sin 40^\circ, \end{aligned}$$

which is just what we wanted.

What we have actually shown is that $\cot x^\circ = \cot 30^\circ$. Since $0^\circ < x^\circ < 180^\circ$, it follows that $x = 30$. Hence $\measuredangle PAC = 30^\circ$. $\square$

Remark. Note the approach that we have used: trigonometric analysis, based on well-known properties of triangles; a guess at the answer, using approximate computations; and then a proof that the answer is correct, drawing on well-known identities. These are common themes in problems of this genre. We will see them again in the future.

The question of a pure geometry solution remains. We shall pass on the challenge to the reader. If you come up with a solution within the boundaries of synthetic geometry, do please share it with us!

List of trigonometric identities used in the article

To the reader: If you have 'insider knowledge' of trig identities, this article will validate many known facts. If you have not, then it may be a good idea for you to try and match which identity has been used in each instance in the article above.

Supplementary angle identities: For any angle x°,

$$\sin(180 - x)^\circ = \sin x^\circ,$$

$$\cos(180 - x)^\circ = -\cos x^\circ,$$

$$\tan(180 - x)^\circ = -\tan x^\circ.$$

Double angle identities: For any angle x,

$$\sin 2x = 2\sin x \cos x,$$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

Addition formula: For any two angles x and y,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Sine rule for triangles: For any $\triangle ABC$ with sides a, b, c:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

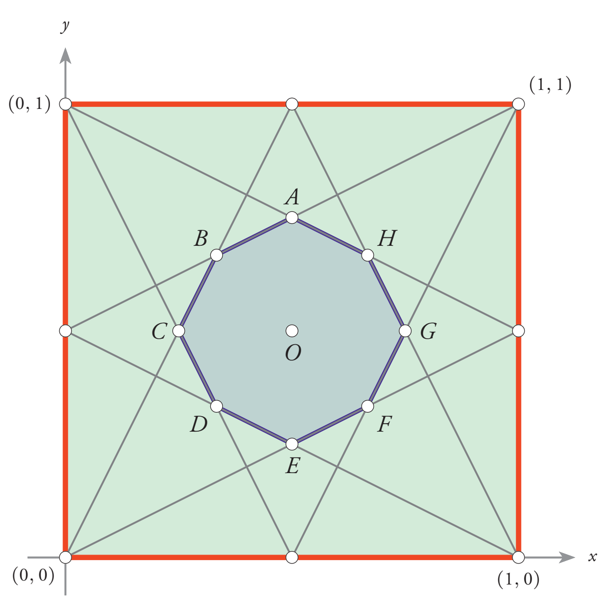
where R is the circumradius of the triangle.

Octagon in a Square

ClassRoom

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

In the November 2015 issue of AtRiA, the following geometrical puzzle had been posed. An octagon is constructed within a square by joining each vertex of the square to the midpoints of the two sides remote from that vertex. Eight line segments are thus drawn within the square, creating an octagon (shown shaded). The following two questions had been posed: (i) Is the octagon regular? (ii) What is the ratio of the area of the octagon to that of the square?



Label the vertices of the octagon A, B, C, D, E, F, G, H as shown. Let O be the centre of the square. Recall that a polygon is said to be regular if its sides have equal length and its internal angles have equal measure. We shall show that the octagon is *not* regular!

We assign coordinates to the plane in such a way that the vertices of the enclosing square have coordinates $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, as shown. The slope of side AB is $1/2$, that of side BC is 2, and that of side CD is -2. Using a well-known formula from coordinate geometry (the one which gives the tangent of the angle between two lines, using their slopes), we compute the tangents of $\measuredangle ABC$ and $\measuredangle BCD$. We have:

$$\tan \measuredangle ABC = -\frac{2-1/2}{1+2\cdot 1/2} = -\frac{3/2}{2} = -\frac{3}{4},$$

$$\tan \measuredangle BCD = -\frac{-2-2}{1+(-2)\cdot 2} = -\frac{-4}{-3} = -\frac{4}{3}.$$

Note what we have found: the tangents of these two angles are not equal! It follows by symmetry that the tangents of the internal angles of the octagon are alternately $-3/4$ and $-4/3$. Hence the octagon is not regular. (For regularity, all the angles would obviously have to have the same tangent value.)

Something curious ...

We note something quite remarkable here. It is easy to see that each of the angles AOB, BOC, COD, ..., GOH, HOA equals 45°. Slightly less obvious but also true is the fact that the sides AB, BC, CD, ..., HA have equal length. For, the equations of the sides are the following:

$$\begin{aligned} \text{Equation of } HA: &\quad y = -\frac{x}{2} + 1, \\ \text{Equation of } AB: &\quad y = \frac{x}{2} + \frac{1}{2}, \\ \text{Equation of } BC: &\quad y = 2x, \\ \text{Equation of } CD: &\quad y = -2x + 1. \end{aligned}$$

Hence we get, by solving pairs of equations:

$$A = \left(\frac{1}{2}, \frac{3}{4}\right), \quad B = \left(\frac{1}{3}, \frac{2}{3}\right), \quad C = \left(\frac{1}{4}, \frac{1}{2}\right),$$

implying that:

$$AB^2 = \frac{1}{6^2} + \frac{1}{12^2} = \frac{5}{144},$$

$$BC^2 = \frac{1}{12^2} + \frac{1}{6^2} = \frac{5}{144}.$$

By symmetry it follows that all the eight sides of the octagon have equal length (equal to $\sqrt{5}/12$ of the sides of the square).

So we see here an instance of an octagon whose sides have equal length, and whose sides subtend equal angles at a point which is symmetrically placed relative to alternate sets of vertices of the octagon, yet which is not regular. Rather a counterintuitive result!

Remark. Another approach to showing that the octagon is not regular is: first compute the coordinates of the vertices (as done above), and then compute the distances from the centre O to the vertices. We have:

$$OA = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$OB = \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{2} - \frac{2}{3}\right)^2} = \frac{\sqrt{2}}{6}.$$

By symmetry we have $OA = OC = OE = OG = 1/4$ and $OB = OD = OF = OH = \sqrt{2}/6$. Since $1/4 \neq \sqrt{2}/6$, it follows that the octagon is not regular.

Computation of area

It remains to compute the area of the octagon relative to the area of the square. There are many ways of obtaining the result, but we shall simply use the sine formula for area of a triangle. We have, since $O = (1/2, 1/2)$:

$$OA = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$OB = \sqrt{\left(\frac{1}{3} - \frac{1}{2}\right)^2 + \left(\frac{2}{3} - \frac{1}{2}\right)^2} = \sqrt{\frac{1}{6^2} + \frac{1}{6^2}} = \frac{\sqrt{2}}{6}.$$

Hence:

$$\text{Area of } \triangle OAB = \frac{1}{2} \times \frac{1}{4} \times \frac{\sqrt{2}}{6} \times \sin 45^\circ = \frac{1}{2} \times \frac{1}{4} \times \frac{\sqrt{2}}{6} \times \frac{1}{\sqrt{2}} = \frac{1}{48}.$$

It follows that the area of the octagon is

$$8 \times \frac{1}{48} = \frac{1}{6}.$$

Hence the area of the octagon is $1/6$ of the area of the square.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

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TechSpace

Desmos Classroom Activities

Play and Learn with Desmos

Desmos (www.desmos.com), the online graphing calculator, is changing classrooms across the world. In my earlier article (Vol. 3, No. 2, July 2014 | At Right Angles), I had taken you through the graphing power of Desmos and had described how Desmos helped to bring out the creativity of my students as they explored the concept of 'Domain and Range of a Function'.

Sangeeta Gulati

This article will highlight another aspect of Desmos, the in-built pre-designed activities, which can be used by the teacher in an interactive manner with students in the mathematics class. These activities, which can be found at https://teacher.desmos.com/, are made by the Desmos team (https://www.desmos.com/team) along with other teachers. A Desmos Class Activity is essentially a sequence of screens, each with a different task, prompt, or question.

You can get started (Fig. 1) by going to the website https://teacher.desmos.com/ and creating an account (if you have one which you used to launch the calculator you can use it to login). To use these activities, the teacher will need to have an account, but students do not require it.

Keywords: *Desmos, freeware, activities, math lab, CCE, mensuration, linear functions*

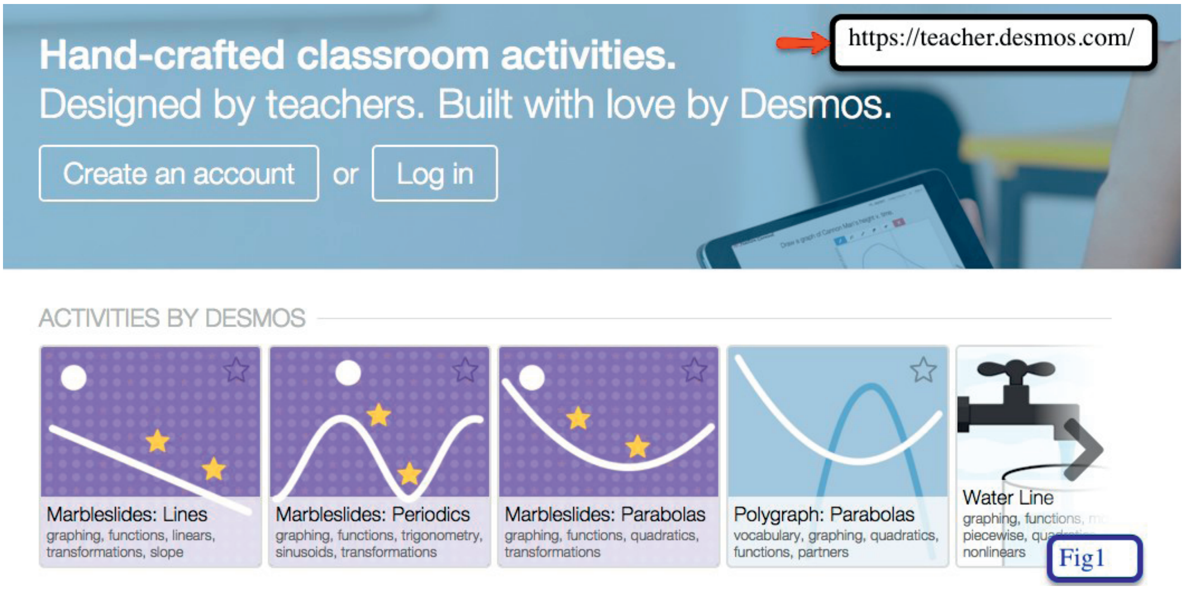


Figure 1

As the teacher selects 'Start a New Session' option for the selected Desmos Activity, a Class Code is generated. Students enter this code on student.desmos.com to get started. As they submit the selected name, the same shows up on the teacher's dashboard; and the fun begins.

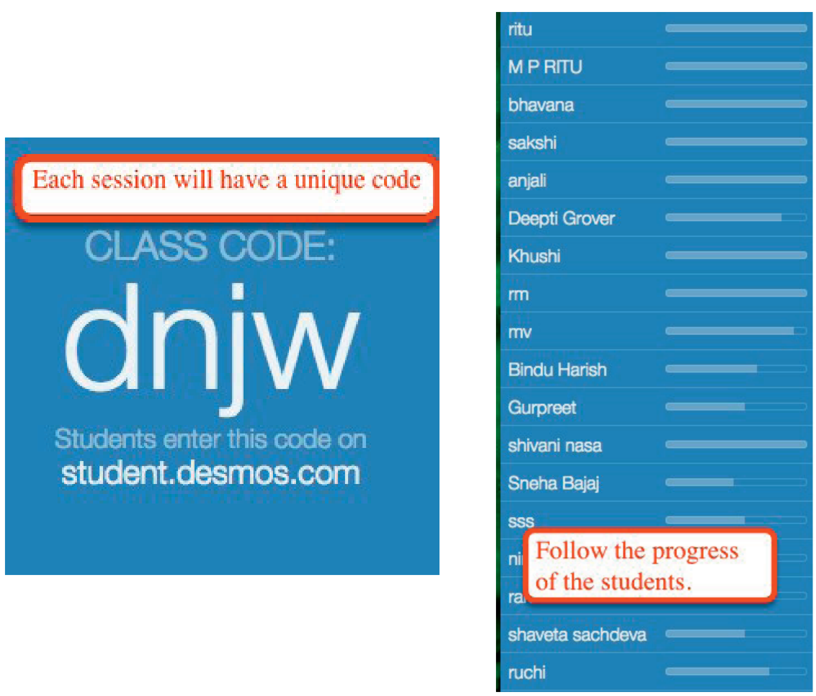


Figure 2

In each activity, there are essentially three screen types: Graph, Question, and Text. The task described in the 'Graph' screens requires an action from the student; this could be 'Plot a line passing through the two given points on the graph' (Match My Line) or 'Drag these dividers to create four spaces of equal width' (Central Park).

The screen with 'Question' gives students space to express their thinking. Prompts such as 'Describe two methods you could use to draw this line' next to a graph require students to type out their ideas which may or may not be made visible to other students of the class, but which you, as the teacher, will get to see in real time. The screen for the 'Text' usually concludes the activity with a message for the students, accompanied by a related image.

Each task in the Classroom Activity section helps to engage students in various ways. As the difficulty level changes at each stage, the student is challenged and nudged to think differently and apply his/her learning. Most of the students start the activities by guessing what the solutions could be, but they soon get into the mode of thinking 'mathematically' as the guesses do not work for the questions in the next few screens.

In this article, I am sharing my experience with two of the Desmos Classroom activities: Central Park and Match My Line.

Central Park

The Central Park Desmos Activity is most suitable for Grade 6 students. This activity helps the student to make the transition from arithmetic to algebra.

Central Park puts the power of algebra in the hands of students by asking them to design parking lots. At first, students place the parking lot dividers by estimating and guessing.

