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At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 1, No. 2
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Lagrange's Theorem

The fine art of Euclidean geometry

Viviani's Theorem

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The stirring story of Srinivasa Ramanujan is well known: his school days in Kumbakonam, his brief college education in Madras (present day Chennai), his discovery of Carr's book *A Synopsis of Pure Mathematics* and the way it opened a completely new world for him, his research which he conducted entirely on his own, there being no one around who could understand the mathematics he was working on, his famous pair of notebooks in which he jotted his findings, his attempts to contact overseas mathematicians, his journey to England in 1913 and his work with the British mathematician G H Hardy, the war years which he spent in England, his return to India in 1919, and his tragic death just a year later (1920).

Today the world has begun to grasp what an extraordinarily fertile mind Ramanujan had, and how deep and path-breaking his work was. The credit for that understanding goes largely to the mathematicians Bruce Berndt, George Andrews and Richard Askey, who have done an enormous amount of work with his unpublished writings.

It was because of the efforts of these individuals and the Indian astrophysicist S Chandrasekhar that the bust of Ramanujan depicted above was commissioned and sculpted by Paul Granlund in 1984; it was presented to Smt Janaki Ammal, the widow of S Ramanujan. Another such bust was commissioned for the Ramanujan Institute (University of Madras) by Mr Masilamani in 1994.

From The Chief Editor's Desk . . .

It is a happy moment for us to release the second issue of *At Right Angles*. It has a rich fare: a lead article on a classically great theorem in number theory which goes by the name "Lagrange's Four Squares Theorem", followed by articles on games of chance; on the axiomatic basis of origami and an unexpected construction possible under the rules of paper folding; on a beautiful theorem of Euclidean geometry called "Viviani's theorem"; and on using a spreadsheet to explore the famous conundrum known as the 'Monty Hall problem'.

In July 2012 ICME 12 took place in Seoul, South Korea. This is one of the premier events in mathematics education, held once in four years, and we have an informally written report on ICME 12 which gives a flavour of an ICME. The authors walk you through the lecture rooms of the mall where the conference was held, and through the many exhibits. Following this we have articles on the role of open-ended questioning in classroom teaching; on pitfalls in the teaching of the method of induction; on how to approach problem solving in geometry. There are two small cameos in geometry: an entry from one of the notebooks of Ramanujan in which a result of geometry is manufactured from an algebraic identity, and an example of a proposition for which one expects the converse to be true but finds it to be not so. In the continuing 'Math Club' column we study a seemingly commonplace problem that lies in the region shared by geometry, combinatorics and the topic of sequences. The 'Pullout' in this issue features the teaching of decimal fractions. The 'Review' page features one of the best used sites in school level mathematics: the site belonging to the NRICH project of the University of Cambridge.

We have received many appreciative letters from readers telling us how happy they are that such a publication has seen the light of day. Two of our readers have sent in thoughtful contributions which have been featured on the 'Letters' page. We hope that such material will continue to flow into our e-mail Inbox.

It is said that mathematics has a secret garden. Perhaps other fields of study have such gardens as well, but mathematics very certainly has one. It is a garden where one finds great and exciting stories: of theorems that were conjectured hundreds of years ago and proved only recently; of unsolved problems; of human struggle and toil; of strange and mysterious connections between different topics of mathematics, and between mathematics and art and music and sculpture. It is our hope that we will always be able to profile stories in this magazine that will help children find this secret garden and uncover its delights.

- Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Features

This section has articles dealing with mathematical content, in pure and applied mathematics. the scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practicing mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

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This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practicing teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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"Tech Space" is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

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Pullout

Teaching Decimal Fractions

Tracing the history of a theorem

Lagrange's Four Squares Theorem

From conjecture to proof

Why are theorems discussed in class restricted to those in the syllabus? Do we fear that students would be intimidated by equations and long proofs? In this article the author traces the development of a famous theorem and its proof. Read the article not just for the theorem but also to pick up tips on how to use numerical examples to understand algebraic equations, how to use historical details to move from conjecture to proof, and how to provide students with sufficient scaffolding to enable them to prove the theorem for themselves.

ANURADHA S. GARGE

Number theory is a branch of mathematics in which we study the integers and rational numbers (e.g., prime numbers; squares; cubes) and concepts derived from them. It is a field with a long past and a rich history, and it has been strongly influenced by the work of mathematical giants like Gauss, Legendre, Lagrange, Fermat, Euler, Ramanujan, etc. This article presents the story of a theorem in number theory which is very easy to state, but it took mathematicians one and a half centuries to write its full proof! It is named after Joseph Louis Lagrange, the French mathematician who gave the first complete proof in 1770.

It is common for a theorem to appear initially in the form of a 'conjecture' which may be simply an intelligent guess concerning the properties of some mathematical object; for example, we have the *Four Colour Conjecture* which went unproven for over a century before it became a theorem. Such conjectures are usually

easy to verify for small instances of the problem. But as the size of the problem gets larger, verification becomes steadily more difficult. So one needs a proof which does not depend on case-by-case analysis. It can be an uphill task to produce such a proof.

The theorem proved by Lagrange concerns a conjecture made by Bachet, a French mathematician, about natural numbers written as sums of squares. Not every natural number is a square. Can every natural number be written as a sum of *two* squares, like $5 = 2^2 + 1^2$? No, we cannot write 3 this way (check!). Can every natural number be written as a sum of *three* squares, like $3 = 1^2 + 1^2 + 1^2$ and $11 = 3^2 + 1^2 + 1^2$? Many numbers can be written this way, but some cannot; e.g., 7. But we can write 7 as a sum of *four* squares: $7 = 2^2 + 1^2 + 1^2 + 1^2$. Can every natural number be written as a sum of four squares? Amazingly, the answer is **Yes**; we never need more than four squares! For example, we have $2011 = 35^2 + 28^2 + 1^2 + 1^2$, and $2012 = 44^2 + 6^2 + 6^2 + 2^2$.

Note that writing a natural number as a sum of squares is not difficult; 1 is our friend! But we are interested in finding the *least* number such that every natural number can be written as a sum of at most that many squares. Experimentation suggests that we never need more than four squares, and this is what the amazing theorem of Lagrange asserts: *Every natural number, however large, can be written as a sum of at most four squares.* (Yes, even big numbers like one lakh (10^5) or one crore (10^7) can be written this way; see if you can find the expressions for these numbers! Experimenting further, try to guess which numbers require three squares and no less, and which numbers require four squares and no less.)

In the century before Lagrange, another brilliant French mathematician had worked on this problem: Pierre de Fermat (1601–1665). He classified those natural numbers which could be expressed as sums of two squares and which are not squares themselves. The final part of the proof was completed by Lagrange in 1770. This article describes how Bachet's conjecture turned into a theorem, and gives a (very) brief idea of Lagrange's proof.



Fig. 1 *Arithmetica* by Diophantus; source: [6]

Bachet's conjecture

The origins of the conjecture lie in the work of Diophantus, a third century AD mathematician from Alexandria (Egypt), who was the first to introduce notation in algebra. He wrote a book called *ARITHMETICA* (see Figure 1) which had a collection of problems based on what are now called *Diophantine equations*. These equations differed from one another only marginally, but a new trick had to be used to solve each one. A significant feature of the problems is their focus on *solutions which are rational numbers*. Here is an example; it shows the level of sophistication of the problems: Diophantus asks for an expression for 13 as the sum of two rational squares each exceeding 6, and gives the following as an answer:

$$13 = \frac{66049}{10201} + \frac{66564}{10201} = \left(\frac{257}{101}\right)^2 + \left(\frac{258}{101}\right)^2.$$

Diophantus was able to solve the equations by making clever substitutions so he had to deal with simpler equations. The sophistication of his approach justifies the title he is sometimes given, 'Father of Modern Algebra'.

In 1621, Bachet (Claude Gaspard Bachet de Méziriac, to give him his full name), a French mathematician, linguist and poet, translated *ARITHMETICA* from Greek into Latin. While doing so, he was led to claim (or perhaps to affirm the claim made by Diophantus) that every natural number can be written as a sum of at most four



Fig. 2 Claude Gaspard Bachet de Méziriac (1581–1638); source: [7]

squares. This therefore came to be called **Bachet's conjecture**.

Fermat's contribution: the two squares theorem

Fermat was a lawyer by profession, but made numerous important contributions to mathematics, in fields such as probability theory, coordinate geometry, maxima-minima, optics and number theory. He often stated theorems without giving proofs. He studied Bachet's translation of Diophantus and worked on its problems. One of his remarkable claims, made in the margin of one of Bachet's books, was that the equation $a^n + b^n = c^n$ has no solutions in positive integers a, b, c, n if $n > 2$. (See [1] for a review of the book [5] by Simon Singh which gives an account of this story.) An important result that Fermat found — which he *did* prove — had to do with natural numbers which can be written as sums of two squares. He had discovered a result now known as *Fermat's two squares theorem*.

The theorem says that *an odd prime p can be expressed as a sum of two squares if and only if it leaves remainder 1 when divided by 4*. For example: the primes 5, 13 and 17 can be written as sums of two squares, but 7, 11 and 19 need more than two squares. (Please check.) This observation had been made earlier (by Albert Girard, in 1632), but Fermat was the first to prove it. He announced the theorem in a letter to Marin Mersenne dated 25 December 1640, and for this reason it is sometimes called *Fermat's Christmas Theorem*.



Fig. 3 Pierre de Fermat (1601–1665); source: [8]

Every natural number can be factored into a product of primes (in a unique way; this is the statement of the *Fundamental Theorem of Arithmetic*). Suppose that every prime in the factorization of a natural number N is a sum of two squares. Is N itself then a sum of two squares? The answer is yes, and this may be shown by using an interesting identity known as the *Brahmagupta identity* (see [10]) which states that for any natural number n ,

$$\begin{aligned}(a^2 + nb^2)(c^2 + nd^2) \\ &= (ac - nbd)^2 + n(ad + bc)^2 \\ &= (ac + nbd)^2 + n(ad - bc)^2.\end{aligned}$$

The special case $n = 1$, which was known to Diophantus, states the following:

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= (ac - bd)^2 + (ad + bc)^2 \\ &= (ac + bd)^2 + (ad - bc)^2.\end{aligned}$$

It shows that a product of two numbers which are sums of two squares is itself a sum of two squares. For example, let $a = 2, b = 1, c = 3, d = 2$. Then from the relations

$$5 = 2^2 + 1^2, \quad 13 = 3^2 + 2^2$$

we find, by substitution, two different ways of writing $5 \times 13 = 65$ as a sum of two squares:

$$65 = 8^2 + 1^2 = 4^2 + 7^2.$$

Fermat's two squares theorem allows us to find an equivalent description of the natural numbers

that are sums of two squares. Let N be any natural number. Write N as k^2m where m is not divisible by any square number greater than 1. (Thus, k^2 is the largest square divisor of N .) Then N is a sum of two squares if and only if every prime divisor of m leaves remainder 1 when divided by 4. We illustrate this with two examples.

- Consider $N = 240 = 4^2 \times 3 \times 5$, for which $k^2 = 4^2$ and $m = 3 \times 5$. The presence of the '3' shows that this number cannot be written as a sum of two squares.
- Consider $N = 765 = 3^2 \times 5 \times 17$, for which $k^2 = 3^2$ and $m = 5 \times 17$. The primes which divide m (namely, 5 and 17) leave remainder 1 under division by 4. And indeed we have two such representations:
 $765 = 27^2 + 6^2 = 21^2 + 18^2$.

Lagrange's proof

It was Joseph Louis Lagrange (1736–1813), a brilliant Italian-born French mathematician and astronomer, who first proved the four squares theorem. Lagrange contributed not only to mathematics but also physics, specially classical mechanics. He served as the director of the Prussian Academy of Sciences for twenty years and won prizes for solving problems in astronomy posed by the French Academy of Sciences. A lunar crater is named after him, and his name appears amongst 72 names inscribed on the Eiffel tower! (See [3] and [9].)

We now mention the main steps in the proof of Lagrange's theorem. Ambitious students may want to complete the proof on their own, using the lemmas. Crucial to the proof is in an amazing

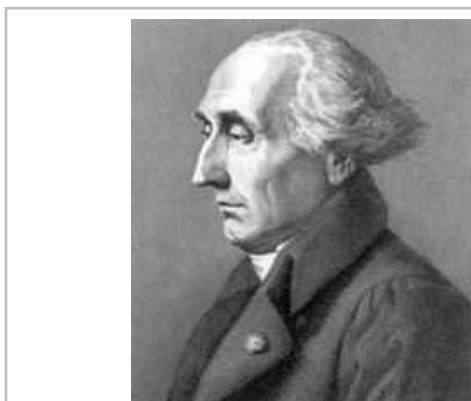


Fig. 4 Joseph Louis Lagrange (1736–1813); source: [9]

identity for sums of four squares which is much like the Brahmagupta identity for sums of two squares. It was discovered by Euler and is called the *four squares identity*. It is easy to verify, but discovering it must have been quite an achievement! We state it as a lemma and leave its verification to you.

Lemma 1. For any numbers a, b, c, d and p, q, r, s we have:

$$\begin{aligned} & (a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) \\ &= (ap - bq - cr - ds)^2 \\ &+ (aq + bp + cs - dr)^2 \\ &+ (ar + cp + dq - bs)^2 \\ &+ (as + dp + br - cq)^2. \end{aligned}$$

The lemma tells us that the product of two numbers which are expressible as sums of at most four squares is itself expressible as a sum of at most four squares. Here is an instance of the identity: $31 = 5^2 + 2^2 + 1^2 + 1^2$, with $(a, b, c, d) = (5, 2, 1, 1)$; $71 = 7^2 + 3^2 + 3^2 + 2^2$, with $(p, q, r, s) = (7, 3, 3, 2)$; $31 \times 71 = 2201$; and $2201 = 24^2 + 28^2 + 21^2 + 20^2$.

Since every natural number can be written as a product of primes, Lemma 1 implies that if you can express each prime number as a sum of at most four squares, then you can express every natural number as a sum of at most four squares. As 2 is a sum of two squares, it only remains to prove that every odd prime p is a sum of at most four squares. This can be done by using the following sequence of lemmas:

Lemma 2. If n is even and is a sum of at most two squares, then so is $n/2$.