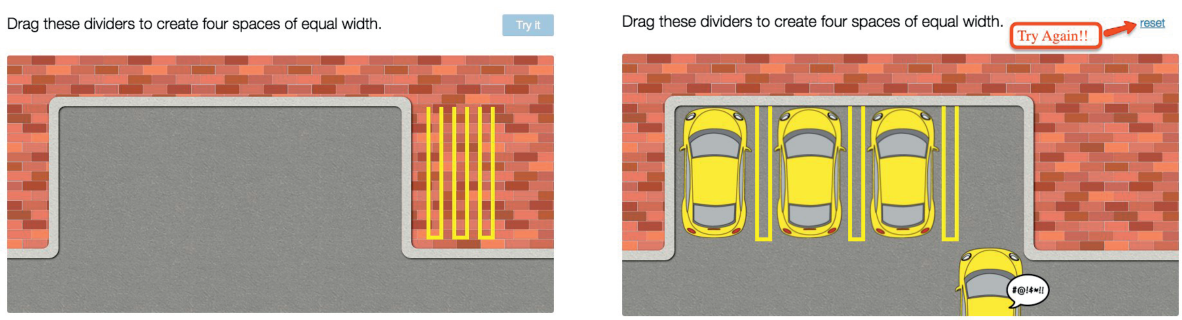


Figure 3

Then they compute the proper placement.

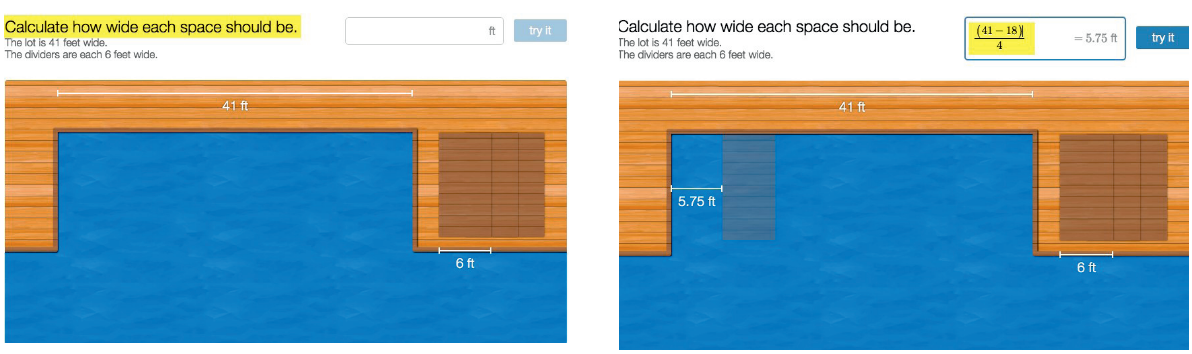


Figure 4

Finally, they write an algebraic expression that places the dividers for many different lots.

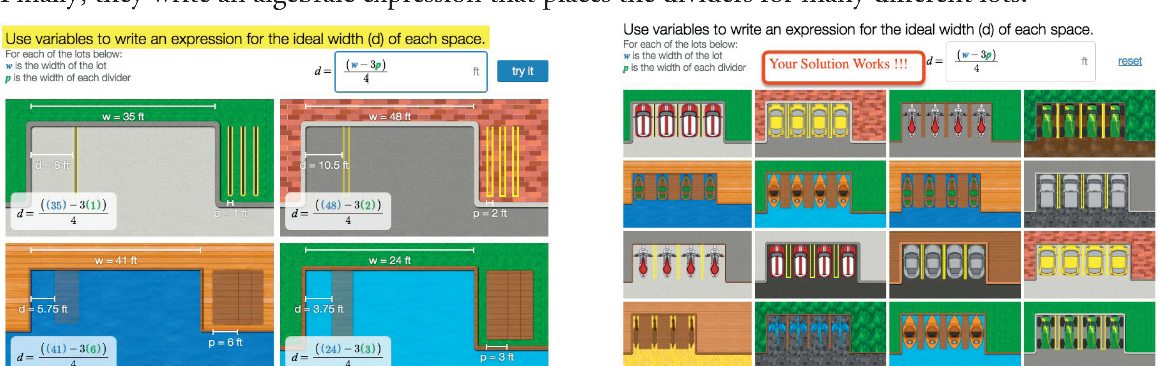


Figure 5

The transition from Estimate to Calculate and then to Algebraic Expression is so smooth that very few students need assistance from the teacher to get the correct expression at the end. The change-over from numbers to variables is also helped by the Question Screen, which asks students to verbalize their thought processes: *Write instructions explaining how to calculate the right width of the parking space for any situation.*

GURPREET
Width of parking space= {total length of wall-n(width of plank)}/n+1

RUCHIKA
(total width-(width of each divider*no of dividers))/no of partitions

SWETA JAIN
subtract the width of the three dividers from the total width and divide the remaining part by4

ANJALI
From the total length of the space, subtract the total width of the dividers and divide the result by 4.

VIKRAM SINGH
1/4(total distance-3*width of bars)

SNEHA BAJAJ
Subtract the total width of the planks from the width of the space and divide it by 4.

RITU
total width - sum of widths of dividers .then divide the difference by number of parking vehicles

KHUSHI
(total space-number of barriers*width of barrier)/ total parts in which the space has been divided

DEEPTI GROVER
add the width of the dividers, subtract it from the total length to be divided. Divide the result by the no. of spaces to be created.

Also at each stage, the student gets instant feedback and is encouraged to Reset and attempt the task again in case the solution does not meet the expectation.

Students spend 30 to 45 minutes in 'playing' the game, and unknowingly they get introduced to the 'algebra'! Most of my students who attempted this activity found it to be 'great fun' while a few termed it as 'challenging', and all wanted some more of such activities!

Match My Line

Match My Line can be used as an introductory activity to begin the topic on Straight Lines in Grade XI. It is a series of graphing challenges designed to build student understanding of linear functions in various forms.

The tasks are designed to prompt and provoke the student to think about the relationship between the given ordered pairs, and to work out the functional relationship which connects them. The ease of plotting and re-plotting graphs to come up with the required expression makes the activities so much more appealing and meaningful.

It is interesting to note that students who were already aware of the various forms of the equations of straight line opted to use the formulae to get the required equations, whereas students who had not learned these results prior to the activity used their observations and related the *x*- and *y*-coordinates of the points to come up with the appropriate expression to graph the lines. It was an 'Aha!' moment for the class to listen to the arguments of these students and appreciate the depth of their understanding.

Here are two different approaches in response to the Challenge #2: Plot six lines. Each of the lines should pass through the black point and one of the blue points (Fig. 6).

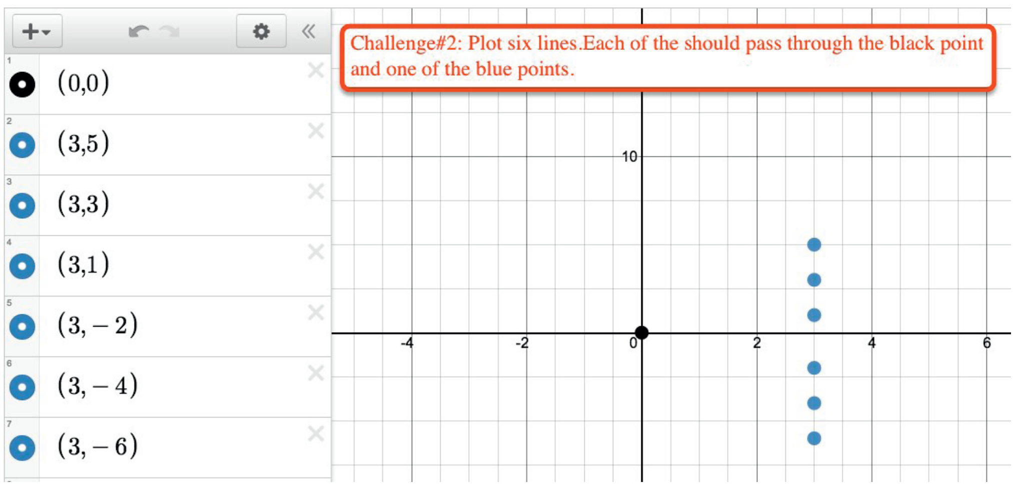


Figure 6

Desmos activities from the Teacher's Perspective

One of the main concerns of the teachers using technology in classrooms is that it is difficult to monitor individual students' work from the front of the class. Here, however, the setup is such that the teacher has a bird's eye view of students' screens. This feature allows the teacher to follow the progress of the class and identify individual students who need help or a word of caution. One can click on the name of the student to see his/her individual progress, or have an overview of the entire class on the screen.