Lemma 3. If n is even and is a sum of at most four squares, then so is $n/2$.

Lemma 4. If p is an odd prime, then there exist integers a and b and an integer k , $0 < k < p$, such that $a^2 + b^2 + 1 = kp$.

Lemma 5. If p is an odd prime and there exists an integer $k_1 > 1$ such that k_1p is a sum of four squares, then there exists an integer $k_2 < k_1$ such that k_2p is a sum of four squares.

Of these, Lemmas 2 and 3 are not difficult to prove. (*Hint.* Simplify the expression $(\frac{1}{2}(x+y))^2 + (\frac{1}{2}(x-y))^2$.) Lemma 4 is proved using ideas from combinatorics (specifically, a principle called the ‘pigeon hole principle’). Lemma 5 is the key step; it is called a *descent* step, as it allows us to ‘descend’ from a higher multiple of p to a lower multiple, and ultimately to p itself. The proofs of Lemmas 4 and 5 are fairly challenging.

Closing remarks

Lagrange’s theorem led naturally to questions about writing the natural numbers as sums of fourth, fifth and higher powers; and this in turn led to a problem now known as *Waring’s problem*, whose full story, spanning more than three

centuries, involves many well known mathematicians including two from India: S S Pillai and R Balasubramanian.

In the computer algebra package *Mathematica* one can just type a command to get all decompositions of a natural number as a sum of squares or higher powers: the command `PowersRepresentations[n, k, p]` gives all representations of n as a sum of k non-negative p -th powers. A challenging exercise is to write this program and to get to know the powerful theorems that lie beneath it. The following theorem proved by Legendre (1752–1833) turns out to be handy: *A natural number is a sum of three squares if and only if it is not of the form $4^k(8m + 7)$.*

Acknowledgements

I thank the editors for giving me an opportunity to write in *AtRiA*. Without Wikipedia, this article would not have got its colour!

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Analyzing games of chance with math . . .

Fair Game

What are the odds that the game of chance you are playing is fair? How does the organizer of a lottery make a profit? How often does a customer win at a casino? Kamala Mukunda analyzes games of chance and brings great expectations within reach of an introductory course to probability.

KAMALA MUKUNDA

Many things in life boil down to a game of chance, actually! Getting insurance, gambling, telling the girl you like that you like her, buying a lottery ticket, riding a motorcycle without a helmet . . . all these risky activities have two key things in common. They have an *outcome* which is usually in terms of winning or losing something. For example, in gambling one can win or lose money; or it may be peace of mind. Also, there is a *probability* associated with each outcome; this can be calculated precisely in some cases, and only be estimated in other cases. Relative frequencies are one way of understanding the meaning of probability: if an experiment is carried out a large number of times, then the relative frequency of the occurrence of an event (in the 'long run') may be described as the probability of that event. An interesting topic in probability is the understanding of 'games of chance', defined broadly. One more useful feature of such games is that they can be played over and over again, which gives the idea of a 'long-run average' (a useful concept as you will see).

Let's start with a very simple game played between two friends A and B, using a coin. They decide that A wins if the outcome is heads, and B wins if the outcome is tails. Then they both put 10 rupees each in the middle, and toss! If it lands heads, A gets to keep the 20 rupees, but if it lands tails, A loses her 10 rupees. If the game is played only once, one of the friends is certain to be disappointed and the other one elated! But here's a question: assuming they play long enough, is this game fair to both players? Clearly, yes. Suppose they toss two coins, and A wins if both are same, while B wins if both are different. Again, the game is perfectly fair, because the four outcomes HH, TT, HT and TH are equiprobable (each of the outcomes has probability $1/2 \times 1/2 = 1/4$).

In fair games, the probability of winning is the same for both players. A nice way to formalize this is to draw a simple table showing the outcomes for each player, with their associated probabilities. For A, the outcome X could be -10 or $+10$ as below.

X	-10	$+10$
P(X)	$1/2$	$1/2$
X.P(X)	-5	$+5$

The third row is the product of X and P(X), and when we add these values for all values of X, we get something called an Expected Value for X, or E(X). E(X) is the long-run average of a person's outcomes over many trials, or in this case, it is the average of A's winnings over many, many repetitions of the same game with B. As you can see in the example, E(X) is zero! Of course it will be the same for B. That's why this is a fair game, because although on any given game only one of them can win, in the long run, neither is expected to win more often than the other.

Another interpretation of E(X) is that each time A and B play, they each should 'expect' to win 0 rupees! Of course no one in her right mind will really expect the winnings to be 0—we know that either you will lose 10 rupees or you will win 10 rupees—but this is what 'expected' means in a probability course. (By similar reasoning, each time you roll a die, you expect to get 3.5, even though you cannot possibly ever get it!)

In an actual run of 100 games, it may happen that A wins 54 times and B wins 46 times; or A wins

45 times and B wins 55 times; or A wins 60 times and B wins 40 times; etc. So A may leave the place with some winnings from B's pocket, or it may happen the other way round. That is part of what is meant by 'expected value'; we expect both A and B to win 50 games each, but we also expect some reasonable variation from that scenario. In this article, I will not go into the meaning of 'reasonable variation' (though it's certainly important and interesting), because a great deal of fun can be had with expected values alone.

Here is an example of an unfair game: A die is rolled, and A wins if the outcome is less than or equal to 4, whereas B wins if the outcome is greater than 4. The table below for B's winnings shows why:

X	-10	$+10$
P(X)	$4/6$	$2/6$
X.P(X)	$-20/3$	$+10/3$

On every game, B expects to win $-20/3 + 10/3 = -10/3 = -3.33$ rupees, or in other words he stands to lose each time.

You can see that the key in these tables is to be able to figure out the probability values, P(X). Tossing a coin and rolling a die seem to yield straightforward calculations . . . but you can quickly make it more difficult. Consider this game: as before, A and B place 10 rupees each on the table. A rolls a single die and if she gets a 3 or a 6, she wins and the game ends. If she does not, B rolls two dice and if he gets a 6 on either die he wins. If he gets no 6s, the game is drawn, and they each take their money back. (See Figure 1.)

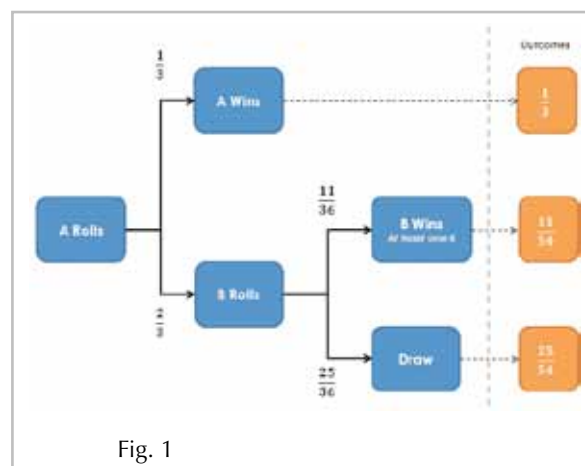
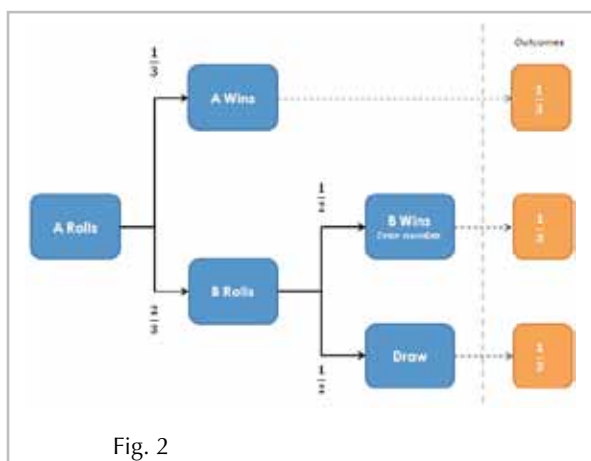


Fig. 1

The table of outcomes for B is below:

X	-10	+10	0
P(X)	1/3	11/54	25/54
X.P(X)	-10/3	+55/27	0

For B, $E(X) = (-10/3 + 55/27) = -1.29$ rupees, so B can expect to lose Rs. 1.29 every time he plays. If they played 100 times, he could expect to lose around 129 rupees, give or take a little. You can make a similar table of outcomes for A, using the probabilities computed above and you will see where the 129 rupees is going! To make this a fair game, we could say that for B to win, he needs to roll a single die and get an even number (probability 1/2). Figure 2 shows the tree diagram for this game; make the tables of outcomes for A and B to convince yourself that this is now a fair game.



A similar game was popular among French noblemen in the 1600s, the assumption being that it was a fair game, because it was thought that the probabilities were equal. They did not know about tree diagrams then, or how to calculate probabilities. It was when they noticed that whoever played in B's position would come out the loser in the long run that they wrote to the mathematician Blaise Pascal, who along with his friend Pierre de Fermat solved the problem and invented probability theory in the process. (I've included the problem at the end of this article for you to solve.)

Now it's time to introduce a third friend, C. He joins the fair game A and B are playing in Figure 2, and says to them, "If neither of you wins, I'll take

the 20 rupees!" The outcomes have now changed, as you can see below. In the tree diagram, you need to replace 'Draw' with 'C wins'.

For A and B, $E(X) = (-20/3 + 10/3) = -3.33$:

X	-10	+10
P(X)	2/3	1/3
X.P(X)	-20/3	+10/3

For C, $E(X) = 0 + 20/3 = 6.67$:

X	0	+20
P(X)	2/3	1/3
X.P(X)	0	+20/3

So this seems like an unfair game, one that no A and B in their right minds would agree to play. But it is *exactly* the kind of game you are agreeing to play when you walk into a casino! (That's what C stands for, by the way.)

Of course, the numbers will not be quite so obviously tilted in favour of C. Instead, you may place 10 rupees on the table, and stand to win 100. If the probabilities are going to be as in the table below, then neither the casino nor you are gaining anything in the long run:

X	-10	+90
P(X)	90%	10%
X.P(X)	-9.00	+9.00

So they have to create a game only slightly different from the above, where the probabilities are more like this for you:

X	-10	+90
P(X)	91%	9%
X.P(X)	-9.10	+8.10

This ensures that you still feel like playing, but in the long run you're quite sure to leave a loser! Your $E(X)$ of -1 rupee for each time you play may seem ridiculously small winnings for the casino, but if many, many people play this game many, many times, then it translates to big winnings for the casino (this is their long-run average working for them). *Remember, this includes the fact that some people will win, "just by chance".* Ten thousand games a day makes 10,000 rupees for them, and with the profits they can easily afford

some fine furniture, flashing lights and free food to make you return for more games!

It's relatively easy to create a game where the probabilities turn out as in the above table. The trick is to make players feel like they really have a chance to win, and one way is to charge relatively little to play, and dangle a sufficiently large win under your nose. Lotteries are like that.

A thousand people buy a one rupee ticket each, and the winner wins, not a thousand rupees but 900 rupees. The remaining 100 is for the organizers of the lottery. If I don't exactly know how many tickets are sold and therefore cannot calculate my probabilities, the ratio 1:900 may persuade me to buy a ticket. My table of outcomes:

X	-1	+899
P(X)	0.999	.001
X.P(X)	-0.999	+0.899

$$E(X) = -0.1.$$

Sometimes people feel if they buy ten tickets their chances are better. Let's see:

X	-10	890
P(X)	0.99	0.01
X.P(X)	-9.90	+8.90

$E(X) = -1.00$! Why did this happen? Look at it this way, if you bought all 1000 tickets, you'd be sure to win the 900 rupees, and your $E(X)$ would be -100! There's always a casino in the background.

A quick look at Insurance Policies

We take a quick look at insurance policies and why they work. By now the language of tables is easy for you to follow, and I will use it in an oversimplified example of a one-year accident insurance for an individual, freely estimating probabilities and costs! In the table, X refers to the money the customer has at the end of a year. It may be negative (the premium paid to the insurance company) or positive (the company paid all costs incurred due to the accident). Let's assume the premium is 1000 rupees, and accident costs come to 100,000 rupees.

X	-1000 (the premium)	100,000 - 1000 = 99,000
P(X)	(as high as) 0.999	(as low as) 0.001
X.P(X)	-999	99

$E(X) = -900$, which means that every year you buy insurance you can expect to lose 900 rupees. You may wonder why anyone would buy an insurance policy under these conditions. To better understand this, you should play around with the table changing the cost and the probability of an accident (make both numbers larger). You will see that the expected value quickly jumps to a positive number. In other words, someone who estimates the probability of having an accident as relatively high, and who also imagines the prospect of a large bill for that accident, is more likely to take out the insurance policy, because they calculate their $E(X)$ to be positive!

X	-1000	200,000 - 1000 = 199,000
P(X)	0.99	0.01
X.P(X)	-990	1990

However, where is the profit for the insurance company going to come from, if each customer's $E(X)$ is positive? The trick is to adjust the premium so that the $E(X)$ is a small negative number for each customer. Insurance companies don't rely on imagination and 'gut feel' to set premiums. Analysis of large sets of data on accident statistics and costs help insurance companies set premiums that ensure they make a profit, while at the same time making the customers feel that it is a good deal for them. For the company, the table of outcomes looks something like this with each customer.

X	- accident costs (say a lakh of rupees)	+ premium amount
P(X)	(as low as) 0.01	(as high as) 0.99
X.P(X)	-1000	+ nearly the premium amount

$E(X)$ just has to be a positive number; remember they have many, many customers, so thanks to long-run averages $E(X)$ for each customer can be small. Therefore in this simplified example, the company can set premium at just over 1000 to make a decent profit and still attract customers.

Many of our daily decisions are in fact the result of mental 'calculations' from tables of outcomes. Depending on our personality, we assign different probabilities to different outcome values, and make our estimates of $E(X)$. Without access to the kind of actuarial data that an insurance company

has, our estimates could easily be misleading! I show my students who have just obtained a two-wheeler license the following table of outcomes for driving without a helmet. And I ask, is this a fair game, would you like to play it?

X	–Life	Discomfort of wearing a helmet
P(X)	(as low as) .001	(as high as) 0.999
X.P(X)	– depends how you value your life	Discomfort of wearing a helmet

Problems

1. The game that inspired Pascal and Fermat to invent probability theory involved rolling a single die four times, or rolling two dice 24 times. If you chose the former, you won on getting a six. If you chose the latter, you won on getting a double six. Which option should you choose? The prevailing assumption was that these two were equally likely to give a win, but their reasoning was wrong. They thought that since the chance of a six on each roll is $1/6$, on four rolls the chance will be $4/6$. They also reasoned that since the chance of a double six is $1/36$, in 24 throws it will be $24/36$, or $4/6$ again, and therefore the game is fair. Though their calculations are wrong, the difference from 50–50 is so slight that it was only upon playing many, many times that it could be detected. In fact the game is not fair. Can you do the calculations correctly?

ANS: The probability of a six with four throws of a single die is $1 - (5/6)^4$ which exceeds 50% (by just a small bit), and the probability of a double six with 24 throws of two dice is $1 - (35/36)^{24}$ which is less than 50% (by just a small bit)!

2. A popular game in casinos and statistics textbooks is *roulette*. It consists of a spinning disc with 38 pockets and a ball that can fall into any one with equal probability. The pockets are numbered 0, 00 and 1 to 36. The 0 and 00 pockets are green, half the rest are black and half red (see the image, taken from <http://www.kanzen.com/genimg/american-roulette-wheel-0-00-abb.jpg>).

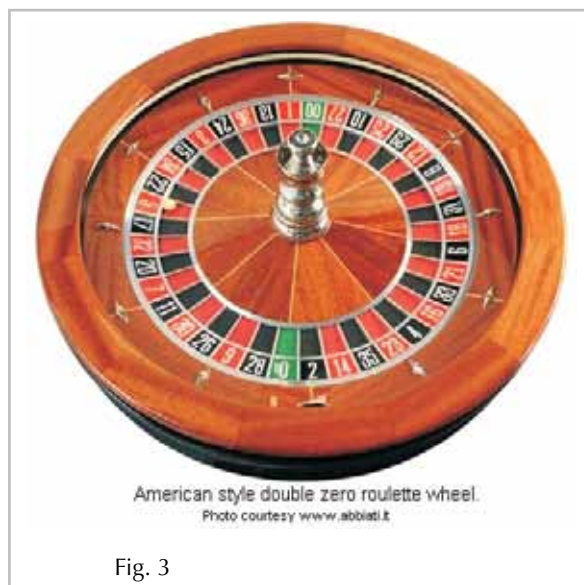


Fig. 3

The wheel is spun, and before the ball lands in a pocket, you can place one of several bets.

- Bet on red (or black), odd (or even; crucially, 0 and 00 are not considered even numbers in roulette!), a number from 1–18 (or from 19–36). The winnings for all these kinds of bets are ‘double your money’—you get back what you paid, plus an equal amount as winnings. Of course if you lose you forfeit whatever you staked.
- Bet on a single number; winning gives you what you staked, plus 35 times that amount.
- Bet on any two consecutive numbers (you win if the ball lands in either); winning gives you what you staked, plus 17 times that amount.
- Bet on any four consecutive numbers (you win if the ball lands in any of them); winning gives you what you staked, plus 8 times that amount.

Make the table of outcomes for a player (or for the casino) for each of these bets, calculating the $E(X)$ values. Having done this it will be clear to you why the bets are arranged as they are. It will also be clear why there are two pockets numbered 0 and 00!

Suppose you wanted to set up a casino and decided to be greedy, offering lower winnings for each bet. For example, you could say that betting on a single number gives 20 times

(instead of 35 times) the staked amount as winnings. As casino manager, your $E(X)$ would shoot up for that particular bet. But why might you *not* do that?

The editors thank Mr Rajveer Sangha of Azim Premji University for preparing the graphics.



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Axioms of Paper Folding

How does an entertaining pastime such as paper folding evolve into a field of geometry? Can there be axioms about making creases on paper? Shiv Gaur talks about the axioms of the ancient but still richly-evolving field of origami, viewed as a 'cousin' of geometry, and then demonstrates its surprising power.

SHIV GAUR

The axiomatic system originated in ancient Greece. Axioms are “self-evident” truths which do not need proof; they serve as the starting points and building blocks upon which a deductive system is based—like **Euclidean geometry**. Some choices are available in formulating the axioms, but they should certainly be *consistent* as well as *independent* of one another; and it is desirable that they be *simple* (‘user friendly’) and *small in number*.

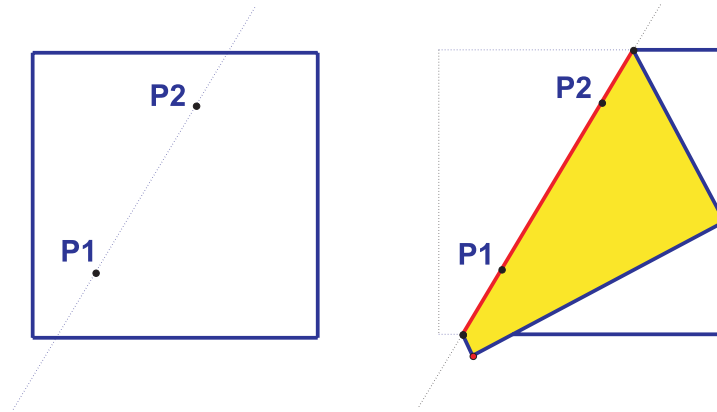
In origami, lines are replaced by *creases*. The question arises whether in paper folding too we can have a set of axioms which govern and tell us what various combinations of points and lines permit us to do.

In 1992 Humiaki Huzita formulated six operations, wherein a single crease could be created by aligning one or more combinations of points and lines on a sheet of paper; these came to be known as *Huzita Axioms*. In 2002 a Japanese origamist

Koshiro Hatori found a single-fold alignment that could not be described in terms of the six axioms, and this became the basis of the seventh axiom. The seven axioms have become known as the **Huzita-Hatori** axioms. Physicist, engineer and origamist Robert Lang proved that these axioms are *complete*, i.e., there can be no other way of defining a single fold in origami using alignments of points and lines.

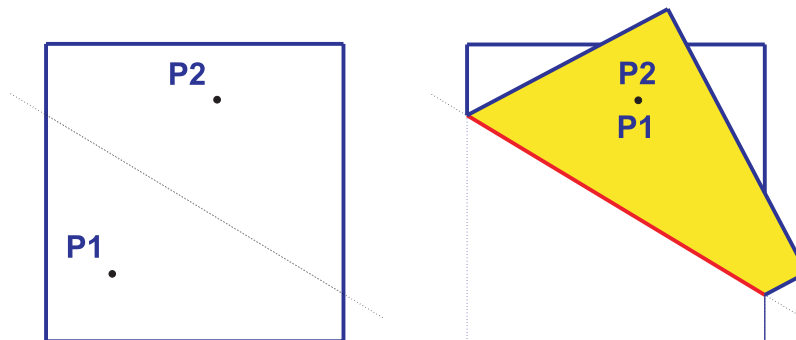
Axiom #01

Given two points P1 and P2, a line can be folded passing through both P1 and P2.



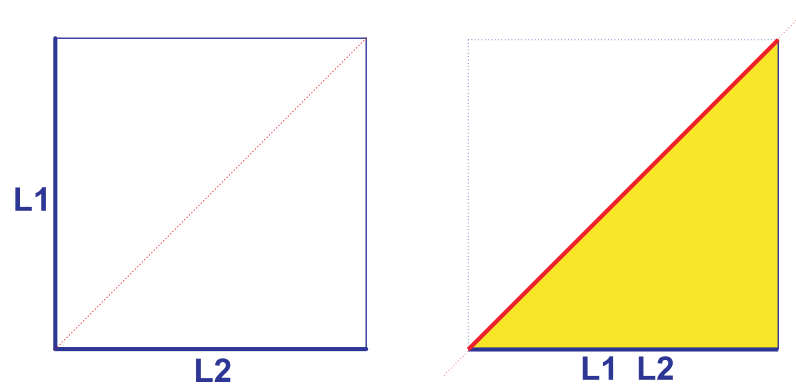
Axiom #02

Given two points P1 and P2, a line can be folded placing P1 onto P2.



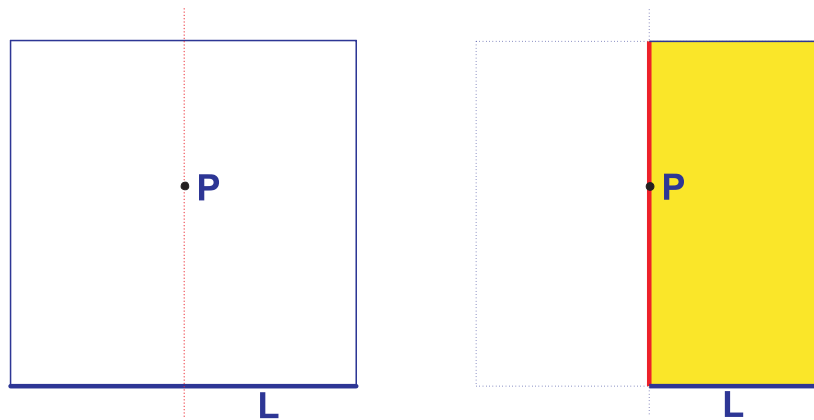
Axiom #03

Given two lines L1 and L2, a line can be folded placing L1 onto L2.



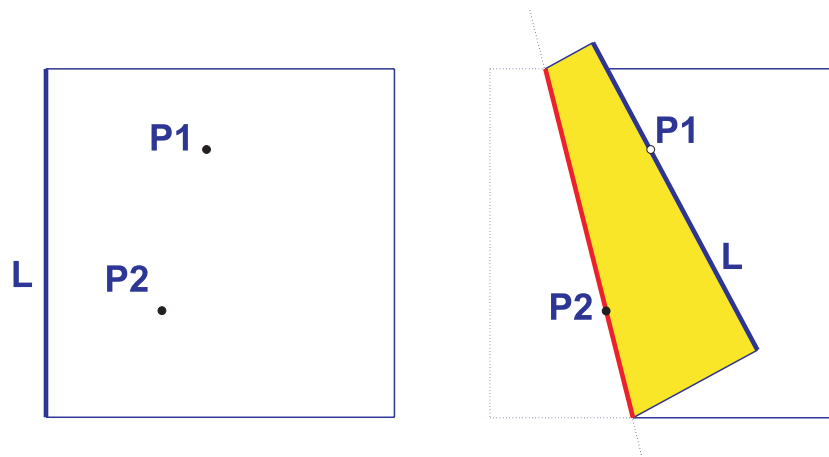
Axiom #04

Given a point P and a line L, a line can be folded passing through P, perpendicular to L.



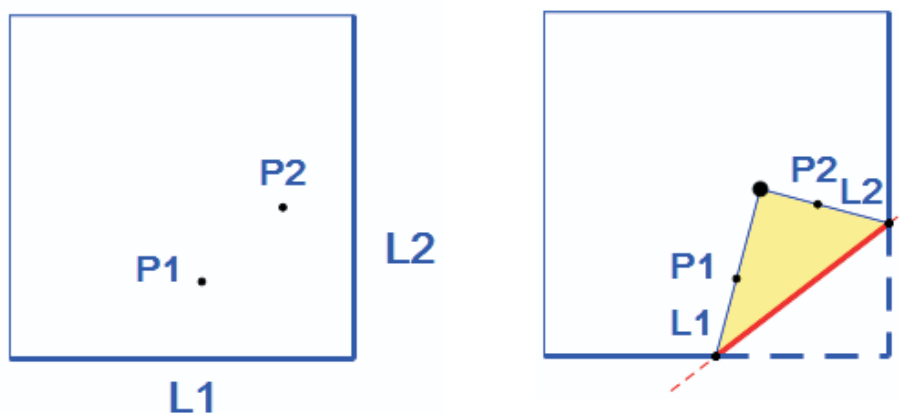
Axiom #05

Given two points P1 and P2 and a line L, a line can be folded placing P1 onto L and passing through P2.



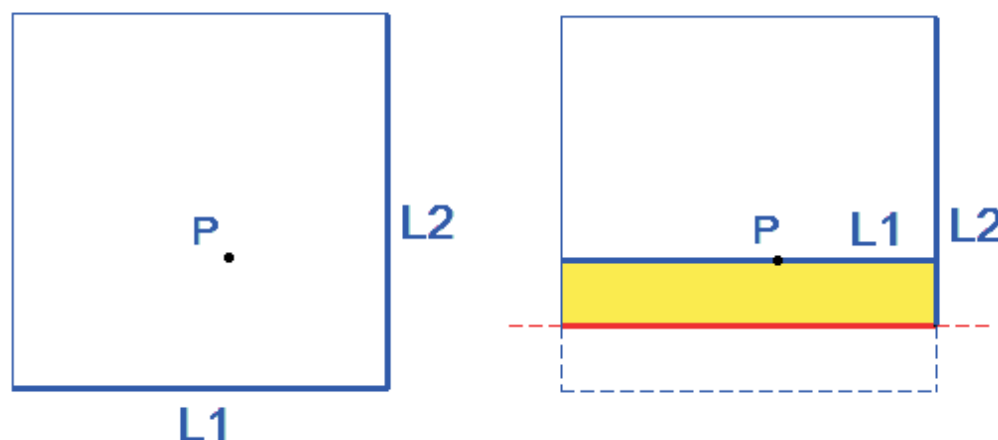
Axiom #06

Given two points P1 and P2 and two lines L1 and L2, a line can be folded placing P1 onto L1 and placing P2 onto L2.



Axiom #07

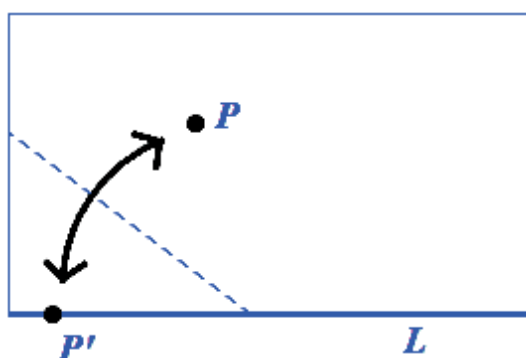
Given a point P and two lines $L1$ and $L2$, a line can be folded placing P onto $L1$ and perpendicular to $L2$.