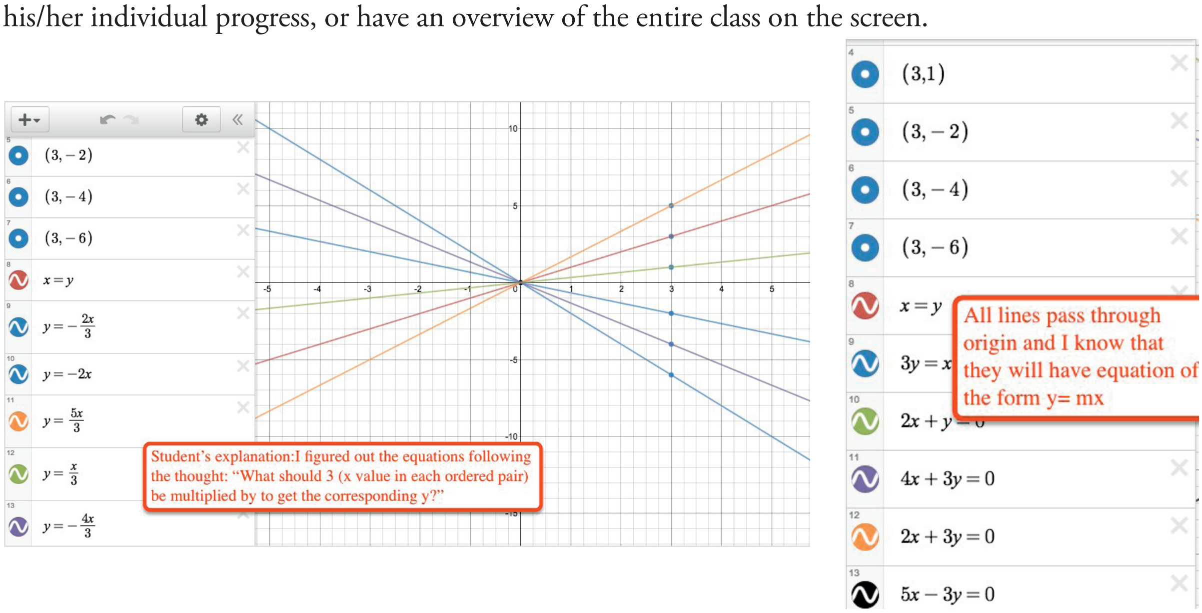


Figure 7

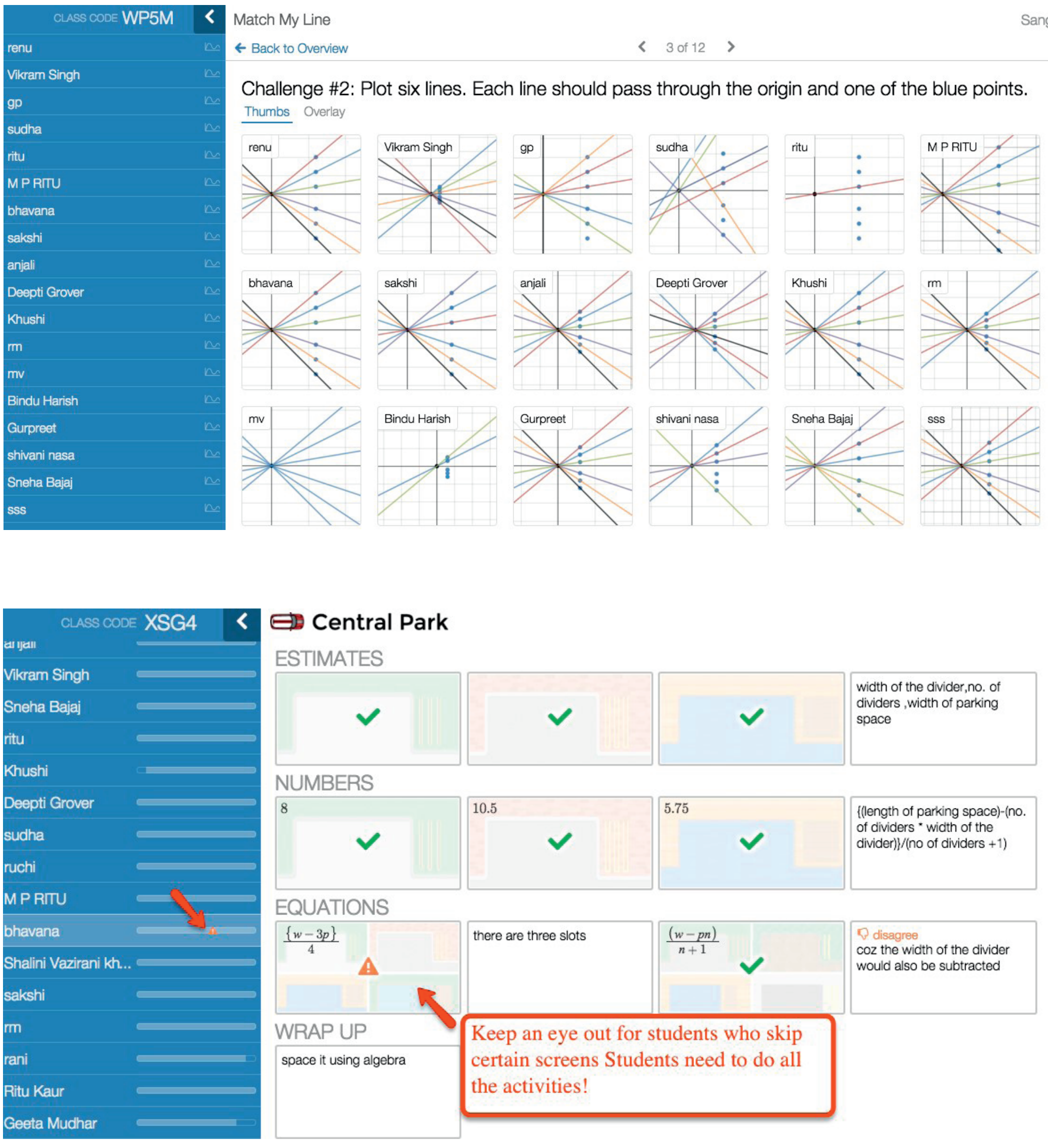


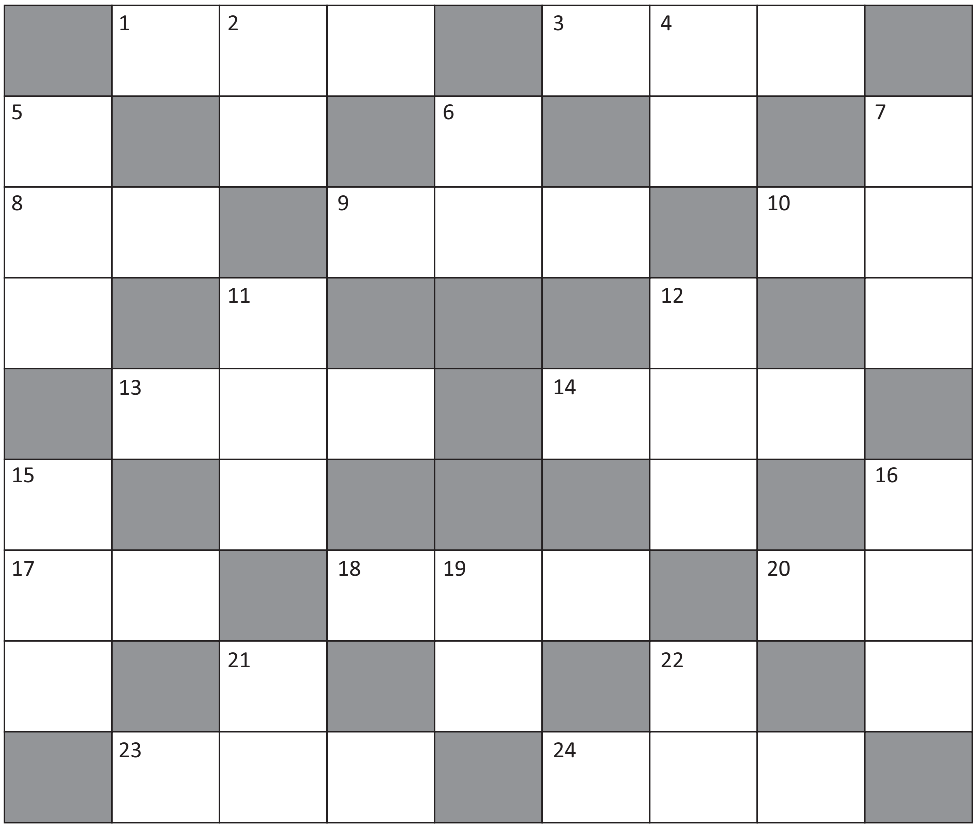
Figure 8

In addition to these two activities, there are several well-designed activities suitable for our classrooms; one can check them out at teacher.desmos.com. I am waiting to try out the latest activity from the Desmos Team with my students; it is called Marble Slides. I found this activity to be both challenging and addictive, with the potential to facilitate learning through hands-on exploration.

SANGEETA GULATI is Head of Mathematics at Sanskriti School, New Delhi. She has taught Mathematics for 25 years and has contributed to profesional development programs aimed at exploring the use of technology in the teaching-learning of mathematics all over India. She conducts workshops on GeoGebra, Geometer's Sketchpad, Google Apps for Education and Online Resources. She has been a regular resource person with NCERT in developing ICT material, and has developed video lessons for classes 11 and 12 with Central Institute of Educational Technology, NCERT. Her wikispace (dynamath.wikispaces.com), which is a product of her action-based research project during Fulbright Distinguish Award in Teaching fellowship in 2011, is a great resource for mathematics teachers. Sangeeta is a Google Certified Innovator and a Certified Edmodo Trainer. She may be contacted at sangeetagulati92@gmail.com.

NUMBER CROSSWORD

Solution on Page 95



CLUES ACROSS

1: 7,14,28,56, -----, 224

3: Circumference of a circle with a radius of 28 units

8: Number of days in February of this year

9: 2D times 22D plus 10

10: 12.5% of 512

13: 1, 2, 5, 26, -----, 458330

14: Exterior angle of an equilateral triangle

17: The second 2 digit prime

18: 15D minus 6D plus 1

20: 14877 divided by 783

23: Average of 573, 439, 811, 1113

24: One hundredth of the number of seconds in a day

CLUES DOWN

2: Difference of square roots of 2601 and 1369

4: A factor of 119 in reverse

5: 17A times 10A minus 4

6: One third of the largest two digit number

7: Three sides of a right angle triangle in ascending order

11: 1A multiplied by 7 minus 10

12: Area of a rectangle of sides 26 and 28

15: Two raised to 9

16: Largest three digit number

19: Largest two digit perfect square

21: Number of weeks in a year plus 1

22: First of three consecutive numbers totaling 51

Problem Corner

Problems for the Senior School

Problem Editors: Prithwijit De & Shailesh Shirali

PROBLEMS FOR SOLUTION

Problem V-1-S.1
What is the greatest possible perimeter of a right-angled triangle with integer sides, if one of the sides has length 12?

Problem V-1-S.2
Rectangle $ABCD$ has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\measuredangle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

Problem V-1-S.3
Let a, b, c be such that $a + b + c = 0$, and let

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}.$$

Determine the value of P.

Problem V-1-S.4
In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\measuredangle BAE = 40^\circ$. If $\measuredangle DAE = \measuredangle DFE$, what is the magnitude of $\measuredangle ADF$ in degrees?

Problem V-1-S.5
Circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume that $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\measuredangle PYX$ in degrees?

SOLUTIONS OF PROBLEMS IN ISSUE-IV-3 (NOVEMBER 2015)

Solution to problem IV-3-S.1
Determine all possible integers N such that $N(N-101)$ is the square of a positive integer.

Let $d = \text{GCD}(N, N-101)$. Then d divides $N-(N-101) = 101$. Hence $d = 1$ or $d = 101$.

If $d = 1$, then $N(N-101)$ is a perfect square if and only if each of N and $N-101$ is a square. Let $N = a^2$ and $N - 101 = b^2$. Then

$$(a-b)(a+b) = 101,$$

so $(a-b, a+b) = (1, 101)$, leading to $a = 51$, $b = 50$ and $N = 51^2 = 2601$.

If $d = 101$, then $N = 101k$ and $N - 101 = 101(k-1)$ for some positive integer $k > 1$. Therefore:

$$N(N-101) = (101^2)k(k-1).$$

But $k(k-1)$ is never a square for any positive integer $k > 1$, because

$$(k-1)^2 < k(k-1) < k^2.$$

Thus $N(N-101)$ is not a square when $d = 101$. It follows that there is just one integer value of N for which $N(N-101)$ is a perfect square; namely: $N = 2601$.

Solution to problem IV-3-S.2
Let R, S be two cubes with sides of lengths r, s, respectively, where r and s are positive integers. Show that the difference of their volumes numerically equals the difference of their surface areas if and only if $r = s$.

For the given condition to hold we must have

$$r^3 - s^3 = 6(r^2 - s^2). \qquad (1)$$

This can be expressed as

$$s^2 = (6-r)(r+s), \qquad (2)$$

which shows that $r < 6$. Similarly $s < 6$. Now observe that equation (1) can be written as

$$(6-r)^2 + (6-s)^2 + (r+s)^2 = 72. \qquad (3)$$

The right-hand side of relation (3) is divisible by 4. Therefore the left-hand side must be divisible by 4. The square of any positive integer is either divisible by 4, or exceeds a multiple of 4 by 1. Thus, in this case, each term on the left-hand side must be divisible by 4, which forces each term to be even. Hence, both r and s are even. If r, s are different, then r, s are $2, 4$ in some order, so $r + s = 6$, and by equation (2),

$$s^2 = 6(6-r),$$

which is false for positive integers $r, s \in \{2, 4\}$. Thus $r = s$. If we know that $r = s$, then equation (1) is clearly true.

Solution to problem IV-3-S.3
Suppose $S = \{0, 1\}$ has the following addition and multiplication rules: $0+0 = 0$, $0+1 = 1+0 = 1$, $1+1 = 0$, $0 \times 0 = 1 \times 0 = 0 \times 1 = 0$, $1 \times 1 = 1$. A system of polynomials is defined with coefficients in S. Show that in this system $x^3 + x + 1$ is not factorisable.

Suppose the polynomial can be factored as $(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \{0, 1\}$. By equating the coefficients of $x^3, x^2, x^1, x^0 = 1$ on the two sides, we see that:

$$ac = 1, \quad bc + ad = 0, \quad bd + ae = 1, \quad be = 1.$$

Thus $a = b = c = e = 1$, which leads on substitution to $1 + d = 0$ and $d + 1 = 1$, which contradict each other. Hence the factorisation is not possible.

Solution to problem IV-3-S.4
Consider all non-empty subsets of the set $\{1, 2, 3, \ldots, n\}$. For each such subset, find the product of the reciprocals of each of its elements. Denote the sum of all these products by a_n. Prove that $a_n = n$ for all positive integers n.

Observe that $a_1 = 1$ and $a_2 = 2$. Suppose $a_k = k$ for some positive integer $k > 1$. Then:

$$a_{k+1} = \left(1 + \frac{1}{k+1}\right) a_k + \frac{1}{k+1} = \frac{(k+1)^2}{k+1} = k + 1.$$

Thus by the principle of mathematical induction, $a_n = n$ for all natural numbers n.

Solution to problem IV-3-S.5

Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real zero.

Let $f(x) = x^8 - x^7 + x^2 - x + 15$. Observe that all the coefficients of $f(-x)$ are positive. Thus $f(x)$ does not have any real negative zero. We can write

$$f(x) = x^7(x-1) + x(x-1) + 15,$$

hence $f(x) > 0$ for $x \geq 1$. Writing $f(x)$ as

$$f(x) = x^8 + (1 - x^7) + x^2 + (1 - x) + 13,$$

we see that $f(x) > 0$ when $0 \leq x \leq 1$. Thus $f(x)$ does not have any real positive zero.

Problems for the Middle School

Problem Editor: ATHMARAMAN R

Problem Corner

PROBLEMS FOR SOLUTION

Problem V-1-M.1

Find two non-zero numbers such that their sum, their product and the difference of their squares are all equal.

Problem V-1-M.2

Prove that a six-digit number formed by placing two consecutive three-digit positive integers one after the other is not divisible by any of the following numbers: 7, 11, 13.

(Adapted from the Mid-Michigan Olympiad in 2014 grades 7–9)

Problem V-1-M.3

If n is a whole number, show that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5.

Problem V-1-M.4

We know that the sum of two consecutive squares can be a square. For example, $3^2 + 4^2 = 5^2$.

a. Show that the sum of any m consecutive squares cannot be a square for $m \in \{3, 4, 5, 6\}$.
b. Can the sum of 11 consecutive square numbers be a square number?

Problem V-1-M.5

a. Which positive integers have exactly two positive divisors? Which have three positive divisors?

b. Among integers a, b, c, each exceeding 20, one has an odd number of divisors, and each of the other two has three divisors. If $a + b = c$, find the least value of c.

Problem V-1-M.6

A group of 43 devotees consisting of ladies, men and children went to a temple. After a ritual, the priest distributed 229 flowers to the visitors. Each lady got 10 flowers, each man got 5 flowers and each child got 2 flowers. If the number of men exceeded 10 but not 15, find the number of women, men and children in the group.

Problem V-1-M.7

There are two towns, A and B. Person P travels from A to B, covering half the distance at rate a, and the remaining half at rate b. Person Q travels from A to B (starting at the same time as P), travelling for half the time at rate a, and for half the time at rate b. Who reaches B earlier?

SOLUTIONS OF PROBLEMS IN ISSUE-IV-3 (NOVEMBER 2015)

Solution to problem IV-3-M.1

A number when increased by its cube results in the number 592788. Find the number.

Let the number be x; then $x + x^3 = 592788$. Since $x > 0$, we have

$$x^3 < x + x^3 < (x + 1)^3,$$

i.e., $x^3 < 592788 < (x+1)^3$. So it suffices to find a pair of consecutive cubes between which 592788 lies. We find that it lies between 84^3 and 85^3, for we have $84^3 = 592704$ and $85^3 = 614125$. Hence $x = 84$, i.e., the required number is 84.

Solution to problem IV-3-M.2

Find the two prime factors of 206981 given that one of them is approximately three times the other.

Let the prime factorisation of 206981 be $206981 = pq$, where $q > p$; then q is close to $3p$. Hence we have: $p \times 3p \approx 206981$, i.e., $3p^2 \approx 206981$, which yields $p^2 \approx 68993$. Hence p is close to the square root of 68993, i.e., p is close to 263. It so happens that 263 is a prime number, and moreover that the quotient $206981/263$ is an integer; indeed, $206981/263 = 787$, and 787 is a prime number! So the required primes are 263 and 787.

Solution to problem IV-3-M.3

How would you distribute 44 pencils to 10 students such that each student receives a different number of pencils?

If we assume that at least one pencil is given to each student, then we must have at least $1 + 2 + 3 + \cdots + 10 = 55$ pencils. But we have only 44 pencils. Hence such a distribution is impossible. So there must be a student who does not get any pencil. But this requires at least $0 + 1 + 2 + \cdots + 9 = 45$ pencils. Hence even this distribution is impossible.

Solution to problem IV-3-M.4

Find the sum of all three-digit numbers $\overline{ABC}$ such that the two-digit numbers $\overline{AB}$ and $\overline{BC}$ are both perfect squares. [Jamaican Math Olympiad 2015]

The numbers $\overline{AB}$, $\overline{BC}$ must belong to the set $\{16, 25, 36, 49, 64, 81\}$. Hence $(\overline{AB}, \overline{BC})$ is one of the following: $(16, 64)$; $(36, 64)$; $(64, 49)$; $(81, 16)$. Hence the possible values of $\overline{ABC}$ are: 164, 364, 649, 816. So the required sum is $164 + 364 + 649 + 816 = 1993$.

Solution to problem IV-3-M.5

The numbers $1, 2, 3, 5, 7, 11, 13$ are written on a board. You may erase any two numbers a, b and replace them by the single number $ab + a + b$. After repeating this process several times, only one number remains on the board. What might be this number? [Adapted from UAB MTS: 2006-2007]

Note that $a + b + ab = (a + 1)(b + 1) - 1$; e.g., $2 + 5 + (2 \times 5) = (2 + 1)(5 + 1) - 1$. Thus, regardless of the order in which the numbers are selected, the result is the product of all the listed numbers increased by 1, from which then 1 is then subtracted. Hence the answer is:

$$(1+1)(2+1)(3+1)(5+1)(7+1)(11+1)$$
$$(13+1)-1 = 2\times 3\times 4\times 6\times 8\times 12$$
$$\times 14-1 = 193535.$$

Solution to problem IV-3-M.6

Between 3 PM and 4 PM, Ramya looked at her watch and noticed that the minute hand was between 5 and 6. When she looked next at the watch, slightly less than two hours later, she noticed that the hour and minute hands had switched places. What time was it when she looked at the watch the second time? [Adapted from "Mathematical Wrinkles" by S.I. Jones, 1912]

Suppose the times when she looked at the watch are x minutes past 3 PM and y minutes past 5 PM, respectively. Let us measure angles in degrees from the 12 o'clock position, in the clockwise direction. The interval between the events is given to be less than two hours long, so $y < x$.

On the first occasion when she looks at the watch, the angles made by the hour hand and the minute hand relative to the 12 position are, respectively:

$$90+\frac{x}{2}, \quad 6x.$$

On the second occasion when she looks at the watch, the angles made by the hour hand and the minute hand relative to the 12 position are, respectively:

$$150+\frac{y}{2}, \quad 6y.$$

Since the hour hand and the minute hand have exchanged positions, we have:

$$90+\frac{x}{2} = 6y,$$
$$150+\frac{y}{2} = 6x.$$

We solve this pair of equations simultaneously for x and y. Doing so in the standard manner (details omitted), we get:

$$x=\frac{3780}{143}=26\tfrac{62}{143}, \quad y=\frac{2460}{143}=17\tfrac{29}{143}.$$

Hence the time when she looks at the clock the second time is $5:17\frac{29}{143}$ PM, i.e., $17\frac{29}{143}$ minutes past 5 PM.

Solution to the "A limerick in disguise"

(page 39)

A dozen, a gross, and a score
Plus three times the square root of four
Divided by seven
Plus five times eleven
Is nine squared and not a bit more.

The 'not a bit more' is a particularly neat touch!

Note: This puzzle has been floating around on the web, but we find it impossible to credit any one source for it.

We invite readers to generate more limericks of this kind (possibly involving more complicated mathematical operations) and to send them to us. Could you add your impression about using such a device in class? Could it have any pedagogical benefits? Please send your responses to atria.editor@apu.edu.in.

Problem Corner

Two Problems

C⊗MαC

We present as earlier a small collection of problems, followed by their solutions. We state the problems first so you have a chance to try them out on your own.

PROBLEMS

1. Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ touches the smaller circle ω at Q. Show that PQ bisects $\measuredangle APB$. What might be a meaningful generalisation of this result? (See Figure 1.)

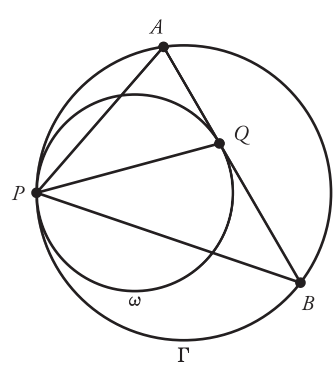


Figure 1. Two tangent circles

2. Show that the function $f(x, y) = \frac{1}{2}\left((x+y)^2 + x + 3y\right)$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ to $\mathbb{N}_0$ bijectively. (Here $\mathbb{N}_0$ refers to the set of all possible non-negative integers. So the problem asks us to show that as x and y take all possible non-negative integral values, $f(x, y)$ takes all possible non-negative integral values, with no repetition.)

Problem 1. *Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ touches the smaller circle ω at Q. Show that PQ bisects $\measuredangle APB$. What might be a meaningful generalisation of this result?*

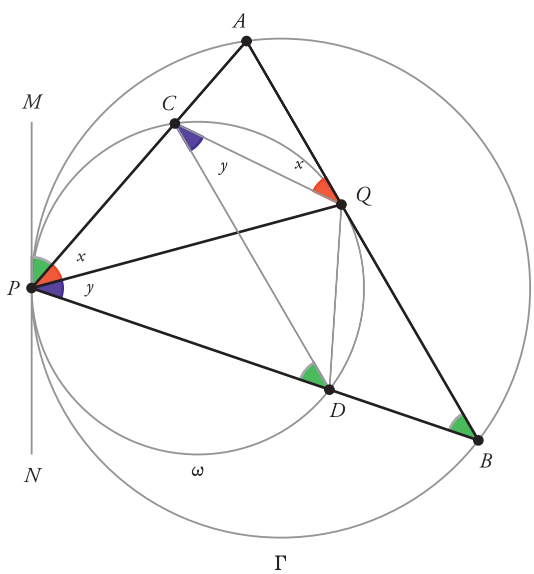


Figure 2

Solution. Let C be the point where PA intersects ω again, and let D be the point where PB intersects ω again (see Figure 2). Join CQ, DQ, CD as shown. The solution will now follow from old-fashioned 'angle chasing.' The two angles marked y are equal ("angles in the same segment of a circle"). So are the two angles marked x (this follows from the well-known theorem about the angle between a tangent to a circle and a chord at the point of contact).

Next we show that CD is parallel to AB. Here is one way of seeing why. Draw the tangent MN at P to ω; it is also the tangent at P to Γ. Using the theorem just quoted above, we see that $\measuredangle APM$ is equal to $\measuredangle CDP$ as well as to $\measuredangle ABD$. It follows that $\measuredangle CDP = \measuredangle ABP$, and hence that $CD \parallel AB$. From the parallelism, it follows that $\measuredangle AQC = \measuredangle DCQ$, i.e., $x = y$. Hence PQ bisects $\measuredangle APB$.

Another way of seeing why $CD \parallel AB$ uses the idea of a geometrical transformation, in this case a dilation centred at P, with scale factor $PA : PC$. The dilation maps ω to Γ and hence C to A and D to B; this implies that $AB \parallel CD$. □

We have been asked to suggest a meaningful generalisation of the above result. Here is one which seems particularly attractive:

> *Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ cuts ω at points Q and R. Show that $\measuredangle APQ = \measuredangle BPR$.* (See Figure 3.)

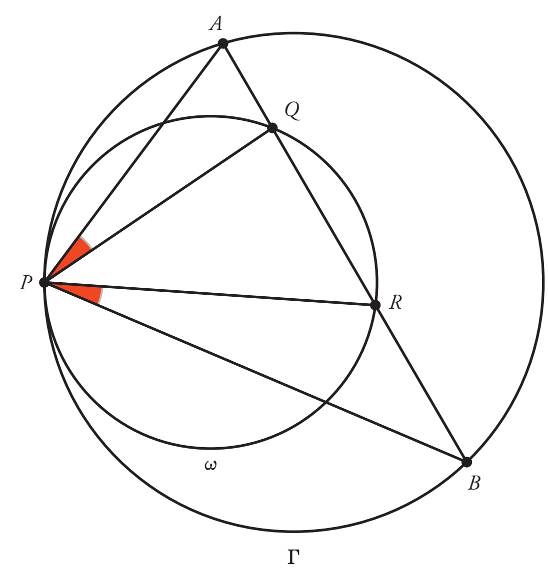


Figure 3. Generalisation of the earlier result

Do you see how this is a generalisation of the earlier result? See if you can find the proof for yourself!

Problem 2. *Show that the function $f(x, y) = \frac{1}{2}\left((x+y)^2 + x + 3y\right)$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ to $\mathbb{N}_0$ bijectively.* ($\mathbb{N}_0$ refers to the set of all non-negative integers.)

Solution. There are actually two parts to this question. Firstly, we must show that the range of f is the whole of the co-domain; i.e., for every non-negative integer n, there exists a pair x, y of non-negative integers such that $f(x, y) = n$. Secondly, we must show that two different pairs of non-negative integers cannot be mapped by f to the same image point. Only if we show both these parts can we claim that f is a bijective map. What we shall do now is to establish these two parts separately.

The expression for f can be written in the following form:

$$\begin{aligned} f(x,y) &= \frac{1}{2}\left((x+y)^2+x+y\right)+y \\ &= \frac{(x+y)(x+y+1)}{2}+y. \end{aligned}$$

Let $x+y=k$; then $f(x,y)=T_k+y$, where T_k is the k-th triangular number. Since $y \le x+y$, we must have $y \le k$. So what we have to prove is the following: if n is any non-negative integer, then we can find a unique pair of integers k, y such that $0 \le y \le k$ and $n = T_k + y$.

We first show that such a pair of non-negative integers can always be found; i.e., every non-negative integer lies in the range of f. Our proof is algorithmic: given n, we show how to find x, y such that $f(x,y)=n$. All we do is to find the largest non-negative integer k such that $T_k \le n$; then we let $y = n - T_k$ and $x = k - y$, and with this choice we have $f(x,y)=n$. An example will illustrate the mechanism. Let $n=50$; since $T_9 < 50 < T_{10}$ and $50 - T_9 = 5$, we get $y=5$ and $x = 9-5=4$. Check:

$$f(4,5) = \frac{(4+5)^2+4+15}{2} = \frac{81+19}{2} = 50.$$

To complete this part of the proof, we must show that $y \le k$. But this is clear, since $T_{k+1} - T_k = k+1$, which implies that if $T_k \le n < T_{k+1}$, then $n - T_k \le k$.

Now for the uniqueness part, we must show that two different pairs of non-negative integers cannot map to the same value. For this it suffices to show the following: if a, b and c, d are pairs of integers such that

$$0 \le b \le a, \qquad 0 \le d \le c, \qquad T_a+b = T_c+d,$$

then $(a,b)=(c,d)$. To see why this is so, we treat a, c as fixed, and b, d as variables, with $0 \le b \le a$ and $0 \le d \le c$. If $a=c$ and $b \ne d$, then clearly $T_a + b \ne T_c + d$. So we may assume that $a \ne c$. Without loss of generality, suppose that $a < c$. The range of values taken by $T_a + b$ for $0 \le b \le a$ is the set

$$\{T_a, T_a+1, T_a+2, \ldots, T_a+a\},$$

and the range of values taken by $T_c + d$ for $0 \le d \le c$ is the set

$$\{T_c, T_c+1, T_c+2, \ldots, T_c+c\},$$

and these sets are clearly disjoint, because

$$T_a + a < T_{a+1} \le T_c.$$

The stated claim follows. □

A Review of

Taming the Infinite

By Ian Stewart

Review

Tanuj Shah

At some point students begin to show interest in the origins of mathematical ideas: How were logarithms discovered? Who was the first person to compute the log tables? When were quadratic equations first used, and how were they solved? Can equations of a higher order be solved in a similar way? Why are representations of complex numbers on a plane called Argand diagrams? Ian Stewart's book *Taming the Infinite: The Story of Mathematics from the First Numbers to Chaos Theory* throws light on questions like these. It is an attempt at mapping the major themes in mathematics through a historical perspective. This compact book of less than 400 pages covers the major topics in Mathematics and is accessible to students in secondary school. The latter part of the book gives a flavour of some areas of college mathematics. An underlying theme of the book is that the modern world owes a great debt to advances in Mathematics, and mention is made of some of the applications of Mathematics. Brief biographical sketches of the major players are given, though there are some odd omissions like Euler and Leibniz. There are boxed items, some with intriguing titles: *What We Don't Know About Primes, What Trigonometry Did For Us*. Though the book is compact, it is ambitious in its scope; but appears thin

Keywords: *mathematics, history, mathematicians, numbers, sets, algebra, trigonometry, calculus*

in some places as a result. A major drawback is that the book does not give much space to non-European strands: Aryabhata, Brahmagupta, Mahavira and Bhaskaracharya are given less than a couple of pages, and Madhava and the Chinese mathematicians do not even figure in the book.

Ian Stewart has tried to order the book thematically and chronologically, but this brings in a few anomalies. Chapter I on Numbers and Chapter III on Number Systems should sit together, but the chronology results in a chapter on Geometry in-between. The book starts by looking at the way numbers developed in two early civilizations – Egyptian and Babylonian. The Babylonians were able to accurately predict celestial events such as solar eclipses and the movements of planets. However, there is evidence that human beings were interested in counting much before these civilizations emerged. Two bones found in Africa, the Lebombo bone and Ishango bone, with tally marks on them, have got mathematicians speculating on their possible origins. Chapter III picks up the story with the background to our present number system which originated in India. A mention is made of some Indian Mathematicians during this period, and the legend of how Bhaskaracharya named one of his books (*Lilavati*) is nicely narrated. This section would have benefited from greater detail; those interested should refer to *The Crest of the Peacock* (reviewed in an earlier issue). Stewart gives a sketch of the life of Leonardo of Pisa, better known as Fibonacci, who introduced the Indian numeration system to Europe through his book *Liber Abbaci*.