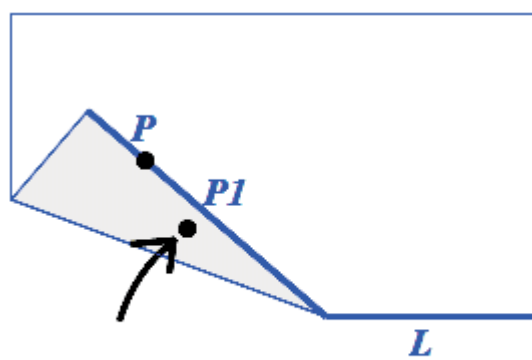
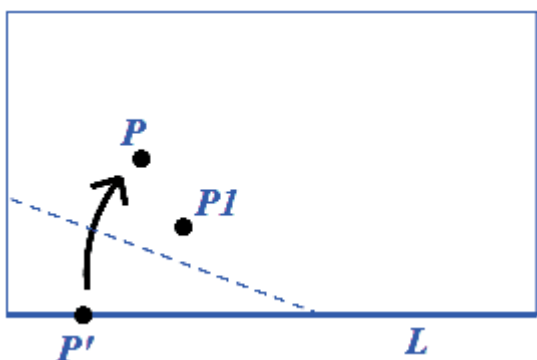
Axiom 6 is essentially about drawing the lines which simultaneously touch two parabolas; as such, it involves the solution of a *cubic equation*. Since a cubic equation can have up to three real roots, there can be up to three such common tangent lines. This means that Axiom 6 allows us to solve cubic equations! Such a possibility does not exist in regular Euclidean geometry, in which the instruments at our disposal—straightedge and compass—permit us to draw only straight lines and circles, and these never give rise to cubic equations. It follows that origami geometry is ‘stronger’ than Euclidean geometry! In particular, one can solve problems such as angle trisection and doubling of the cube, which are impossible using straight edge & compass.

As a small exercise, by actual folding of paper or by simulating with any dynamic geometry software, try experimenting and folding all the possibilities present in Axioms 5 and 6. Specifically one can try these:

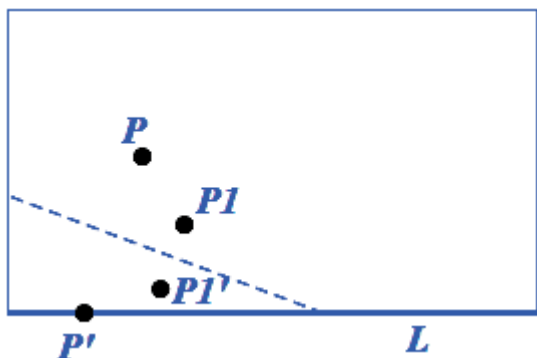
1. On a sheet of paper consider one side as L . Mark a point P anywhere on the paper and keep folding L to P and crease the paper along L . Do this repeatedly (choose different points on L each time you map P to L). What shape emerges from doing this?



2. Next, repeat this experiment by drawing a circle on the paper, which now represents L , and repeat the procedure with P first inside and later outside the circle. What do you find?
3. Again, on a sheet of paper consider one side as L . Mark a point P anywhere on the paper. Take a second point P_1 anywhere. Fold P to L and using the resulting crease mark the reflection of point P_1 . Choosing different points on line L repeat this procedure again and again. What curve emerges from the marked points?



With a dark marker (which bleeds through the paper) mark here

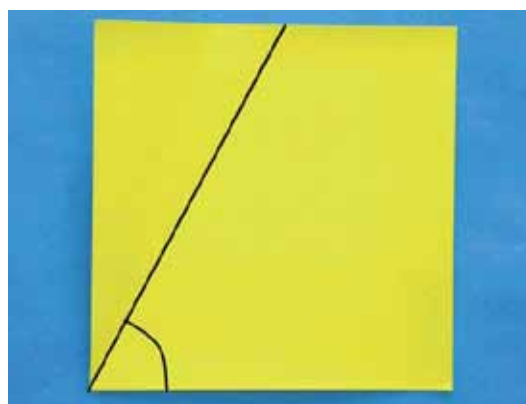


Any dynamic geometry software such as GeoGebra or Geometer's Sketchpad can simulate the above mentioned activities with the help of the 'Trace' function.

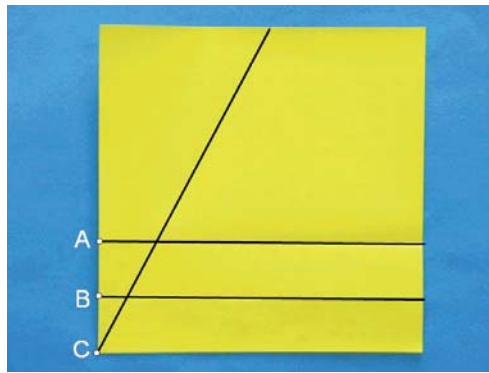
Trisecting an Angle

Now let's have a look at the steps involved in the trisection of an angle using the above mentioned axioms. This method is by H. Abe (from "Trisection of angle by H. Abe" (in Japanese) by K. Fusimi, in Science of Origami, a supplement to Saiensu (the Japanese version of Scientific American), Oct. 1980).

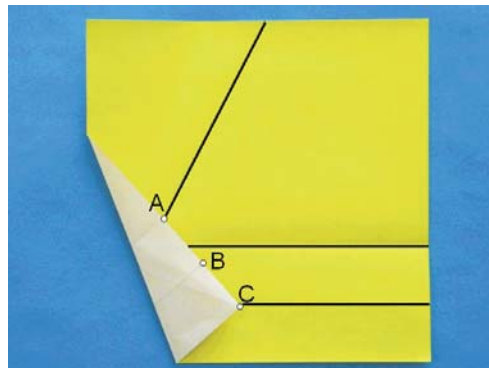
- Step 1: Fold a crease on a square paper creating an angle at the corner (to be trisected).



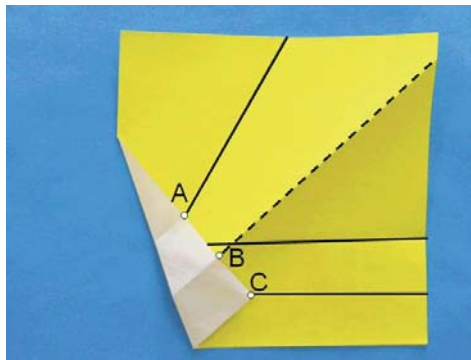
- Step 2: Make two horizontal creases at A and B by folding twice, so that $AB = BC$. Here C lies at the corner of the sheet.



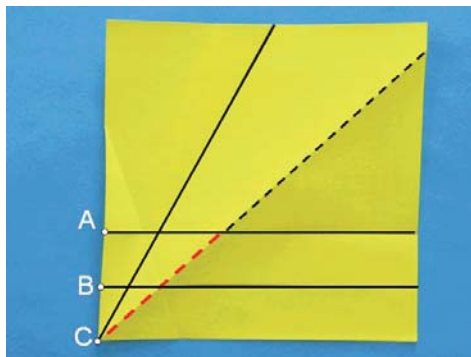
- Step 3: Fold point A onto the crease of the given angle, and point C onto the bottom crease.



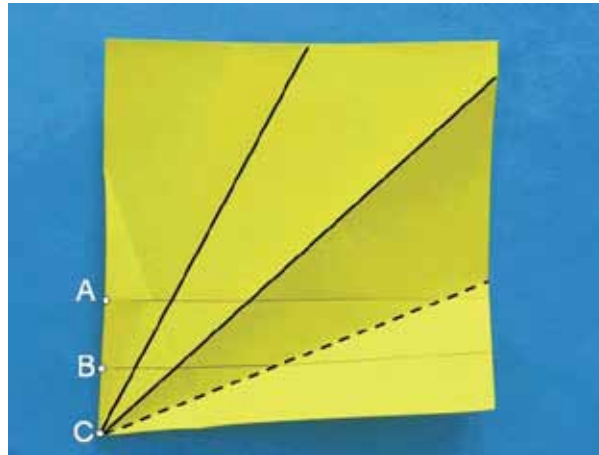
- Step 4: Extend the pre-creased line from B towards the top right corner (dashed line).



- Step 5: Unfold; extend the crease (dashed line) to the bottom left corner. This is one of the trisectors!



- Step 6: Folding the base to the trisector just found yields the other trisector of the angle.



We have thus drawn the two trisectors. (We leave the proof to you!) Such a construction is not possible in Euclidean geometry.

References

- [1] http://en.wikipedia.org/wiki/Huzita%E2%80%93Hatori_axioms
- [2] <http://kahuna.merrimack.edu/~thull/omfiles/geoconst.html>



A B.Ed. and MBA degree holder, Shiv Gaur worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 12 years, and is currently teaching the IGCSE and IB Math curriculum at Pathways World School, Aravali (Gurgaon). He is deeply interested in the use of technology (Dynamic Geometry Software, Computer Algebra) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was a guest speaker at IIT Bombay, TIME 2009. Shiv is an amateur magician and a Modular Origami enthusiast. He may be contacted at shivgaur@gmail.com.

The Fine Art of Euclidean Geometry

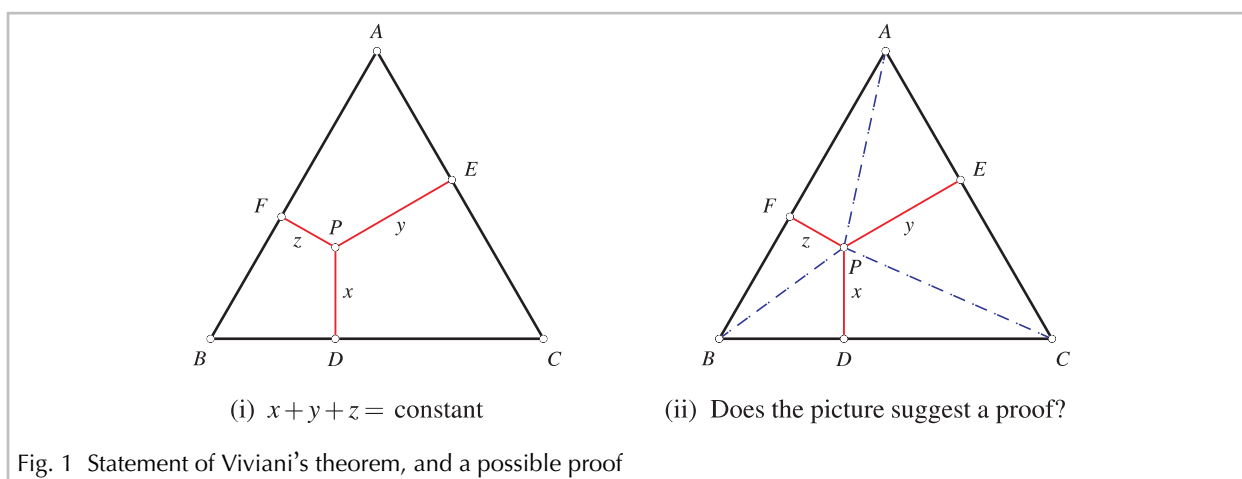
Viviani's Theorem . . . And A Cousin

"I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions — they are not just repetitions of each other." Sir Michael Atiyah, interview in European Mathematical Society Newsletter, September 2004.

SHAILESH A SHIRALI

Viviani's theorem is one of those beautiful results of elementary geometry that can be found experimentally even by young children. It is appropriate from a pedagogic point of view, because it allows us to illustrate what it means to 'do mathematics': it gives us the opportunity to find a proof, and the opportunity to experience the pleasure of generalization.

The theorem is named after Vincenzo Viviani (1622–1703), an Italian mathematician-scientist and a disciple of Galileo during the last few years of his (Galileo's) life. (Readers may recall that Galileo was under 'house arrest' for the last several years of his life. Viviani was with him during part of that period, and helped in the compilation of the important book, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*.)



Here is the statement of the theorem. Let $\triangle ABC$ be equilateral, and let P be any point in its interior (Figure 1 (i)). Then: *The sum of the distances from P to the sides of the triangle is a constant.* Thus, if perpendiculars PD , PE , PF are drawn from P to the sides BC , CA , AB , and their lengths are x , y , z , respectively, then the sum $PD + PE + PF = x + y + z$ is the same for all positions of P .

A way of proving the theorem is indicated in Figure 1 (ii). We draw segments PA , PB and PC (shown dashed). All we need to do is compute the area of $\triangle ABC$ in two different ways and study the resulting expressions; the constancy of $x + y + z$ follows. We discover as well what the 'constant value' is: *it equals the altitude of the triangle.* (Which makes sense: think of the different positions that P can occupy.) We invite you to complete the proof.

A Geometric Proof

A cardinal rule in the teaching-learning of mathematics is not to be satisfied with a single solution or proof — however pretty it is, or however satisfying! In that spirit we look for other proofs of Viviani's theorem. As a part motivator for this, we note that the proof suggested above, based on computation of area, is 'algebraic'. At a crucial step we factorize an expression and divide by one term, exploiting the fact that the three sides have equal length. For this reason we describe the proof as *essentially algebraic*, and we ask: *Is there an essentially geometric proof?* We now present such a proof.

In Figure 2 we have drawn a segment B_1C_1 through P , parallel to side BC (with B_1 on side AB ,

and C_1 on side AC). It is clear that for all positions of P on B_1C_1 , the distance of P from BC remains the same. Let us now show that the sum of the distances from P to sides CA and AB is the same for all positions of P on B_1C_1 . Accordingly, consider another point P_1 on B_1C_1 , and drop perpendiculars P_1E_1 and P_1F_1 from P_1 on AC and AB . We must show that $PE + PF = P_1E_1 + P_1F_1$. Drop perpendiculars $PQ \perp P_1F_1$ and $P_1R \perp PE$. In moving from P to P_1 , the distance to side AB has increased by P_1Q , while the distance to side AC has decreased by PR . So we must show that $P_1Q = PR$.