Chapter II on Geometry begins with the Babylonians, who were familiar with what is now known as the Pythagorean Theorem. Mention is made of Archimedes in the context of the problem of finding good rational approximations to pi, and of finding formulas for the volume and surface area of a sphere. There is a biography of Hypatia, the woman mathematician who met a tragic end at the hands of a Christian mob in Alexandria. Euclid's *Elements*, and the concepts of *theorem, proof*, *postulate* and *axiom* are explained lucidly with examples; it will help the readers in putting the school geometry in context.

Chapter IV looks at the development of Algebra. Some readers will find it surprising that quadratic equations were being solved by the Babylonians in much the same way we do today. Stewart speculates that the Babylonians may have come upon this solution by depicting the equation pictorially – a form which students will find appealing. Next, Stewart takes us through a tour of attempts to solve the cubic equation, starting with attempts by the Greeks using the conics sections; much later, the same approach was formalised by the Persian Omar Khayyam, author of the poem *Rubiyat*. In the middle of the 16th century, an algebraic solution was found by Italian mathematicians. Readers may be surprised to find that many symbols in common use today were not in use at that time. For example, the '+' and '–' symbols appeared only around the 16th century. The 'equals to' sign was invented in 1557 by the English mathematician Robert Recorde, who said that he could not think of any two things more alike than a pair of parallel lines!

Ian Stewart gives the following example from an algebra book written in the 16th century (*Ars Magna*):

qdrat actur 4 rebus p: 32.

In modern notation this would be written as:

$$x^2 = 4x + 32.$$

Chapter V looks at how trigonometry and logarithms developed. Trigonometrical tables were developed in different ways in many places over the years; they arose from the needs of astronomers. The first trigonometric tables were derived by Hipparchus around 150 BC. In India, Aryabhata and Brahmagupta developed trigonometric concepts using the notion of the half-chord in a circle. The Arab mathematician Nasir-Uddin combined trigonometry in the sphere and the plane. This is one area where the mathematics done by the ancients is more complex than that learnt in school today. Plane trigonometry as currently taught in schools developed around the 15th century. On the other hand, logarithms developed from the need to make calculations easy; however, this aspect of logarithms has now become obsolete. The enormous effort put in by John Napier in constructing the logarithm tables eased the work of mathematicians and scientists. The number *e* which is closely related to logarithms makes an appearance here, though no one is credited with its discovery.

Today we take the idea of latitude and longitude for granted, but it took a long time for such ideas and for the notion of coordinate geometry to develop. In the next chapter, Stewart begins with Fermat who in the 17th century was the first to try and relate a geometric curve to an equation and to represent it on an oblique coordinate grid. The modern rectangular coordinate system was introduced by Descartes in an appendix to his book *Discourse de la Methode*; he showed how geometric curves can be represented using algebraic equations. Fermat extended the coordinate system to three dimensions. Jacob Bernoulli developed the idea of polar coordinates, where position is defined using an angle and a radial distance. The chapter ends by looking at some applications of coordinate geometry, including the use of GPS which is ubiquitous in today's world.

Chapter VII looks at Number Theory, a topic dealt with superficially in school mathematics. The early mathematicians mentioned here are Euclid and Diophantus; the absence of Asian mathematicians is glaring. Euclid proved many properties of primes; for example, that every number can be expressed as a product of prime numbers in a unique way. Stewart illustrates this with an example in a note, *Why Uniqueness Of Primes Is Not Obvious*.

The next big idea was the invention of Calculus, and it had an enormous impact on the development of mathematics. It seems to have arisen out of many unrelated investigations: instantaneous change in velocity, finding maxima and minima, finding tangents to curves, finding areas of planar shapes. The breakthrough was made by Leibniz, who was the first to realise that finding tangents to a curve was the inverse operation to finding area under the curve. He was the one who gave us nearly all the symbols we now associate with calculus. He published his work on calculus in 1684, though not many understood the significance of his work at the time. Newton had been developing his ideas on calculus at nearly the same time, but using different symbols; he published his ideas in 1687. Stewart gives a vivid description of the controversy that erupted regarding who was the "first to discover calculus", and the sizeable rift it caused between European and British mathematicians. The development of the planetary laws of motions, which can be regarded as a prime mover behind the development of calculus, is explained in detail, from Ptolemy's system of epicycles based around a stationary earth, to the Copernican sun-centred solar system, to the observations of Brahe and Kepler which allowed Kepler to come up with his laws of planetary motion; all these together with Galileo's work laid the foundations for calculus.

Calculus in its early days lacked a clear logical foundation, as the idea of limits had not been developed; that concept would take another century to develop. Leibniz used the term *infinitesimal* to describe a number close to zero, and Newton came up with the term *fluxion*. Since calculus bestowed such power in the hands of its users, most mathematicians ignored this lack of rigour. A significant exception was Bishop Berkeley who pointed out that it was meaningless to divide by a quantity that is later set equal to zero. How this conundrum was later resolved is described in chapter XI.

The next chapter looks at a topic familiar to students: Complex Numbers. Stewart explains that the early mathematicians ignored answers involving the root of a negative number. The first recorded manipulation of imaginary numbers is by Rafael Bombelli in his book *L'algebra* (1557). In the context of finding cube roots, he came across imaginary numbers and operated on them as if they were ordinary numbers. In 1673, John Wallis invented a way to represent imaginary numbers on a plane. Wallis's work was explained more clearly by Caspar Wessel in 1797, but his work went by unnoticed. The French mathematician Jean Robert Argand came up with the same idea independently in 1806; today we call his way of representing complex numbers *Argand diagrams*.

This was the start of the development of the field of complex analysis; ultimately it led to Cauchy's theorem which extends calculus to the complex plane; Gauss had come upon the same idea earlier but had not published it.

The rest of the book – apart from the chapter on Probability and Statistics – is devoted to topics that students study in college. Stewart has made a great effort at making the key ideas accessible to high school students. There are three chapters on non-Euclidean geometry. One such geometry which emerged from the work of the Renaissance artists is *projective geometry*. The search for a proof of Euclid's fifth postulate ultimately resulted in the development of a different kind of geometry. Gauss was convinced from an early stage that it ought to be possible to come up with a logically consistent system of non-Euclidean geometry (the geometry of curved space), but he was fearful of publishing his work as he felt that people were too conditioned to the geometry of Euclid and would ridicule his work. The chapter *Rubber Sheet Geometry* considers operations not studied in traditional geometry. For example, a shape like a square can be bent, stretched and twisted into a triangle or circle; likewise a coffee cup can be moulded into a doughnut shape. These objects which can be moulded into each other can be regarded as congruent, the only invariant aspect being *connectedness*. Stewart mentions the Königsberg bridges problem and discusses notions like the Möbius band and the Riemann Sphere.

The chapter titled *The Shape of Logic* deals with matters which are central to mathematics; it looks at how mathematicians started questioning the very foundation of mathematics. Dedekind was unsettled by the fact that obvious properties of real numbers had not been proved, for example, $\sqrt{2}\times\sqrt{3}=\sqrt{6}$, and he expressed his thoughts in a book he published in 1872. In another book in 1888, he exposed gaps in the foundations of the system of real numbers and proposed a new approach: Dedekind cuts. He found a way of defining the properties of real numbers solely in terms of rational numbers. This led to the question: How do we know that the properties for numbers hold true? In 1889, Giuseppe Peano proved the basic operations of arithmetic by creating a list of axioms for whole numbers, the most important ones being that there exists a whole number, 0, and each number n has a successor $s(n)$, which is 1 more than the previous one. As mathematicians grappled with these ideas, they began to explore the meaning of 'number'. In this context, sets were introduced as a 1-1 correspondence with numbers by Gottlob Frege. Unfortunately for Frege, his work was rendered worthless by George Cantor's work, and Bertrand Russell pointed out a paradox in his work just as his book was being published. Russell tried to fill the gap in Frege's work with his *theory of types*, but this was equally contentious. In his three-volume book, the definition of 2 comes at the end of volume I, and $1 + 1 = 2$ is proved on page 86 of volume II! There were more strange discoveries on numbers with Cantor's theory of transfinite numbers and different sizes of infinity. The chapter ends with Hilbert's ambitious project to put the whole of mathematics on a sound footing, but this was ruined by Gödel's shattering discoveries.

The final chapter of the book is on *Chaos Theory*, one of the new branches of Mathematics which challenges the deterministic, clockwork universe of Newton. Its beginnings were in 1886 when King Oscar II of Sweden offered a prize to solve the problem of stability of the solar system. Henri Poincaré, working on the three-body problem, realised the complexity of the problem; his work led to the development of chaos theory. In 1926, a Dutch engineer while simulating the heart using an electronic circuit realised that under certain conditions the resulting oscillations were irregular. However, it was another forty years before chaos theory began to be seriously studied. Meteorologist Edward Lorenz set out to model the atmospheric convection by approximating the complex equations with much simpler ones. He discovered that if the initial conditions varied even slightly, the differences became amplified and the final solutions looked very different from each other; this came to be known as the 'butterfly effect'. Parallel to these developments, a group of mathematicians towards the beginning of the 20th century were coming up with bizarre shapes: a curve that fills an entire space (Peano and Hilbert), a curve that crosses itself at every point (Sierpinski), a curve of infinite length that encloses a finite area (Koch). In the 1960s, Benoit Mandelbrot realised that these monstrous shapes reflected irregularities in nature and came up with the notion of a *fractal*. Mandelbrot was able to show many examples of fractals in nature, and in unexpected contexts like stock market prices.

Ian Stewart's book is an excellent resource for teachers who want to inspire their students; it can be equally enjoyed by students at the +2 level. It need not be read in a linear way. One can dip into any part of the book and make sense of it without having read earlier portions. There are many applications interspersed through the book. It is also a good read for those who wish to develop a coherent picture of modern mathematics as a whole, in terms of how the fundamental ideas relate to each other.

TANUJ SHAH teaches Mathematics in Rishi Valley School. He has a deep passion for making mathematics accessible and interesting for all and has developed hands-on self learning-modules for the Junior School. Tanuj Shah did his teacher training at Nottingham University and taught in various schools in England before joining Rishi Valley School. He may be contacted at tanuj@rishivalley.org.

SOLUTIONS

NUMBER CROSSWORD

Page 81

	1 1	2 1	2		3 1	4 7	6	
5 8		4		6 3		1		7 3
8 2	9		9 2	3	4		10 6	4
8		11 7				12 7		5
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The Closing Bracket . . .

The history of Mathematics has been irrevocably linked with the history of mankind. Mathematics has for years been the common language for classification, representation and analysis. Learning mathematics is an integral part of a child's education. Yet, for many reasons, it is traditionally perceived as a difficult subject and is a cause of distress for many students.

One of the reasons for this is, that the emphasis in mathematics learning is on rote memorization of procedures, paper-pencil-drills, manipulation of symbols and on learning tricks to solve problems. This is not to undermine the importance of procedures in mathematics learning but an overemphasis on procedures leads to a lack of relevance for the student who perceives the subject matter as abstract and far removed from everyday life. At the school level, few opportunities are provided in the classroom for visualization and exploration. In fact in the vast majority of classrooms in the country there remains a significant gap between content and pedagogy.

This brings us to the question of technology which has been the buzzword for most aspects of life, today. One of the most fundamental impacts of technology perhaps has been its contribution in expanding the boundaries of knowledge in almost every field. And in this age of innovations, is it any wonder that technology should reach the classroom, especially the mathematics classroom? After all if we agree that learning entails freedom – freedom to explore, experiment, question and visualize, then for all these modes of learning, technology, if properly used, can be a great enabler. It is therefore imperative that new and innovative methods of teaching enabled by technology increasingly find relevance in today's mathematics classroom.

World over, the advent of technological tools – handhelds in the form of graphic calculators, or computer software such as computer algebra systems (CAS), dynamic geometry software (DGS), spreadsheets and others have brought about a change in the way mathematics is taught. These tools have been evolving and over the years have succeeded in enabling the processes of visualization, exploration and experimentation. Many research studies in mathematics education have pointed to the positive impact of technology, highlighting its role in visualization of concepts, in exploration and discovery, in promoting multi-representational approach, in focusing on applications, in redefining the teacher's role and in helping sustain students' interest.

However, in India, the integration of technology especially in mathematics teaching has met with much resistance and scepticism. People have expressed the opinion that use of technology will increase the student's dependence on the tool and will be a detriment to the development of her thinking skills. She will no longer be able to do the paper-pencil calculations by hand. This argument is flawed since technology should not be used as a means to replace paper-pencil methods. Rather it should be used to develop the students' mathematical thinking by giving her opportunities to explore ideas and access higher level concepts or processes which cannot be explored using paper and pencil. Further, much of the computational work can be 'out sourced' to the tool, so that student (and teacher) may focus on developing insights. The position paper on teaching of mathematics of the NCF 2005 recommends that there should be a shift in mathematics learning from 'content to processes' and that the curriculum should provide the student with ample opportunity to engage in the processes of abstraction, quantification, estimation, guess and verify, and optimization. We need to examine the role of technology in facilitating this shift from content to processes.

Notwithstanding the benefits provided by the use of technology, its integration into the traditional mathematics classroom poses its own challenges particularly in a country like India with its vast and diverse student population. Firstly, technology must be cost effective and easy to deploy in order to achieve large scale integration in schools. Secondly, technology requires infrastructure. The sheer lack of infrastructure in schools which cater to a vast number of under privileged students is indeed a pressing problem. Beginning with frequent power outages to lack of connectivity, the problems are many. This makes it difficult to use technology in any form. Another challenge is that the present curriculum does not readily lend itself to integration of technology. The curriculum will need to be redesigned so that the possibilities offered by technology can be utilised to the maximum. The goals of mathematics learning and assessment will also need to undergo a major shift in paradigm in a technology integrated mathematics curriculum.

The fact is that technology by itself is neither good nor bad. However, when used appropriately, it can significantly and positively impact mathematics learning. This has important implications for teacher preparation. Teacher education programmes must be designed where student teachers are taught mathematics using various technological tools. In-service teacher training programmes must focus on changing teachers' mindset towards technology and on developing their pedagogical content knowledge. Teachers must be convinced that technology can empower them to transform their classroom teaching and make mathematics learning a meaningful experience for their students.

Thus, a great deal of work needs to be done in terms of redesigning the curriculum, teacher preparation and developing the infrastructure to reap the benefits of technology in mathematics classrooms. But all the effort will be worth it if it brings about a change in mathematics education especially at the school level. Let us begin by opening up our minds to the possibilities that technology has to offer.

Jonaki Ghosh

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

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Azim Premji University

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TEACHING Word Problems

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A PRACTICAL APPROACH

At Right Angles
A Resource for School Mathematics

Word problems become a stumbling block for many children, including those who are adept at operational and procedural skills. Many children develop an approach to tackling word problems based on looking for cue words – such as altogether, difference, sum and so on; but this has a very limited value. Too often, such children resort to guesswork while figuring out an operation. These children experience significantly greater math anxiety when they are confronted by word problems. Why is this?

Primary reasons

Here are some reasons which lie behind such math anxiety:

1. Lack of exposure to problem situations and problem contexts during the introductory and teaching phase.
2. Lacunae in the usage of concrete materials as an aid in the visualisation of the problem.
3. Insufficient training in representation of problems through drawings and other means of reconstruction.
4. Difficulty in following multiple statements and instructions at the same time.
5. Inadequate stress on vocabulary and weak linkages or connections between concepts and associated words.
6. Absence of discussion and conversation around the questions (whether in English or in the mother tongue).
7. Lack of recording of the solution by the children in their own words. Most teachers follow rigid ways of writing statements for word problems. Writing in the initial years must come from the child's own experience and understanding. It need not be structured according to any norms; on the contrary, it needs to be personal.

All of these reasons point to poor teaching practices.

In conjunction with this is the fact that many textbooks are not particularly child-friendly. By the time the child reaches class 4 or 5, he or she would have basic literacy skills. Yet very few children read the textual material for the following reasons:

1. The language used is not close to the child's experience.
2. The word problems are not based on real life and familiar situations.
3. They are not phrased in a sufficiently interesting way, and do not draw the child 'into' the problem.
4. They are not accompanied by drawings (this is crucial for non - English-speaking learners).
5. They are often limited in variety and repetitive, and thus hold no challenge.

Often the problems are not posed in a properly graded sequence.

Keywords: *Math anxiety, thinking skills, logic, reasoning, exploring, pattern, conjecture, experiential*

Language matters

Language plays an important role in learning mathematics. Several reasons can be listed to see why:

1. In the context of mathematics teaching, a teacher uses language to communicate concepts and processes, to provide explanations, to compare alternatives, to discuss causal and dependent relationships, and to justify answers.

   In a country like ours, the language used in the classroom may not be the mother tongue of the child. Hence, particularly in the primary school, the mathematics teacher may need to translate to aid in the comprehension of the problem.

2. It is important for the child to articulate his understanding through language so that the teacher gains a window into the child's interpretation and understanding of the problem. Language and discussion enable the child to reconstruct the problem.

3. Words tend to acquire special meaning in the context of a math problem. Words like measure, share, part, product, equal, face, table and volume carry other meanings in normal, everyday language, but in a mathematics classroom may have a somewhat different connotation. Some words have more than one meaning in mathematics, e.g.: base, difference, square. Children need to become familiar with their usage in math contexts.

   Teachers need to use correct language while teaching. For some reason, in India there is a practice of using 'into' instead of 'times' while teaching multiplication table. So '2 multiplied by 3' should be read aloud as 'two times three' (and not '2 into 3' which implies division).

   Also, in problems involving subtraction, the right word to use is 'exchanging' tens for units or hundreds for tens, etc, and not 'borrowing'.

4. Mathematical symbols have to be initially understood through words, and children need to acquire clarity about them. One of the frequently misunderstood symbols is the 'equals to' symbol. It does not imply 'write the answer'. However, given the problem $2 + 5 = \_\_ + 4$, many children may fill the blank with a '7'.

5. The child has to learn to read, comprehend and interpret the text and word problems.

An important point I wish to emphasise is that an inability or weakness in attempting word problems is not an isolated problem. The teacher needs to realise that the root of the problem lies in the teaching process itself. Often when children struggle with a word problem, it serves as a pointer to the fact that either the concept or the math terminology itself has not been explained adequately. As a teacher, when I introduce a topic, I need to use varied real life contexts and situations to highlight the connections between the real world and the problems posed in the class. I need to verbalize the multiple ways of looking at a problem or analysing the structure of a problem. As I explain procedures or draw diagrams, I need to give a running commentary to show the relationship of my action to the wording of the problem. Most importantly, I need to help the child to 'get inside' the problem.

ACTIVITIES

When does one start helping children to feel their way into word problems?

Apart from the exposure that happens at home, it is almost from day one in preschool and primary school. Preschool classrooms usually have equipment which lends itself to pretend-play activities (dressing up, cooking, shopping), and these can be used to introduce mathematical vocabulary. "Arya has a bigger glass than Disha", "Warad is wearing a longer kurta than Sarthak", "Mitali's shop has more flowers than Shreyan's", etc.

Number rhymes with math contexts are to be acted out so that the child sees the connection between the words he or she is reciting and the action associated with it. Rhymes which teach number concepts, ordinal numbers, increasing or decreasing sequences of numbers can be taught as action songs to demonstrate the underlying simple concepts.

Real life situations which occur during the school day of a child (lining up for assembly, library time, games time, recess time when snacks are served) should be made use of to teach the appropriate vocabulary and to introduce mathematical concepts. These everyday situations can be later referred to and incorporated into the class conversations.

Measurement-related vocabulary (big, bigger, biggest), the vocabulary of comparison (more, less), the vocabulary of quantity (numbers) and other such words are taught through activities and actions.

The notion of time is complex to grasp, and the related vocabulary (yesterday, today, tomorrow) is best learnt when children begin to share their experiences in the class and the teacher reinforces the concepts by talking about the planned activities for the day and the following day, etc.

It is very important to create enjoyable action settings, where children feel secure and free to ask questions. This provides children the opportunity to communicate, to comment on the activity they are performing, and to formulate questions. The teacher must step in to supply them with the right mathematical vocabulary as they struggle to express themselves.

GAME SETTINGS

Games provoke talk, reactions and arguments. They offer excellent platforms for reinforcing concepts and associated vocabulary.

Bogie game: Write some random numbers on the floor tiles.

Figure 1

Children can move from one bogie to the next by making a statement using appropriate mathematical words. The child standing on the first tile says 'I need seven more to go to the next bogie', the child standing on the second tile says 'I am 3 more than the next bogie', the child standing on the third tile says 'I will give away 4 to go the next bogie' and so on. As the focus of the game is on using language, a child who makes a mistake in his or her calculation can be corrected and allowed to move on.

I share here some approaches I have used in helping children to attempt and solve word problems. In all the approaches, one essential aspect is the slowing down of the reading of the word problem. Children find it difficult to hold multiple statements all at once in the mind. These approaches help in breaking it down for them. They learn how to codify the facts and interpret each statement separately. Through the process of reconstruction, they begin to understand the problem as a whole and the way the parts relate to each other.

I have illustrated the various approaches through examples selected from different levels of lower primary school.

Dramatization and mime

Most children enjoy drama and role playing. It doesn't take very much time or resources to act out a word problem set in a context. Blocks (interlocking cubes are more useful as they stack up well and can be easily used for comparisons by stacking) can be used to represent small numbers, and can easily be scattered or collected in an action scene.

Note: Larger numbers can also be represented by using a bigger block to stand for ten.

LEVEL: CLASS 3, 4

The children or the teacher can read out the problem to the class. For young children, the teacher may need to read out the problem or guide them in their reading.

Meher has three guavas. Aryaman has four more guavas than Meher. Their friend Adway brings two guavas. They share the guavas equally amongst themselves. How many guavas does each one get?

The problem is about guavas which are a great favourite with children. They love to climb guava trees and pluck guavas. The teacher needs to initiate a discussion to stimulate interest in the problem before expecting children to enact out a story. The discussion can begin with a question: 'What is your favourite fruit?', 'Shall we bring some guavas from a fruit garden?' and so on. It is important to draw the children into the context of the problem for it to become real.

Let three children act out the whole scene. They pretend to be in a garden, plucking guavas from the trees. Initially Meher shows her three guavas with blocks and states 'I have three guavas'. Aryaman has to show seven blocks and state 'I have four more guavas than Meher'. The remaining children can check whether he has got the right number. If the problem has not been read carefully, or if the child has not understood the statement correctly, he may show only four blocks. At this point, either the other students may point out the error, or the teacher can intervene. Now Adway can pretend to bring in two guavas. Together, they now need to figure out how to share them equally. They can finally state 'We have twelve guavas altogether. Each of us gets four guavas. Three fours make twelve.' The focus of this activity is the following: (i) To interpret the statements correctly; (ii) to make appropriate statements corresponding to their actions.

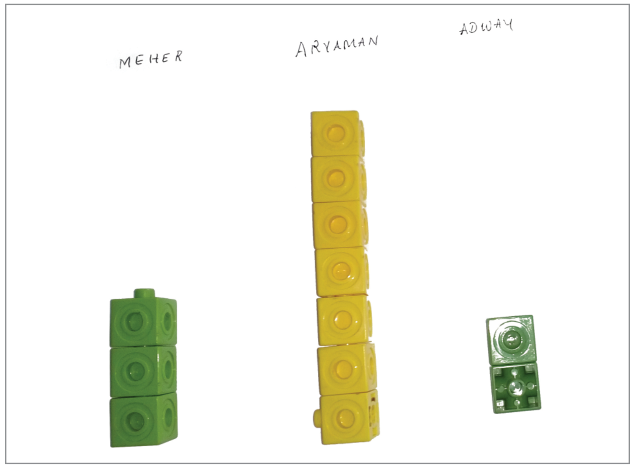


Figure 2

Modification of the question: Having spent time drawing the children into the garden theme, the teacher can play around with the question by changing the numbers or the operations involved in the following way.

'What if Aryaman finds worms inside two of them and throws them away? How many more does Aryaman now have than Meher?' (Both can stack up their fruits to find the answer.)

'Will they now have enough guavas to share equally?'

'If they need to take 15 guavas back to their class, how many more guavas do they need to pluck?'

What does dramatization of a problem achieve?

- It aids in the comprehension of the problem.
- It helps children whose reading levels are low.
- It creates a sense of participation among the children.
- It can also help a teacher to assess a student's comprehension quickly
- Teacher can initiate remedial measures quickly by simplifying a problem or raising the challenge level.

Using concrete materials

Solving problems with the use of concrete materials shares some aspects of dramatization as well as mathematical modelling. However, it is particularly relevant for word problems which require special equipment for modelling; for example, place value materials, cardboard clock, play currency or geometric shapes.

LEVEL: CLASS 3

Vaishnavi and Ameya had 16 straws to make triangle shapes. They used three straws for each triangle. How many more straws do they need if they want to make 6 triangles altogether?

Children can use sticks to model the problem. It happens on occasion that as children begin the process of modelling, they begin to visualise the problem and are able to work out the solution even before completing the act of modelling. In such cases, it is important to let the children state the answer at that point itself, and get them to verify the answer by completing the act of modelling.

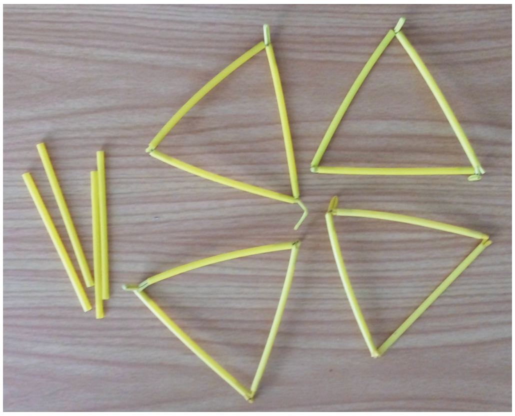


Figure 3

LEVEL: CLASS 4

Samriddhi made a string of beads, one red bead followed by two yellow beads. What will be the colour of the thirteenth bead?

A child may use coloured counters to model such a problem and find the answer.

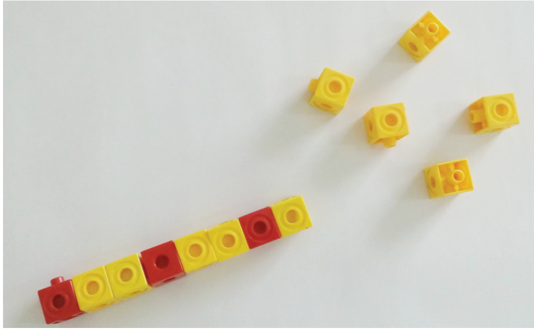


Figure 4

As the children play around with the models, the teacher can suggest trying out new number combinations. What if you make a string with two red beads followed by three yellow beads? What will be the colour of the tenth bead? They can model this situation and record the result in a sentence form.

My string has two red beads followed by three yellow beads. The thirteenth bead is ______.

Children can try out their own combinations and record the result in sentence form.