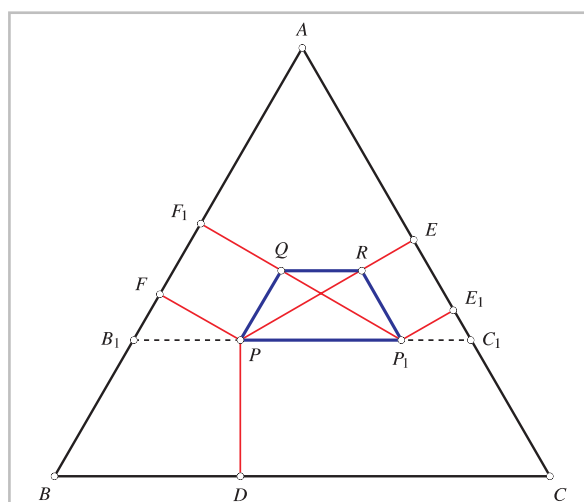


Fig. 2 Proof with a geometric flavour

But this follows readily. Since $PQ \perp P_1Q$ and $P_1R \perp PR$, points P , Q , R , P_1 are concyclic. Further, PR and P_1Q both subtend angles of 60° at the circumference of the circle. Hence they have equal length. (Note that $PQR P_1$ is actually an isosceles trapezium.)

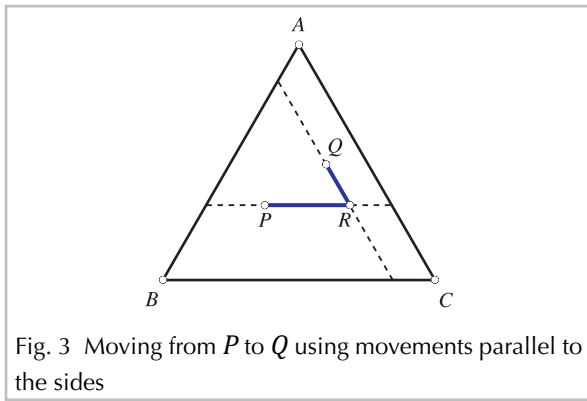


Fig. 3 Moving from P to Q using movements parallel to the sides

It follows from this that if P moves along a path parallel to any side, its sum of distances to the sides remains constant. Since one can move from any point within the triangle to any other point through movements parallel to the sides (in fact, just two such movements are needed; Figure 3 shows how to move from one interior point P to another one Q via the intermediate point R , using the path coloured blue), it follows that each point within the triangle has the same distance sum.

A Proof Using Vectors

It is possible to devise a proof using vectors based on the following principle. Let PB be a segment and let ℓ be a line passing through P (see Figure 4). Let the projection of PB on ℓ be PD . Then $PD = \overrightarrow{PB} \cdot \mathbf{u}$, where \mathbf{u} is a unit vector along the line ℓ (oriented suitably).

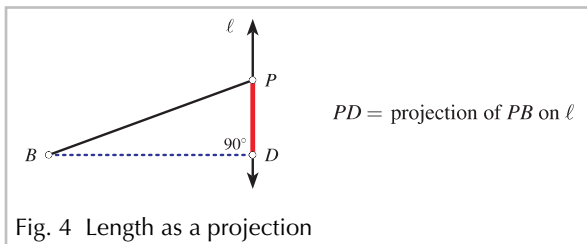


Fig. 4 Length as a projection

We apply this to the proof of Viviani's theorem. Let P be a point within the equilateral $\triangle ABC$. Let \mathbf{u} , \mathbf{v} and \mathbf{w} be unit vectors perpendicular respectively to the sides BC , CA and AB , and oriented from the centre O of the triangle towards the midpoints of the sides (see Figure 5). From symmetry considerations we see that $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is zero. Making use of the 'projection principle' noted above, we infer that

$$PD = \overrightarrow{PB} \cdot \mathbf{u}, \quad PE = \overrightarrow{PC} \cdot \mathbf{v}, \quad PF = \overrightarrow{PA} \cdot \mathbf{w}.$$

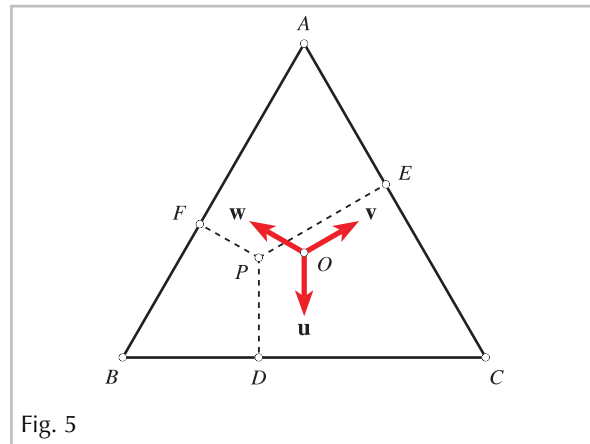


Fig. 5

Hence we must show that $\overrightarrow{PB} \cdot \mathbf{u} + \overrightarrow{PC} \cdot \mathbf{v} + \overrightarrow{PA} \cdot \mathbf{w}$ has a constant value, independent of P . Let Q be a second point within the triangle. The distance sum associated with Q is then $\overrightarrow{QB} \cdot \mathbf{u} + \overrightarrow{QC} \cdot \mathbf{v} + \overrightarrow{QA} \cdot \mathbf{w}$. Hence the difference between the two distance sums is

$$\begin{aligned} & (\overrightarrow{PB} \cdot \mathbf{u} + \overrightarrow{PC} \cdot \mathbf{v} + \overrightarrow{PA} \cdot \mathbf{w}) \\ & - (\overrightarrow{QB} \cdot \mathbf{u} + \overrightarrow{QC} \cdot \mathbf{v} + \overrightarrow{QA} \cdot \mathbf{w}) \\ & = (\overrightarrow{PB} - \overrightarrow{QB}) \cdot \mathbf{u} + (\overrightarrow{PC} - \overrightarrow{QC}) \cdot \mathbf{v} \\ & \quad + (\overrightarrow{PA} - \overrightarrow{QA}) \cdot \mathbf{w} \\ & = \overrightarrow{PQ} \cdot \mathbf{u} + \overrightarrow{PQ} \cdot \mathbf{v} + \overrightarrow{PQ} \cdot \mathbf{w} \\ & = \overrightarrow{PQ} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) = 0, \end{aligned}$$

since $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is zero. Hence the difference between the distance sums is 0, which means that the distance sums associated with P and Q are the same. Since this relation is true for any two points P and Q , it follows that the distance sum is a constant.

A Cousin of Viviani's Theorem

The clinching condition in the above proof is the fact that $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the zero vector. This simple observation allows us to find another theorem which looks much like Viviani's theorem but is different from it. We shall call it a 'cousin' of Viviani's theorem.

What the vector proof shows is that if P is a point within an equilateral $\triangle ABC$, and \mathbf{u} , \mathbf{v} , \mathbf{w} are any three fixed unit vectors such that $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is the zero vector, then the sum of lengths of the projections of PA , PB , PC on \mathbf{u} , \mathbf{v} , \mathbf{w} (respectively) will be a constant for all such points P . (The reason for this claim should be clear.)

But we get infinitely many theorems from this statement, because we can choose the three unit vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in infinitely many ways! All we need to ensure is that their (vector) sum is zero. Here is one possibility: Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be unit vectors along the directions $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ respectively. It is obvious that the sum of these three unit vectors is zero (see Figure 6). This gives rise to the following theorem.

Theorem. Let $\triangle ABC$ be equilateral, and let P be a point in its interior. Let perpendiculars PD, PE, PF be dropped to BC, CA, AB respectively. Then $BD + CE + AF$ has the same value for all positions of P .

It is easy to deduce what the constant value of $BD + CE + AF$ must be. Let P lie at the circumcentre of $\triangle ABC$; then $BD = CE = AF = a/2$ where a is the side of the triangle. Hence an equivalent claim is: $BD + CE + AF = 3a/2$ for all positions of P .

This yields yet another form of the claim! For, if $BD + CE + AF = 3a/2$, then we also have $CD + BF + AE = 3a/2$. So: $BD + CE + AF = CD + BF + AE$ for all positions of P . We give an alternate proof of this claim, based on the Pythagorean theorem.

Since $PB^2 = PD^2 + BD^2$ and similarly for PC^2 and PA^2 , we have: $PB^2 - PC^2 = BD^2 - CD^2 = (BD - CD) \cdot (BD + CD) = a(BD - CD)$.

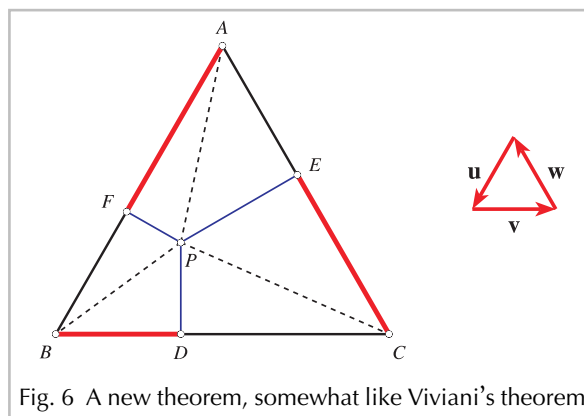


Fig. 6 A new theorem, somewhat like Viviani's theorem

We have similar relations for $PC^2 - PA^2$ and $PA^2 - PB^2$. Hence:

$$\begin{cases} PB^2 - PC^2 = a(BD - CD), \\ PC^2 - PA^2 = a(CE - EA), \\ PA^2 - PB^2 = a(AF - FB). \end{cases}$$

The sum of the three quantities on the left is 0, and so therefore is the sum of the quantities on the right. It follows that

$$(BD - CD) + (CE - EA) + (AF - FB) = 0,$$

and hence that

$$BD + CE + AF = CD + BF + AE.$$

It follows that

$$BD + CE + AF = 3a/2 = CD + BF + AE.$$

Questions to Ponder

In closing we leave the reader some questions to ponder.

- Q1: Viviani's theorem requires that P be a point within the equilateral $\triangle ABC$. Can a modification be found in the statement of the theorem which will make it applicable to points outside the triangle?
- Q2: Can there be an 'inequality form' of Viviani's theorem (presumably, for triangles which are not equilateral)?
- Q3: What generalization can be made of Viviani's theorem to polygons with a larger number of sides? Is there a class of polygons with the property that the sum of the distances from an interior point to the sides of the polygon is the same for every point? (It seems highly plausible that the property will be true for any regular polygon. Could it extend to polygons which are not regular?)



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Complete Family of Pythagorean Triples

From Random to Systematic Generation

In Part I of this article we presented a few methods for generating Primitive Pythagorean Triples (PPTs). You will recall that they were all 'piece meal' in character. Now we present two more approaches which offer complete solutions to the PPT problem. Both are based on straightforward reasoning and simple algebra. And no PPT is left out: we capture the complete family in each case.

$\mathcal{C} \otimes \mathcal{M}\alpha\mathcal{C}$

At the start we recall the definition: a Pythagorean triple is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. The triple is called 'primitive' if a, b, c have no common divisor exceeding 1; we call such a triple a 'Primitive Pythagorean Triple' (PPT for short). For example, $(5, 12, 13)$ is a PPT, while $(6, 8, 10)$ is a Pythagorean triple which is not a PPT.

Remark. We make the following number theoretic observation about PTs which are not PPTs. *If two numbers in a PT share a common factor exceeding 1, this factor divides the third number as well.* For example, $(9, 12, 15)$ is a PT, and its numbers 12 and 15 share the factor 3; this factor divides 9 as well. To see why this claim of divisibility will always be true, suppose that in the PT (a, b, c) , both b and c are divisible by some integer k . Then k^2 divides both b^2 and c^2 , hence k^2 divides a^2 , since $c^2 - b^2 = a^2$; hence k divides a as well. This logic works no matter which two of a, b, c are divisible by a common factor. Hence, to check that a

PT is a PPT, it is enough to pick any two of its entries and check that they are coprime; the nice thing is that it does not matter which two entries we pick!

Generating the Full Family of PPTs By Solving Equations

Let (a, b, c) represent a PPT. We write its defining relation $a^2 + b^2 = c^2$ in the form

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1. \quad (1)$$

Let $u = a/c$ and $v = b/c$. Then u and v are positive rational numbers, and they have the same denominator (because no ‘cancellation’ can take place in either of the two fractions). Also, they lie between 0 and 1, and they satisfy the equation $u^2 + v^2 = 1$.

To solve this equation we transpose the terms and write it in the form $u^2 = 1 - v^2$. In this form it immediately looks more familiar, because we are able to make use of the well known ‘difference of two squares’ factor formula. Write the equation $u^2 = 1 - v^2$ as

$$u \cdot u = (1 - v) \cdot (1 + v), \quad \therefore \frac{u}{1 - v} = \frac{1 + v}{u}. \quad (2)$$

Denote the common value of $u/(1 - v)$ and $(1 + v)/u$ by t (in terms of the original quantities a, b, c we have $t = a/(c - b)$; note that t is a positive rational number, for it is the ratio of two positive rational numbers):

$$\frac{u}{1 - v} = t, \quad \frac{1 + v}{u} = t. \quad (3)$$

By cross-multiplication and transposing terms, we obtain a pair of simultaneous equations in u and v :

$$\begin{cases} u + tv = t, \\ tu - v = 1. \end{cases} \quad (4)$$

Treating t as a fixed quantity, we solve for u and v in the usual way (we do not give the steps here; please check the answer we have given); we obtain:

$$u = \frac{2t}{t^2 + 1}, \quad v = \frac{t^2 - 1}{t^2 + 1}. \quad (5)$$

Recall that t is a positive rational number. Let $t = m/n$ where m and n are positive, coprime

integers. Since $u = a/c$ and $v = b/c$ we get, by substitution:

$$\frac{a}{c} = \frac{2 \cdot m/n}{(m^2/n^2) + 1} = \frac{2m n}{m^2 + n^2},$$

$$\frac{b}{c} = \frac{(m^2/n^2) - 1}{(m^2/n^2) + 1} = \frac{m^2 - n^2}{m^2 + n^2}.$$

Hence:

$$a : b : c = 2m n : m^2 - n^2 : m^2 + n^2. \quad (6)$$

It is easy to verify that if a, b, c satisfy these ratios then they satisfy the Pythagorean relation, because of the identity $(2m n)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2$. So: $(2m n, m^2 - n^2, m^2 + n^2)$ is a PT for every pair of coprime integers m, n with $m > n$.

Note that we only said ‘PT’, not ‘PPT’ — it could happen that the triple is a PT but not a PPT. Here are some examples of both kinds:

- $(m, n) = (8, 3)$ yields the triple $(48, 55, 73)$ which is a PPT.
- $(m, n) = (7, 3)$ yields the triple $(42, 40, 58)$ which is not a PPT as all its numbers are even. But note that we can recover a PPT from it by dividing all the numbers by their gcd which happens to be 2; we get the PPT $(21, 20, 29)$.
- $(m, n) = (5, 3)$ yields the triple $(30, 16, 34)$ which is not a PPT but yields the PPT $(15, 8, 17)$ on division by 2.

So $(m, n) = (8, 3)$ yields a PPT whereas $(m, n) = (7, 3)$ or $(5, 3)$ do not. If you experiment with various coprime pairs (m, n) , and we urge you to do so, you will find that *you get a PPT precisely when m and n have opposite parity* (i.e., when one of them is odd, and the other one even; this may be expressed compactly by writing: $m + n$ is odd). Please experiment on your own and confirm this finding.

How do we prove this? The condition is clearly needed; for, if m, n have the same parity (which means in our context that they are both odd, as they are supposed to be coprime and so cannot both be even), then $2m n, m^2 - n^2$ and $m^2 + n^2$ will all be even numbers.

We now prove that if m and n are coprime and have opposite parity, then $2m n, m^2 - n^2$ and $m^2 + n^2$ are coprime. For this, it is enough if we

show that $m^2 - n^2$ and $m^2 + n^2$ are coprime. (Recall the remark made at the start to see why.) Let k denote the gcd of $m^2 - n^2$ and $m^2 + n^2$. We present the proof that $k = 1$ as follows.

- Since m and n are coprime, so too are m^2 and n^2 .
- Since k divides both the numbers $m^2 - n^2$ and $m^2 + n^2$, it divides their sum (which is $2m^2$) as well as their difference (which is $2n^2$); so k divides both $2m^2$ and $2n^2$.
- Since m and n have opposite parity, m^2 and n^2 have opposite parity. Hence $m^2 + n^2$ and $m^2 - n^2$ are odd, and k , being their gcd, is odd.
- Since k divides $2m^2$ and $2n^2$, and k is odd, it must be that k divides both m^2 and n^2 .
- But m^2 and n^2 are coprime. Hence $k = 1$.

Thus $m^2 - n^2$ and $m^2 + n^2$ are coprime, as claimed, and the PT is a PPT. We conclude: *If m, n are positive coprime integers of opposite parity, and*

$$a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2, \quad (7)$$

then (a, b, c) is a PPT. Table 1 lists some PPTs along with their (m, n) pairs.

A stronger claim

We can make a stronger statement: *The above scheme generates every possible PPT (a, b, c) in which a is even and b, c are odd.* Let us show why.

Let (a, b, c) be a PPT in which a is even, and b, c are odd. Let the fraction $t = a/(c - b)$ be written in its simplest form as m/n (so m, n are coprime). Working as shown above, we find that $a : b : c = 2mn : m^2 - n^2 : m^2 + n^2$. We now show that m, n have opposite parity. Suppose that m, n are both odd (obviously, they cannot both be even). Then $2mn$ and $m^2 + n^2$ are both of the form $2 \times \text{an odd number}$, whereas $m^2 - n^2$ is a multiple of 4. Dividing through by 2 we find that it is b rather than a which is an even number. However we had supposed that a is even and not b . Hence it cannot be that m, n are both odd. So they must have opposite parity. But if m, n are coprime and have opposite parity, then $2mn, m^2 - n^2$ and $m^2 + n^2$ are coprime; we had shown this earlier. Now from the equalities $a : b : c = 2mn : m^2 - n^2 : m^2 + n^2$ and the fact that a, b, c are coprime as well as $2mn, m^2 - n^2, m^2 + n^2$, we can conclude that $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$, as required.

Example: Consider the PPT

$(a, b, c) = (48, 55, 73)$. Here $t = a/(c - b) = 48/18 = 8/3$; so we take $m = 8$ and $n = 3$. Now check that $(m, n) = (8, 3)$ generates the PPT $(48, 55, 73)$.

A number theoretic approach

To round off this discussion we shall derive the formula (7) in a completely different way, number theoretic in flavour. The key principle we use is the following proposition.

Proposition. *If r and s are coprime positive integers such that rs is a perfect square, then both r and s are perfect squares.*

For example, the product of the coprime numbers 4 and 9 is a perfect square, and each of these numbers is a perfect square. We invite you to prove the proposition.

Let (a, b, c) be a PPT in which a is even (and therefore both b and c are odd). From the relation $a^2 + b^2 = c^2$ we get $a^2 = c^2 - b^2 = (c + b)(c - b)$. We write this relation as follows:

$$\left(\frac{a}{2}\right)^2 = \frac{c+b}{2} \cdot \frac{c-b}{2}. \quad (8)$$

Since $a, c + b$ and $c - b$ are even numbers, the quantities $\frac{1}{2}a, \frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ are integers. We claim that $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ are coprime. To see why, suppose that d is a common divisor of $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$; then d must divide their sum ($= c$) as well their difference ($= b$). Hence d divides c as well as b . But we know that b and c are coprime. Hence $d = 1$, and $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ too are coprime.

From (8) we see that the product of the coprime numbers $\frac{1}{2}(c + b)$ and $\frac{1}{2}(c - b)$ is a perfect square. Hence each of them is a perfect square! Let $\frac{1}{2}(c + b) = m^2$ and $\frac{1}{2}(c - b) = n^2$. By addition and subtraction we get $c = m^2 + n^2$ and $b = m^2 - n^2$. From (8) we get $a = 2mn$. Hence there exist coprime integers m and n such that $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$.

We illustrate this step with an example. Take the PPT $(a, b, c) = (48, 55, 73)$ in which a is even, and b and c are odd, as required. For this PPT we have: $\frac{1}{2}(c + b) = \frac{1}{2}(73 + 55) = 64$ and $\frac{1}{2}(c - b) = \frac{1}{2}(73 - 55) = 9$. Observe that $\frac{1}{2}(c + b)$

Table 1. A list of some (m, n) pairs and the PPTs they yield.

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
2	1	3	4	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
6	5	60	11	61
7	2	28	45	53
7	4	56	33	65
7	6	84	13	85
8	1	16	63	65
8	3	48	55	73
8	5	80	39	89
8	7	112	15	113

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
9	2	36	77	85
9	4	72	65	97
9	8	144	17	145
10	1	20	99	101
10	3	60	91	109
10	7	140	51	149
10	9	180	19	181
11	2	44	117	125
11	4	88	105	137
11	6	132	85	157
11	8	176	57	185
11	10	220	21	221
12	1	24	143	145
12	5	120	119	169
12	7	168	95	193
12	11	264	23	265

and $\frac{1}{2}(c - b)$ are perfect squares. Hence $m = \sqrt{64} = 8$ and $n = \sqrt{9} = 3$. Please check that by using these values of m, n in (7) we get the same PPT with which we started, (48, 55, 73).

It remains to show that m, n have opposite parity. But we leave the task to you.

Remark. The approaches we have presented above are only two of many different ways of

tackling the Pythagorean equation.

Here are some other directions we could have taken: (i) the double angle formulas of trigonometry, (ii) complex numbers, (iii) coordinate geometry and quadratic equations. The reassuring thing is that all these different approaches give exactly the same general result. We shall come back to some of these approaches later.



The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach sector of the Rishi Valley Education Centre (AP). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments, schools and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com.

Seoul searching at ICME 12

Close encounters of the 'Math ED' kind

GEETHA VENKATARAMAN
AND SHREYA KHEMANI

Held every four years, the International Congress of Mathematics Education (ICME) is organised under the auspices of the International Commission on Mathematics Instruction (ICMI), started in 1908 as the International Commission on the Teaching of Mathematics with this aim: "To make an inquiry and publish a report on current trends in the secondary teaching of mathematics in various countries" ([1]). Over the years, ICMI activities have contributed to the development of a new discipline: research in mathematics education.

Every fourth year starting with 1972, ICME has brought together math educators, researchers, teachers, policy-makers, students and mathematicians to collaborate on issues and challenges of math education. The aim is to present trends in math education research and the practice of math teaching at all levels. It serves as a meeting space for the international math education community, provides an opportunity for discussion, debate and the presentation of new research and theory. ICME-12, held this year between 8 and 15 July, in Seoul, South Korea, was attended by over 4000 people from over 100 countries. From India, there was a delegation of 25 individuals drawn from across the country.

At Right Angles shares first hand accounts of ICME-12 as reported by Shreya Khemani and Geetha Venkataraman.

Geetha: This was a journey involving many firsts: the first ICME I was attending, and the first time I was visiting a place as far east as Seoul. This essay is not just about mathematics education or about visiting Seoul, it is also a collage of impressions of ICME-12: the talks, the Indian National Presentation, people and places in Seoul, and other vignettes. ICME-12 was held at Coex Mall, a huge mall several floors of which were devoted to ICME-12. Soon after settling into our accommodation on July 8, we headed to the Conference venue, which turned out to be about 45 minutes away by walk and subway. It was soon to become our daily routine for the next seven days.

Shreya: Sessions were held in parallel, making it very difficult to choose one lecture or event over another. There were 7 Plenary Events, 5 Survey Teams, 78 Regular Lectures, 37 Topic Study Groups, 4 National Presentations, 47 Workshop Sharing Groups, 17 Discussion Groups and reports of various ICMI Studies, and we often found ourselves dashing across rooms or across floors, trying to catch as many events as possible. Most people attached themselves to a single Topic Study Group (TSG) of their field of interest. I was a part of TSG 7—Teaching and learning of number systems and arithmetic—focusing on primary education, where I presented a paper (co-authored with Jayasree Subramanian) on our work, *Tackling the Division Algorithm*. The study group consisted of a core population of around 15 people, while others floated in and out. Papers presented differed greatly in scope and method. Unlike the larger events at the Congress, the study group provided a more intimate atmosphere where thoughts

and questions could be shared openly. Similar concerns, a growing familiarity over the course of the week and the small size of the group allowed for lively discussions and a meaningful exchange of ideas.

Geetha: The journey began in early 2011 when plans for the Indian National Presentation (INP) at ICME-12 started taking place. India was one of several countries that were given the opportunity to make a presentation on the state of math education in their country. The aim was to cover different aspects of math education in India, to critically review the situation at the different levels: primary, middle, secondary and tertiary, through the dual lens of curriculum and pedagogy. Several regional conferences and one national conference later, the INP began to take shape.

There were many topics and viewpoints that the INP planned to represent through different media. In addition to a book, audio-visual presentations were to be showcased in short clips. A video film was to be made and an exhibition organized, and all this had to be done with people collaborating from far and wide, across India.

With July approaching there were mad deadlines that everyone was trying to meet, and bouts of panic because of the Air India strike. But despite it all, the team arrived in Seoul. The book ([2]) was ready, as were individual team members' presentations ([3]), the video film ([4]) on mathematics education was canned in time, and charts, posters and display items ([5]) for the exhibition were assembled.

Shreya: An interesting feature of an ICME is the large number of Workshop Sharing Groups (WSG). These are informal small group activities designed to “exchange and discuss relevant mathematical experiences” ([6]). No formal presentations are made; rather, groups are invited to share their experiences of a project they have worked on and open the floor for discussion. I attended the WSG on the Urban Boundaries Project: Mathematics and the Struggle for Survival. It described a project led by a varied group of individuals (architects, biologists, physicists, teachers, math educators) working with two communities in the outskirts of Lisbon, Portugal — an ancient Portuguese



fishing community, and an immigrant population, consisting of people from different ethnic groups living on agricultural land where settlement is deemed illegal. Both communities face problems of social inclusion, and the project seeks to address their educational needs. Coming from within the discipline, one rarely thinks about the relationship between mathematics and politics. One thing that struck me was the difference in the way that mathematicians and math educators view mathematics. As Lyn Steen observes: “To a mathematician, mathematics is singular — a Platonic paradigm in which there are . . . unquestionable criteria for distinguishing right from wrong and true from false. But to math educators, mathematics is plural. Mathematics, among other things, offers a lens through which one can look at the world. In math education the direction is reversed — one looks at mathematics through the lens of the learners [and the teachers]” ([7]). I had never conceived of mathematics as plural. Nor had I ever imagined being at a conference on math education where it would be relevant to ask about the immigration laws of Portugal!

Geetha: An event that left an impression was the Regular Lecture by Alan Schoenfeld (Klein Medal awardee), *How we think: A theory of human decision-making, with a focus on teaching* ([8]). The Abstract seemed to suggest that the speaker was working on a theory that could explain why teachers took particular decisions in class. It seemed to apply to any kind of goal-oriented decision making activity.

Alan had started his career as a mathematician, and a reading of Polya’s *How To Solve It* ([9]) got him thinking about ‘heuristics’ and strategies that mathematicians use to solve problems. This led him to the obvious question as to whether it is

possible to teach students to be better problem solvers and to enjoy the profound beauty of mathematics. From here it was a natural step to turn his attention to teachers and teaching. Eventually this led to his research on goal-oriented decision making of which teaching is an example. The aim was to build a theory to help model goal-oriented decision making tasks like teaching, problem solving, cooking or brain surgery, which could explain and even predict decisions taken in the classroom by a teacher, in the kitchen by a cook, or on an operating table by a surgeon.

The talk was a sell-out. There was no standing space; even the aisles in the auditorium were packed! Interested readers should refer to [10] for details.

Shreya: Both the Plenaries and the Regular Lectures featured prominent scholars in the field, providing us an opportunity to hear at first hand the people whose work we admire. I particularly enjoyed Freudenthal’s *Work Continues* by Marja van den Heuvel-Panhuizen. Her talk was about some recent

Hans Freudenthal, a prominent topologist of his time, was President of ICMI from 1967 to 1970, and it was under his initiative that the first ICME was launched in 1969. He worked extensively for reform in math education and wrote widely on the subject. In 1971 the Freudenthal Institute was established, with Freudenthal as its first Director. At its heart was the ‘Wiskobas’ project (Mathematics in Primary School) which laid the foundations for the development of Realistic Mathematics Education, an approach rooted firmly in Freudenthal’s interpretation of mathematics as a ‘human activity’.

projects in elementary mathematics education carried out at the Freudenthal Institute (Netherlands). In describing each project, she looked back to the work of Freudenthal and his collaborators on Realistic Mathematics Education (RME) which has influenced and inspired math educators and researchers around the world ([11]). What I found particularly interesting was how she placed each project in a larger historical perspective of the institute’s history and Freudenthal’s philosophy.