What does usage of concrete material achieve?

- It facilitates tactile learning and development of kinesthetic intelligence.
- It simplifies the problem and aids in the comprehension of the problem.
- It serves as a visual aid in solving the problem.
- Associated actions like combining things or comparing things will help the child to figure out the operation involved.
- The focus is not on the solution itself, but on modelling and comprehending the problem.

Using drawings

Some word problems are better understood by representing them through drawings.

LEVEL: CLASS 3

Diva was making a border pattern with squares and circles. She drew 10 squares and made 2 circles between every 2 squares. How many circles did she make?

LEVEL: CLASS 5

It takes 10 minutes to saw a log of wood into two pieces. How long does it take to saw the same log into four pieces?

In some of these problems, the children will need to make suitable pictures to understand the problem. They need to perceive the patterns and relationships before they can find the answer.

Making pictorial representations helps in comprehending a problem. However, children may benefit from help in learning how to use pictures as an aid to solving a problem. A teacher can show different ways of representing word problems through drawings. The choice of type of drawing will be based on the context of the problem. Here are a few models.

Pictorial drawings:

LEVEL: CLASS 2

Jhanvi has black and grey marbles. There are 10 marbles in all. If 6 of them are black marbles, how many are grey?

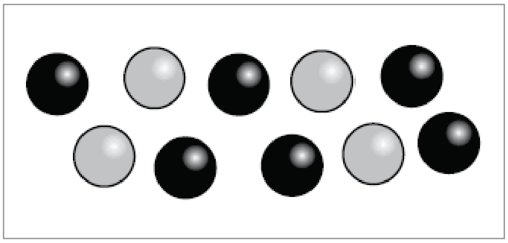


Figure 5

The blue box is bigger than the yellow box. The blue box is smaller than the black box. The black box is smaller than the red box. Which is the biggest box and which is the smallest box?

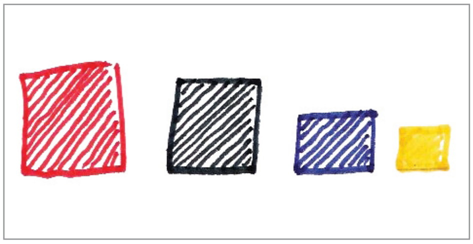


Figure 6

Children can be encouraged to make symbolic pictures for these problems before trying to solving them.

Since children typically will not be able to read at this age, in the follow-up sessions the teacher can supply the children with well-illustrated worksheets.

LEVEL: CLASS 4

In the school dining hall, there are 8 tables with 12 people on each table, 6 tables with 4 people on each table, and 2 tables with 9 people on each table. How many tables are there? How many people are there altogether?

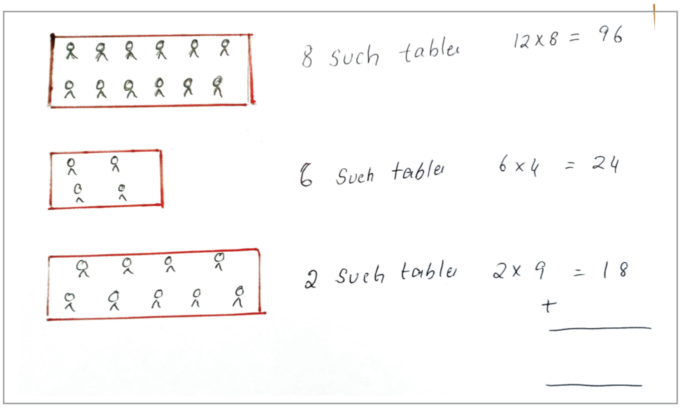


Figure 7

Number line drawings: Horizontal and vertical lines can be used in many kinds of problem settings.

LEVEL: CLASS 4

Rahul is standing behind Yagya in the queue. Rahul is the fourth child in the queue from the front. Yagya is the seventh child from the back of the queue. How many children are there in the queue?

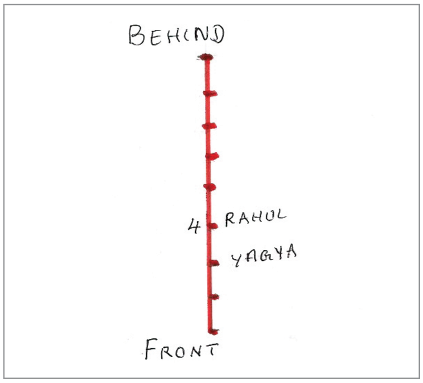


Figure 8

LEVEL: CLASS 4

Aum is four years old. His sister Shreya is seven years old. After five years, how much older will Shreya be than Aum?

This is a question based on common sense. But many children resort to totalling the numbers and they come up with incorrect, meaningless answers as they do not comprehend the problem properly. A number line will make the problem less abstract, and help them realise that the age gap remains the same.

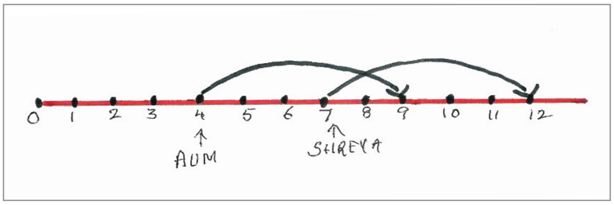


Figure 9

LEVEL: CLASS 4

Sujoy's house is on the third floor of an apartment block. Between every two floors there is a flight of 15 steps. How many steps must Sujoy climb to go from the ground floor to his home?

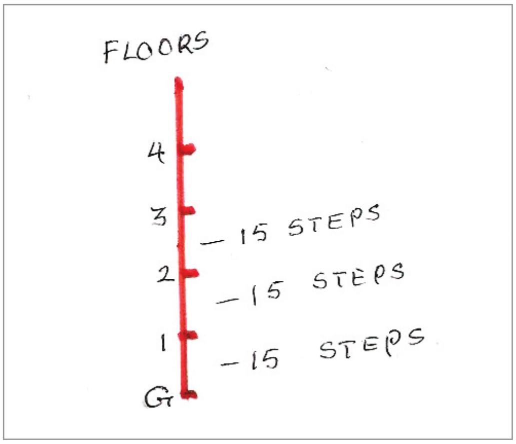


Figure 10

LEVEL: CLASS 5

A bookshelf has five levels. The third level has ten more books than the second level. The second level has four fewer books than the first. The fourth and fifth levels together have the same number of books as the third level. If the first level has 18 books, how many books are there in all in the bookshelf?

Flowchart drawings:

Some problems lend themselves well to flowchart drawing as shown in Figure 11.

LEVEL: CLASS 5

I think of a number. I multiply it by 2, divide by 3, and subtract 4. The result is 2. What is my number?

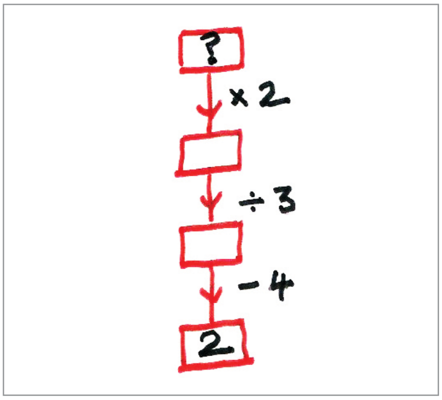


Figure 11

LEVEL: CLASS 5

Chinmayee has four 1 rupee coins, two 2 rupee coins and a 5 rupee coin. In how many ways can she pay for a fruit that costs Rs. 9?

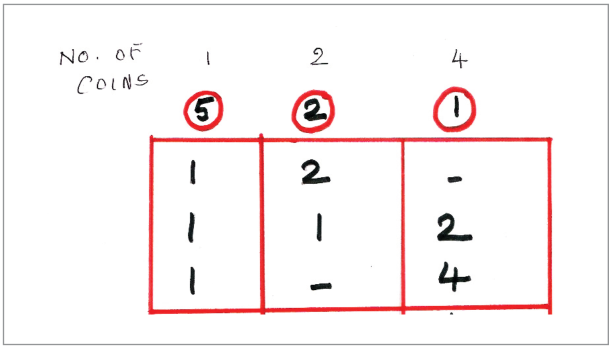


Figure 12

Branching drawings:

Samarth and his friends are making a pyramid formation for their school Sports Day. Samarth balances himself by standing on two students. Each of those two students stands on two other students, and each of those students stands on two other students. How many students are there in all in the whole formation?

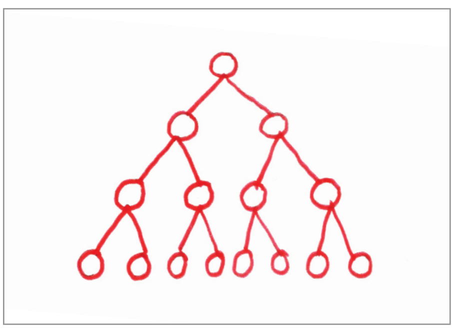


Figure 13

Bar drawings:

A bar drawing is a very useful tool for problem solving which can be used at various levels, right up to high school. It is highly versatile as it can be adapted to various types of problems. Students should be introduced to this method in a graded manner as detailed here. They should practice with several problems at each stage to make the transition smoother.

Stage 1: Use linear pictures to show an operation.

Ex. Addition

Ashwin has 4 cupcakes, and Akriti has 6 cupcakes. How many cupcakes do they have altogether?

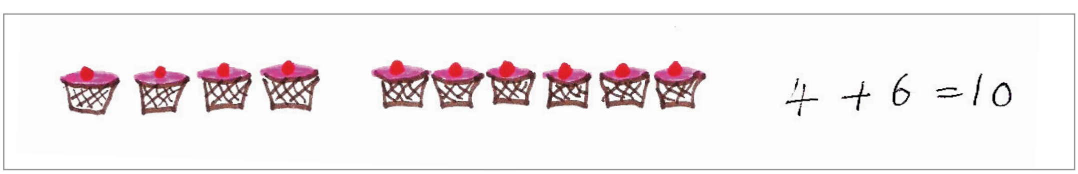


Figure 14

Tabular drawings:

Some problems are better comprehended by using a tabular approach. All possible combinations can be drawn using a table format as shown in Figure 15.

Stage 2: Draw bars around the pictures. The part-whole relationship is brought into focus. The parts have 4 cupcakes and 6 cupcakes. Together they make the whole which is 10 cupcakes.

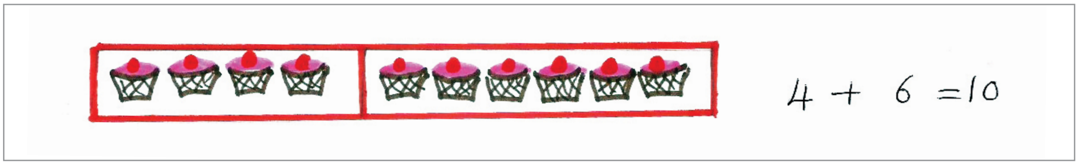


Figure 15

Stage 3: Replace pictures with dots. Dots are a symbolic representation for pictures. Draw arrows below the bars to focus on the whole.

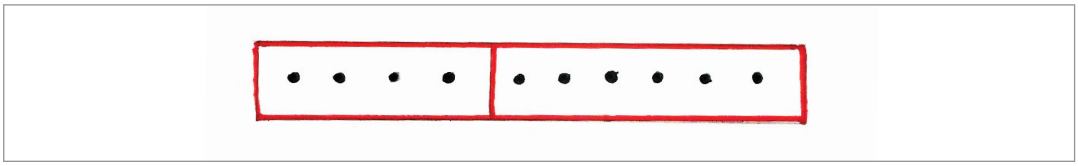


Figure 16

This is to help make the transition from the semi-concrete (picture) to number form.

Stage 4: Replace dots with numbers for the parts. The arrows below the bars represent the whole.

At this point take care to draw bigger bars for bars whose value is high. This will aid in the visualisation.

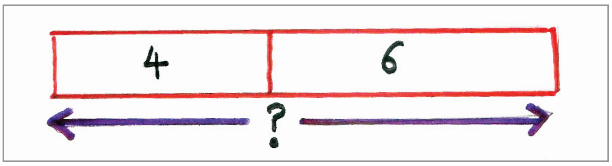


Figure 17

Stage 5: Draw multiple bars to show comparisons. The arrows can also be used to point to portions of the bars. Totalling or summing can also be indicated clearly as shown in these figures.

Ex. Comparison

Kriti has 8 toffees. Nitya has 7 more toffees than Kriti. How many toffees does Nitya have?

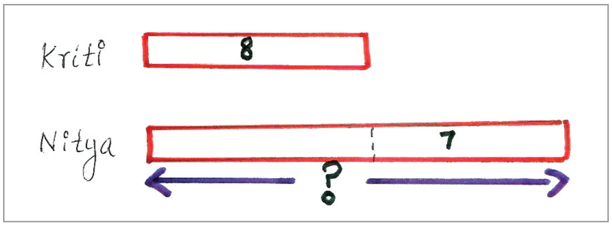


Figure 18

Kriti has 8 toffees. Nitya has 7 more toffees than Kriti. How many toffees do they have together?

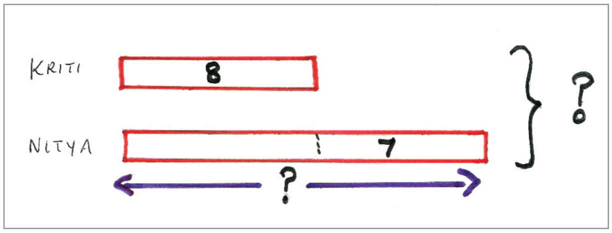


Figure 19

Tanya and Riddhi have 24 coloured pencils altogether. Riddhi has 6 pencils. How many more pencils does Tanya have than Riddhi?