Geetha: One of the plenaries was a panel discussion titled “Mathematics Education in East Asia” which focused on the situation in Japan, Korea and China. It was well presented, with practiced remarks thrown in at various points. For example,

there were video clips shown of classroom situations in USA and China; in the former, students' faces were blurred to protect privacy, whereas no such means were adopted for the latter. The 'remark' then was to the effect that education was a public enterprise in the East and not about individuals.

The presentation seemed to suggest that the culture and societal practices of the East had resulted in a 'war footing' with which mathematics education is approached. There were statistics presented on the millions of dollars spent by students in private coaching, to climb the relentless ladder of success in mathematics. (I may mention here that while returning each day in the metro, we routinely met children returning home from various coaching classes, as late as 9 pm.) There was an attempt to show the contrast between practices in the West (USA) and East: between the focus on the individual learner in the former, and on the entire class in the latter. The fact that many countries of the East had done well in international studies on math achievement (TIMSS [12], PISA [13]) was highlighted.

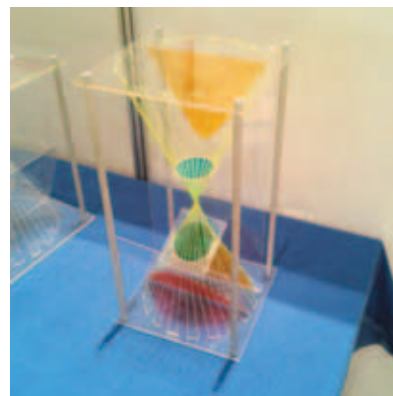
What struck me later was that they seemed so sure that their way of learning and doing mathematics was the right way. I wondered if this is actually the case, or whether there were other critical voices that did not find place in the presentation. Surely, if there is one thing that worries so many of us in India, particularly when policy decisions are made regarding the teaching-learning of mathematics, is whether we are on the right path or not.

Of course, we tilted the scales in the other direction during the INP! Each presentation looked closely and critically at some aspect of mathematics or mathematics education in the country. In our desire to acknowledge that there is so much yet to be done, we tend to be over-critical. But it is important to recognise the many positives that have been achieved.

The INP took place on 10 July. It covered a broad spectrum of topics: glimpses of history of mathematics and math education in India; curriculum and pedagogy for primary, middle and secondary school mathematics; assessment of math learning; math education, nurture and enrichment initia-

tives at the undergraduate level; teacher education and development; and research in math education. Three short films were shown. The preparation that had begun with the creation of NIME (National Initiative on Mathematics Education ([14]) had finally borne fruit. Of course, there were some lessons to be learnt as well. We should have publicised the INP better and tried to reach a larger audience. Our exhibition needed a dedicated team and seemed under-par compared to other National exhibitions. But one of the games exhibited in the stall — *pallankuzhi* — proved to be a great hit, especially with children. (We learnt that variants of this game are played all over the world.)

Shreya: The organisers also hosted a Math Carnival, filled with exhibits and fun activities. Visiting teachers and children spent hours playing mathematical games, pondering Escheresque tessellations, climbing dodecahedrons, and challenging their friends by rolling the Silla Square. It was lovely to see so many young people running around in the midst of a serious academic conference.



On 12 July, participants were taken on excursions through the city of Seoul. We walked through some historic parts of the city, visited the beautiful Gyeonghuigung palace and the Seoul History museum, attended a kimchi cooking class, and had a wonderful traditional Korean meal. Seoul is a large, striking megapolis, grid like and modern in its architecture, and surrounded by mountains. The Han River flows through the centre of the city, dividing it into two halves and separating the Northern, older part of the city from the Southern modern metropolis. July is a monsoon month; while the temperature and rain feel like Mumbai, the temperate vegetation and tall pine trees contradicted and confused what I associate with warm rain. We walked around in the evenings to take in the sights, sounds and smells. Street markets are vibrant and large, and stay open through the night. Filled with ingenious kitsch and delicious food, the markets come alive at night. The older part of



the city still has some of the ancient Hanoks that survived the Japanese colonial invasion, and more recently the real-estate mafia. Stumbling onto an ancient Buddhist temple on a lane just behind a 50-storey glass building leaves you wondering how the ancient and modern co-exist so seamlessly in this bizarre and wonderful city. People are kind and helpful; from missing a train to help you navigate the subway, to walking you to your destination when you ask for directions, Korean kindness—resonant of a Confucian past—is something I cannot forget. I was standing one evening outside a market place when it began to rain. Two young girls came and asked if I had an umbrella. When they found out I didn't, one of them promptly pulled hers out and handed it to me, got under her friend's umbrella, wished me a good evening and waved goodbye! It was difficult to make sense of Korea — with the myth of reunification, the growth of pop-culture, the abundance of 4G de-

vices and kindness—but I more than enjoyed my time in the wonderful city that is Seoul.

On the final day of the Congress, we were treated to a captivating performance by the dance troupe Noreum Machi that performs a percussion music called samulnori.

Geetha: There was also the presentation of the big ICMI (International Commission on Mathematical Instruction) awards: the Felix Klein and Hans Freudenthal Medals. "ICMI awards the Felix Klein Medal to a person who has shown consistent, and outstanding lifetime achievements in mathematics education research and development, and the Hans Freudenthal Medal to a person who has developed a theoretically well-conceived and highly coherent research programme which has had a significant impact on the community" ([15]). The awardees of the medals for 2009 and 2011 were felicitated at the ICME-12 inaugural ceremony. The 2009 and 2011 awardees of the Klein medal were Gilah Leder (Australia) and Alan Schoenfeld (USA), while the Freudenthal medals went to Yves Chevallard (France) and Luis Radford (Canada) for 2009 and 2011 respectively.

Shreya: As we said goodbyes and exchanged email IDs, and I walked off into the rain with my newly-gifted umbrella, I reflected on all I had heard and seen. I found myself faced with two questions, one pertaining to the relationship of mathematicians to math educators; the other, to the relationship of theory to practice in the world of math education. In his plenary address, "Whither the mathematics / didactics interconnection?" Bernard R. Hodgson spoke of the long-standing tradition of eminent

In the late 19th Century, the great mathematician Felix Klein published an important study of the icosahedron. Around the same time, he was busy initiating a reform of secondary mathematics education in Germany, focusing in particular on teacher education, to deal with the problem he described as the double gap — "the discontinuity between school mathematics and university mathematics and the double forgetting of the respective knowledge: first one had to forget school mathematics upon beginning one's university studies and later as a teacher one had to forget university mathematics and return to school mathematics." He believed that 'the whole sector of mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organised as an organic whole', and that without this view, 'even the purest scientific research would suffer, inasmuch as, by alienating itself from the various and lively cultural developments going on, it would be condemned to the dryness which afflicts a plant shut up in a cellar without sunlight'.

mathematicians being involved in education, but warned of a growing 'opaqueness' between educators and mathematicians. I wondered if Felix Klein could have foreseen this drift.

One of the striking things for me about ICME-12 was how senior researchers, Ph D students, mathematicians, school teachers, practitioners and people working in the field were all given a common platform to present their work, raise their concerns and talk about issues in mathematics

education. I remember how, at a conference I once attended, a famous mathematician requested for a woman mathematician in the room to be asked to leave as she was accompanied by her little child who made a sound. In contrast, ICME-12 allowed so many voices to be heard, so many people to be present, so many narratives to be told. For me it was a platform that served to include rather than exclude.

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Stimulating student learning

Open-Ended Questions

A class-room strategy for promoting divergent thinking

Can anything beat the feel-good factor of finding the perfect response to a question in the classroom? This article describes how giving students the opportunity to explore and study a concept through open-ended questions gives them a variety of paths to understanding and bursts the myth that the shortest road to mathematical success is the 'right answer'.

R. ATHMARAMAN

This writer has a mobile phone, which when unlocked, displays the question, "How are you today?" This is one of the simplest examples of an *open-ended question*. We can think of other such examples, such as: "What did you feel after reading this article?", "How interesting was your mathematics class yesterday?" These are in direct contrast with what are known as "*closed-ended*" questions, such as "What is the colour of the silk-sari you purchased yesterday?" "How much increment did you receive in your salary?"

For a maths teacher, open-ended questions offer several advantages, one of which is that they encourage students to speak and express themselves at length, and this is absent in traditional teaching. We illustrate this with a small example of a question for a 7-year old.

Teacher A: What is 7 plus 6?

Learner: 13.

The question is specific and the respondent has simply to state a 'fact'. The question is straightforward, and the answer is simple and predictable. Teacher A, in fact, indirectly controls the response of the learner. This variety of question is closed-ended.

Here is teacher B who uses the same fact in a different way.

Teacher B : Give me two numbers that add up to 13.

The question provides for an assortment of correct and sensible answers. A learner comes out with the answer "9+4" and this ignites an admirable discussion with other learners. They enthusiastically suggest more pairs of numbers with the total 13. The same question will receive a different kind of response when asked to a 12-year old. The student may then give the combination $5.6+7.4$, or even $2^3 + \sqrt{25}$. Often in a situation such as this, one can find learners competing with one another to exhibit their perception, comprehension and awareness. There is a lot of 'why' and 'how' from the novice. The query encourages students to dig into their understanding and impressions. The reason is that the question is open-ended. It presents a challenging situation to the student, who thereby has control over the response, quite unlike the situation created by a closed-ended question.

An enterprising teacher will bring into play adequate number of open ended questions to motivate, introduce or clarify concepts. Such questions encourage *Divergent and Reflective Thinking*. When they are employed in the mathematics class, the instructor can expect a range of responses and can thereby make progressive cognitive demands on students. They help learners put their heads together to make sense of mathematics.

The learners recognize the defining characteristics of the underlying concept, discuss various ideas, reason mathematically and ready themselves to conjecture, invent and solve problems.

Concepts of mathematics get connected to their areas of application. This is a consequence of the fact that when learners respond to open-ended questions, they look into the background of the underlying concept.

By their intrinsic nature, the open-ended questions are a versatile tool for teachers handling any level of mathematics. The modus operandi of teacher B, who asked for a pair of numbers with a sum of 13, is so strikingly simple that it can be adopted at any stage of instruction for any grade. Here are some examples:

- Find two numbers whose product is 1.5
- The difference of two fractions is $\frac{4}{5}$. What could the fractions be?
- Find an algebraic expression which has $(2x - 3)$ as a factor.
- Give the measures of a pair of angles that are supplementary.
- List two vectors whose scalar product is 10.
- The probability of an event is $\frac{2}{3}$; what could be the event?
- The sine of an angle is $\frac{1}{2}$. Find the angle.
- Provide an instance of a situation where L'Hopital's Rule will be needed.
- Give an example of a non-commutative group.

Observe that the responses to the above questions demand not only the comprehension of concepts but also a command of the processes and skills for applying and manipulating them. Additionally they train the learner's mind to logically justify his or her viewpoint and the solution. This remarkable advantage makes the open-ended questions superior to other varieties.

Teachers are quite familiar with the technique of asking traditional type of closed-ended questions such as "Find the LCM of 12 and 15", "What is the arithmetic mean of 5, 13, 26 and 103", etc. With a little more planning and innovation, one can 'create' quite a variety of open-ended questions just by slightly altering the traditional presentation. Let us list some examples:

Here are some more examples:

Closed-Ended	Open-Ended
Add the first three natural numbers which are not multiples of 3.	The sum of three natural numbers, none of which is a multiple of 3, is 20. What are the numbers?
What is the HCF of 24 and 36?	Can 6 be the HCF of 24 and some number n ? Can 7 be the HCF of 24 and some number n ?
How many lines of symmetry does a trapezium have?	Give an example for a quadrilateral that has no line of symmetry.
The perimeter of a rectangle is 28 cm and its length is 8 cm. Find its area.	The perimeter of a rectangle is 28 cm. What might be its area?
Round 23.45 to the nearest tenth.	What number when rounded to the nearest tenth will give 23.5? Substantiate your solution.
Which is bigger, $\frac{1}{\sqrt{3}}$ or $\left(\frac{1}{\sqrt{3}}\right)^2$?	Can the square of a number be smaller than the number itself? Justify your answer.
Draw a rectangle and the middle lines of its sides. Then colour 75% of it in red.	Draw a rectangle and colour 75% of the rectangle. Do you get a unique answer? Explain.
Draw a triangle whose sides are 5 cm, 6 cm and 7 cm in length.	Two of the sides of a triangle are 5 cm and 6 cm long. Draw the triangle. Argue how your construction is appropriate.
What is the shape of a manhole cover?	Why are manhole covers circular? List a few objects or tools around you whose shapes directly relate to their uses.
State the line of symmetry of the quadratic function $x^2 + 4$	Find a quadratic function whose line of symmetry is the y axis.
Show that the roots of the quadratic $x^2 - 9$ are equidistant from the origin	Find a quadratic function whose roots are equidistant from the origin.

- For a quadratic function, what is the connection between the following two properties: "Line of symmetry is the y axis" and "Roots are equidistant from the origin"? What kinds of quadratics have these properties?
- Let $p(x) = x^3 - x^2 + ax + b$. When $p(x)$ is divided by $x - 2$ the remainder is 12. Can a and b be found using this information? If not, what further information would suffice to find a and b ?
- Let A and B be two points in the complex plane corresponding to the points $\{1 - i, \frac{i}{1 - i}\}$. Find a complex number z such that if C is the point corresponding to z , then ΔABC is right angled.

From the examples, one might have noticed the following: *An open-ended problem may yield multiple answers*. Such a problem, requiring divergent thinking, may be solved by many different methods. There will be a great need for investigative and reflective thinking and decision making, to justify the process and the product.

Open-ended questions are not to be confused with 'opening' questions. Opening questions are simply starting points to probe into the background knowledge in the topic to be introduced, the past experiences and the recall of the learner. They are mostly closed-ended, although some starters could be open-ended. However, experience tells us that commencing a class with open-ended questions can spark mathematical communication.