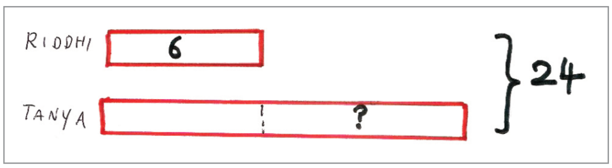


Figure 20

Ex. Multiplication

Sahil has 16 shells. Aman has 3 times as many shells as Sahil. How many shells does Aman have?

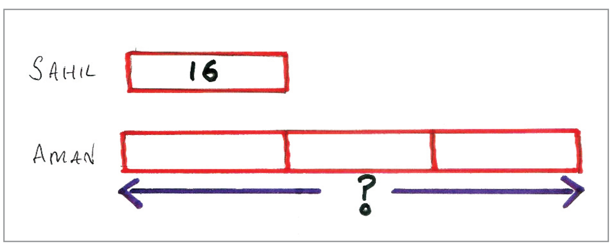


Figure 21

Ex. Division

Ayaan has 18 biscuits. He places 3 biscuits in each plate. How many plates did he fill?

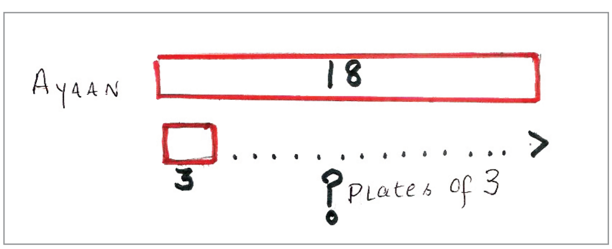


Figure 22

Ex. Fraction

One quarter (1/4) of a number is 8. What is the number?

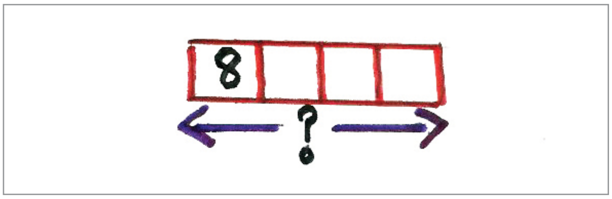


Figure 23

Stage 6: Many problems refer to a whole and its parts. An appropriate drawing can bring out the relationship between the parts and the whole clearly. For complex problems which require further partitioning of bars, it may be better to write the numbers outside the bars.

Ex. Multiple operation

Pranathi made 24 cupcakes. She ate 4 of them and gave 12 cupcakes to her friends. How many cupcakes does she have now?

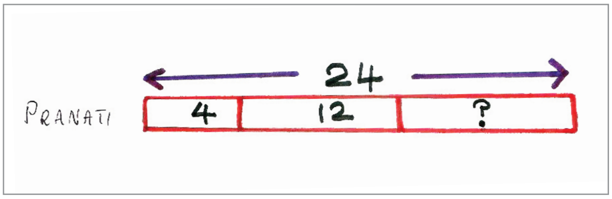


Figure 24

Two thirds (2/3) of a number is 12. What is the number?

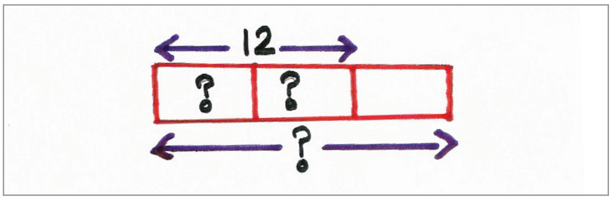


Figure 25

Class 5 has 80 students who are distributed in three sections. There are 22 students in Section A. Section C has 6 more students than Section A. How many students are in Section B?

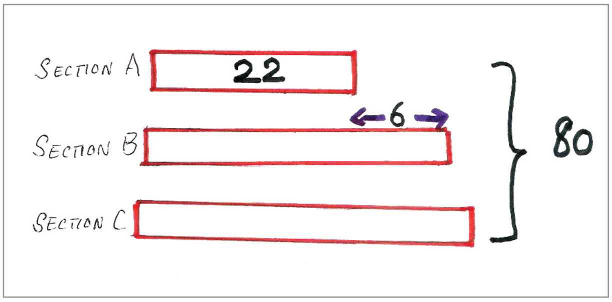


Figure 26

There are 8 more oranges than apples in a basket. There are 24 fruits in all in the basket. How many apples are there in the basket?

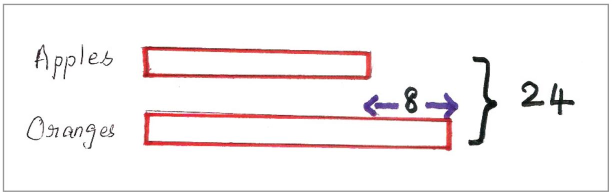


Figure 27

Akriti had 32 marbles. She gave half of the marbles to her brother. After that, she gave half of the remaining marbles to her best friend. How many marbles does she have now?

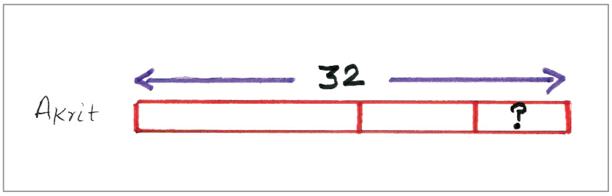


Figure 28

Dhruv's age is three times Aditya's age, and Hari is twice as old as Aditya. The sum of their ages is 30. How old is each boy?

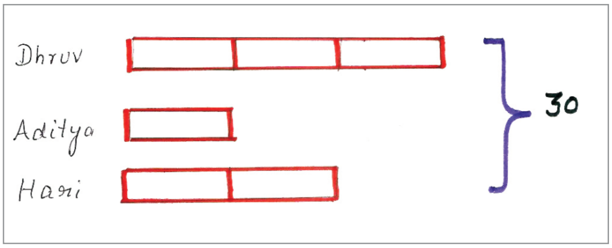


Figure 29

This problem is in fact a precursor to the algebraic way of thinking. However by using this approach, a child will be able to solve it well before he arrives at abstract thinking.

Mrs. Kapoor bought four large mangoes at Rs. 6 each, and five small mangoes at Rs. 4 each. She gave the fruit seller a fifty-rupee note. How much change does she receive from the fruit seller?

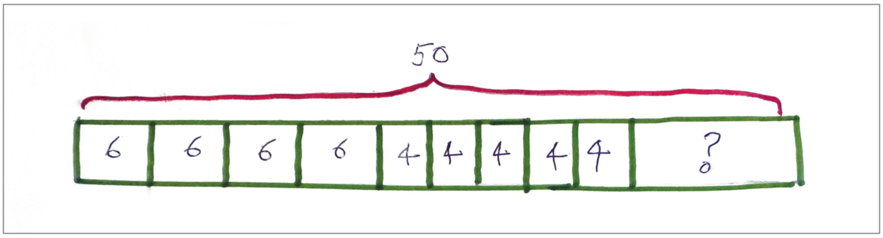


Figure 30

Thinking aloud approach

Some problems lend themselves well to thinking aloud. If the teacher can solve the problem by thinking aloud slowly, children will also pick up better problem-solving techniques as well as the ability to articulate and share their thinking process. Even though young children may have difficulty in expressing themselves, with a few carefully thought-out lead questions from the teacher, they will start to verbalize their thinking. Listening to their peers often helps children notice multiple ways of looking at a problem. For the teacher, it reveals the child's understanding of the concepts and processes, or the child's misconceptions.

Level: Class 2

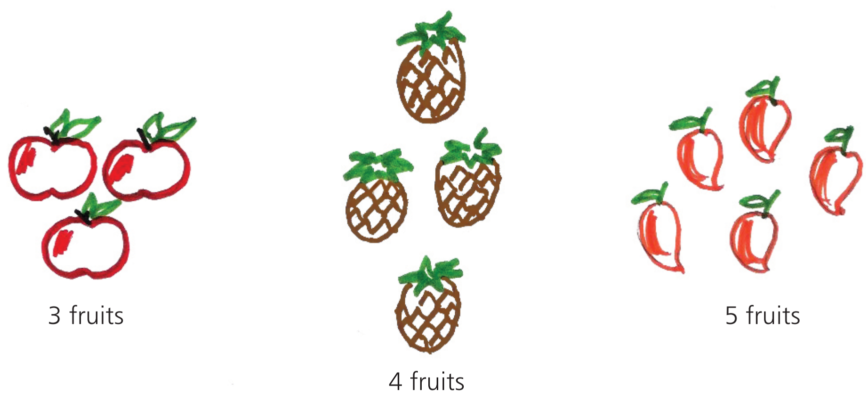


Figure 31

Here are three groups of fruits. Which two groups together have 8 fruits?

Teacher thinks aloud.

Will the first and second group add up to 8? No; 3 and 4 make 7.

Will the second and third group add up to 8? No; 4 and 5 make 9.

What about the first and third? Yes; 3 and 5 do add up to 8.

Level: Class 3

Figure 32 - 6 fruits in the first basket, 10 fruits in the second basket

How many fruits from the first basket should be moved to the second basket so that the number of fruits in the second basket is three times the number in the first basket?

What will happen if I move 1 fruit from the first basket to the second?

There will be 5 in the first basket and 11 in the second basket. 11 is not three times 5.

What if I move 2 fruits from the first basket to the second basket?

There will be 4 in the first and 12 in the second. 12 is three times 4.

A farmer has more than 14 but less than 20 eggs. If he counts the eggs in twos, there is one egg left over. If he counts the eggs in threes, there are two eggs left over. How many eggs does the farmer have?

Can the farmer have 15 eggs? If he counts in twos, he will be left with an extra egg. But if he counts in threes, there will be no eggs left over. So it cannot be 15.

Can it be 16 eggs? And so on.

Five

Writing word (story) problems for pictures and for expressions with number operations

The teacher can give a picture which depicts a problem situation. The children can make up a story problem to match the picture. If the children are not yet in a position to write, they can narrate the story orally and the teacher can write it out for them on the board. When children create and pose a problem for a picture, they come up with different stories. They will have multiple ways of looking at it. This approach potentially has great value. It places the child in the position of a problem poser.

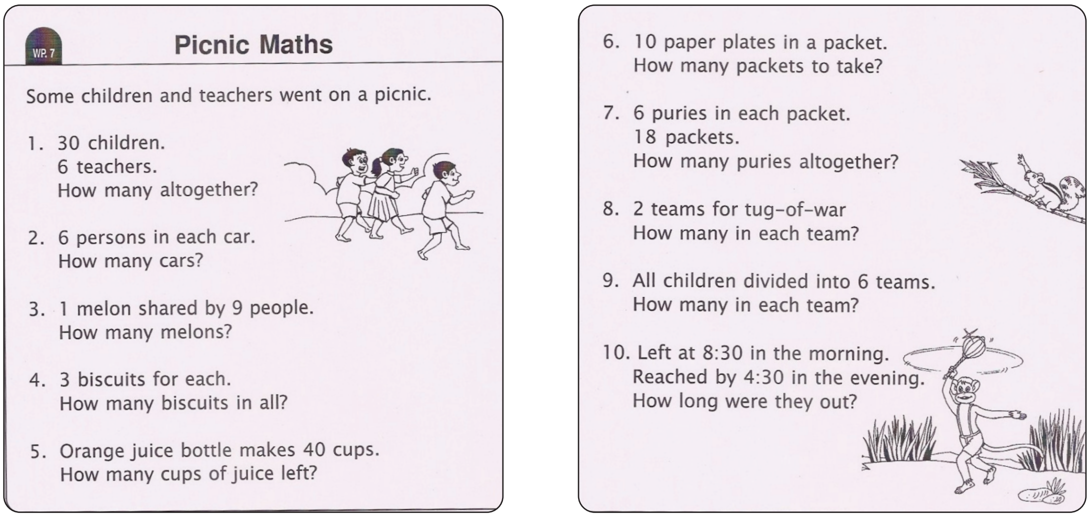
The teacher can also give simple expressions involving two or more operations and ask children to create story problems for them. For example:

Write a story problem for $15 - (5 + 7)$ and another story problem for $15 - 5 + 7$. Discuss both the stories in the class.

Figure 33

In what way do they differ?

It is also good to create theme-based word problems which are child friendly and incorporate various operations.



WP. 7

Picnic Maths

Some children and teachers went on a picnic.

1. 30 children.
   6 teachers.
   How many altogether?
2. 6 persons in each car.
   How many cars?
3. 1 melon shared by 9 people.
   How many melons?
4. 3 biscuits for each.
   How many biscuits in all?
5. Orange juice bottle makes 40 cups.
   How many cups of juice left?
6. 10 paper plates in a packet.
   How many packets to take?
7. 6 puries in each packet.
   18 packets.
   How many puries altogether?
8. 2 teams for tug-of-war
   How many in each team?
9. All children divided into 6 teams.
   How many in each team?
10. Left at 8:30 in the morning.
    Reached by 4:30 in the evening.
    How long were they out?

Figure 34

Six Oral word problem approach

Oral word problem approach

On a daily basis, it is good to pose simple one-line word problems to give practice in math vocabulary and word problem comprehension. Here are a few such one liners.

1. Give me two numbers whose sum is 10.
2. How can you get a product of 12 using two dice?
3. Think of a number whose multiple is 18.
4. What is the smallest two-digit factor of 24?
5. Name some numbers which can be divided by 7 without remainders.
6. I am an odd number between 10 and 20. I am a multiple of 3. Who am I?

Write a single- or double-digit number on the board. "Tell me anything that you can think of about this number using math words." Children can use mathematical terms they have learnt: odd, even, factor, multiple, prime, composite, square number, one less than a square number, etc. Encourage them to find as many ways as describing it.

Game: Twenty questions. The teacher thinks of a number between 1 and 100. Children are allowed to pose questions (which will be answered only Yes or No), making use of math vocabulary. They should try to figure out the answer within twenty questions.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com