A teacher should use a judicious combination of closed-ended and open-ended questions. Closed-ended questions alone may not provide a real assessment of instruction. *It is necessary for the teacher to wait for the responses of the students*

when an open-ended question is asked, and not to hurry the student. Without this allowance of time, the teacher may miss opportunities to spot learning difficulties as well as patterns of valid but divergent thinking in the learners.

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A math connect across the centuries

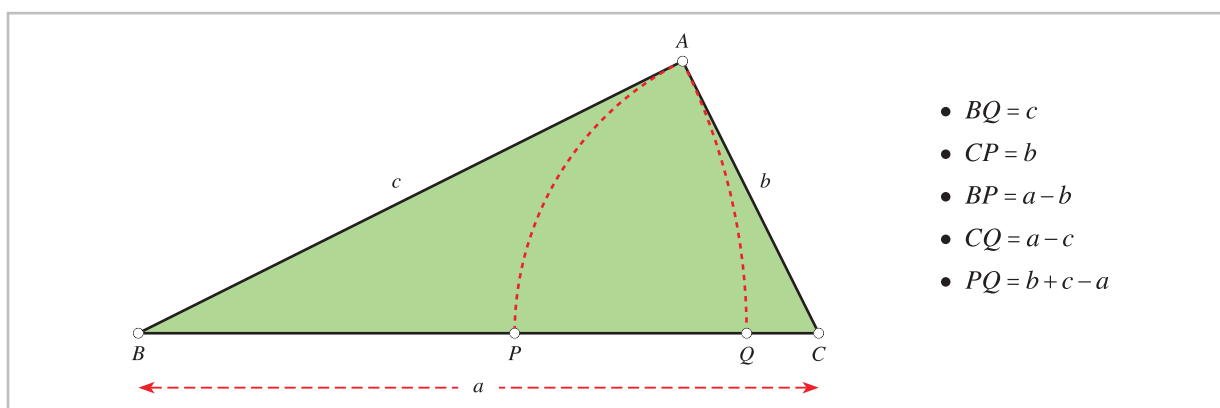
Ramanujan and Pythagoras!

An interesting extract from Ramanujan's notebooks which makes for a great classroom exercise in geometry, with a dash of algebra thrown in. An enterprising teacher could do this proof in stages — starting from showing students the figure and asking them to prove the theorem; if they can't, providing them with enough scaffolding to help them complete the proof.

$C \otimes M\alpha C$

What connection could there possibly be between Ramanujan and Pythagoras, when they lived twenty five centuries apart? Here is one such: an entry in one of Ramanujan's famous NOTEBOOKS, about a right angled triangle, which turns to be a consequence of Pythagoras's theorem. (See Remark 1, below, for some information about these notebooks.)

In the figure we see a right $\triangle ABC$, with $\angle A = 90^\circ$. An arc is drawn with C as centre and radius CA , cutting BC at P , and an arc is drawn with B as centre and radius BA , cutting BC at Q .



Here is Ramanujan's claim about this diagram:
 $PQ^2 = 2BP \times CQ$. See if you can prove it for yourself, before reading on.

Proof. We have $BP = a - b$, $CQ = a - c$, $PQ = b + c - a$. So Ramanujan's claim is:

$$(b + c - a)^2 = 2(a - b)(a - c).$$

We must verify this equality. Expanding the terms and subtracting the quantity on the right side from the quantity on the left, we get the following:

$$\begin{aligned} & (b + c - a)^2 - 2(a - b)(a - c) \\ &= (a^2 + b^2 + c^2 + 2bc - 2ab - 2ac) \\ &\quad - (2a^2 + 2bc - 2ab - 2ac) \\ &= b^2 + c^2 - a^2. \end{aligned}$$

Hence the claim that $PQ^2 = 2BP \times CQ$ is identical to the claim that $a^2 = b^2 + c^2$, which is nothing but the PT. So Ramanujan's claim follows from the PT.

Remark 1. The entry we have described here is one of the few entries in Ramanujan's NOTEBOOKS that deal with geometry. Most of the entries deal with topics in algebra and trigonometry (identities and systems of equations, continued fractions), number theory (properties of various functions, solutions of some equations) and analysis (summations of series). Some entries also deal with magic squares. Probably these were written when he was very much younger. You will find more information on the NOTEBOOKS on this page: http://en.wikipedia.org/wiki/Srinivasa_Ramanujan.

It is not easy to access these notebooks. You can view individual pages at: <http://www.imsc.res.in/~rao/ramanujan/NotebookFirst.htm>. And here is the page where you find the ' $PQ^2 = 2BP \times CQ$ ' entry: <http://www.imsc.res.in/~rao/ramanujan/NoteBooks/NoteBook2/chapterXXI/page11.htm>

Remark 2. On studying the entry closely, one gets the clear impression that Ramanujan *first* discovered the underlying algebraic identity, and then 'cooked up' a theorem based on the identity! For, just below the figure one finds the following statement:

$$\begin{aligned} & (a + b - \sqrt{a^2 + b^2})^2 \\ &= 2(\sqrt{a^2 + b^2} - a)(\sqrt{a^2 + b^2} - b). \end{aligned}$$

Now this is an algebraic identity — a 'stand-alone' relation which does not need to rest on any geometric result either for its meaning or for its proof. It can be verified independently. (Please do try it.) But what is *most* interesting is the statement that appears immediately below this one in the notebook:

$$\begin{aligned} & (\sqrt[3]{(a + b)^2} - \sqrt[3]{a^2 - ab + b^2})^3 \\ &= 3(\sqrt[3]{a^3 + b^3} - a)(\sqrt[3]{a^3 + b^3} - b). \end{aligned}$$

Examining the two relations we see an amazing resemblance between them. But it is clearly not a *logical* connection, for neither relation implies the other one. What seems plausible is that Ramanujan found the first relation the 'ordinary' way, which (perhaps) others could have done, and then in a leap of intuition he 'saw' the second relation too. (The relation is far from obvious! Try proving it.)

How To ...

Solve a Geometry Problem – I

A Three Step Guide

An informal, short guide on solving geometry problems. Ajit Athle describes some strategies which help in solving geometry problems and demonstrates how these strategies are used in solving two intriguing problems.

Ajit Athle

Problem solving in geometry poses special difficulties. Unlike problems in arithmetic or algebra, where one simply starts 'at one end' and proceeds to the 'other end' in a smooth, linear manner, the solution of a geometry problem often gives rise to a blank feeling. One does not know where to start! Often the solution requires the drawing of auxiliary lines and angles, and the figure itself gives no hints. Additionally, one is faced with the task of spotting relationships between pairs of triangles, or pairs of angles, disentangling them from a maze of lines and shapes, and to do so needs a keen eye indeed. This is a skill which can be difficult to cultivate. In short, problem solving in geometry is tough!

In this many-part article we share some thoughts on how to approach this challenging task.

1. Two problems

We first list two problems and invite the reader to spend time on them before reading on.

- (1) Outside a given triangle ABC we have a point D (Figure 1) such that $AB = BD = DA$, $\angle ACD = 10^\circ$, $\angle DAC = x^\circ$, and $\angle DBC = (x + 30)^\circ$. Problem: Find x .
- (2) Figure 2 shows a circle with a chord AC ; the midpoint of arc AC is D . Let B be any point on arc AC such that arc $AB >$ arc BC , and let DE be drawn perpendicular to AB , as shown. Prove that $AE = EB + BC$. (This is the famous *Broken Chord Theorem*, discovered and proved by Archimedes.)

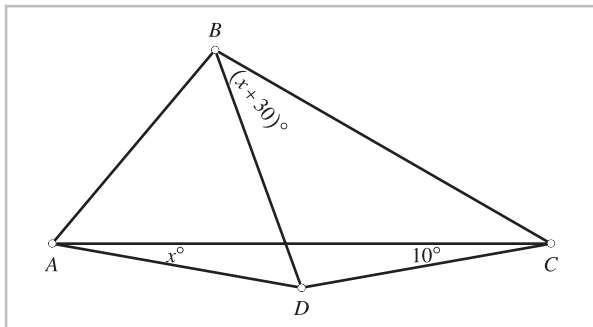


Figure 1

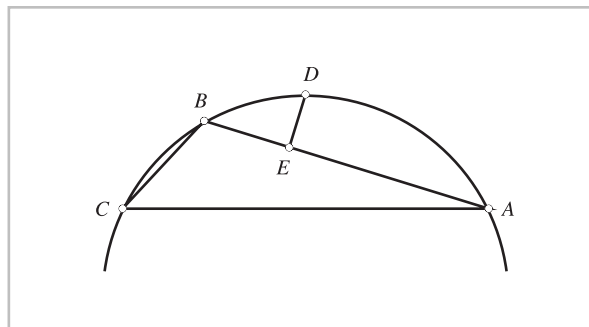


Figure 2

Solutions to the problems

Our approach in general should be to:

- (i) Understand the problem well, and record the given information accurately in a (reasonably large) diagram which is a copy of the given figure, making a note of what is required to be determined or proved;
- (ii) ascertain whether there is any hidden information in the problem statement, and if so, to note this too in the diagram;
- (iii) draw conclusions which lead us to finding the solution to the problem.

Solution to Problem 1. Let us apply the principles listed above to the problem at hand. Here we have, $AB = BD = DA$ (see Figure 3). What does this imply? Well that is clear: $\triangle ABD$ is equilateral, so $\angle BAD = 60^\circ$. Hence: $\angle BAC = (60 - x)^\circ$ and:

$$\angle ABC = (60 + 30 + x)^\circ = (90 + x)^\circ,$$

$$\angle BCA = 180^\circ - (60 - x)^\circ - (90 + x)^\circ = 30^\circ.$$

We see that $\angle ADB$ is twice $\angle ACB$. Recalling the result that the angle subtended by the chord of a circle at the centre is twice the angle subtended at the circumference, we deduce that if a circle is drawn with D as centre, passing through A and B , then the circle passes through C as well.

This means that D is the centre of a circle passing through points A, B, C . Hence $DA = DC$, both being

radii of the circumcircle, and so $\triangle DAC$ is isosceles. It follows that $x = 10$.

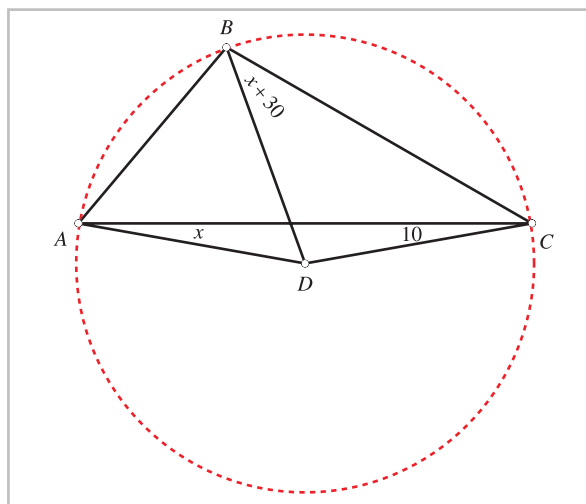
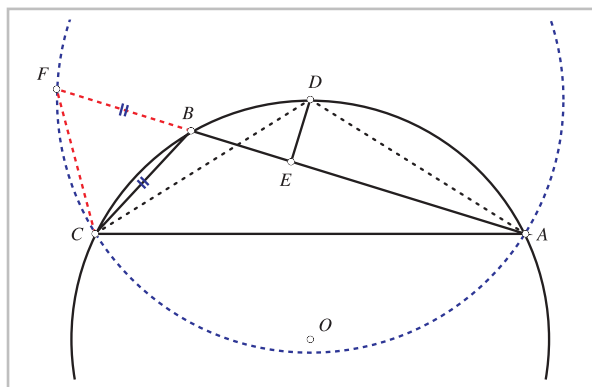


Figure 3

Thus the answer was arrived at in six simple steps using theorems or properties known to all school children. There is much beauty in this simplicity.

In a problem like this, one would try and figure out all angles in the diagram and then see if a special relationships can be observed which lead us to some useful inference. One would not know in advance if any particular angle will be more useful than any other. But that would vary from problem to problem, would it not?

(Broken Chord Theorem). In Figure 4, AC is a chord of a circle; D is the midpoint of arc AC ; B is a point on arc AC such that arc $AB >$ arc BC ; $DE \perp AB$. We must prove that $AE = EB + BC$.



As the segments EB and BC are not in the same line, the problem suggests a natural construction: *Extend AB to F such that $BF = BC$.* We now need to prove that $AE = EF$; i.e., that E is the midpoint of AF .

AC (since D is the midpoint of arc AC), and that AC subtends an angle of $2x$ at D while it subtends an angle of x at F . We infer that D is the circumcentre of the circle passing through points A, C, F . And since DE is perpendicular to AF , it follows that E is the midpoint of AF , which is what we had set out to prove after our construction.

Closing remarks

We shall present and solve more such problems in future editions of this column.



Vol. 1, No. 2, December 2012 | At Right Angles

Pitfalls in ...

The Method of Induction

The perils of teaching by example

Constructive teaching encourages students to recognise patterns and build appropriate theory with its roots in conjecture. What are the dangers in this method? What precautions should the teacher take to avoid conveying the impression that a claim is true simply because it has been observed to be true in all the examples considered?

ARUN NAIK

Inductive and deductive methods are emphasised in the teacher education (B.Ed.) curriculum, in 'methods of teaching mathematics'. I taught the method of induction passionately during my stint as a teacher educator and witnessed many lessons of mathematics taught by student teachers. It was employed by our student teachers whenever they dealt with generalisations and, on occasion, it led to the development of formulae.

How is 'inductive thinking' used in the teaching of mathematics? Typically a teacher provides specific instances one by one, asks children to observe and note the pattern, and to extend this pattern to unknown cases; this leads to a generalisation. For example, if the teacher were teaching the laws of indices, she would take up the following (or similar) examples, repeatedly ask questions, elicit responses from the students, collate responses on the black board systematically, and thus arrive at the law.

$2^3 \times 2^4$	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	2^7	2^{3+4}
$3^4 \times 3^5$	$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$	3^9	3^{4+5}
$0.5^3 \times 0.5^2$	$0.5 \times 0.5 \times 0.5 \times 0.5 \times 0.5$	0.5^5	0.5^{3+2}
$a^4 \times a^2$	$a \times a \times a \times a \times a \times a$	a^6	a^{4+2}
$y^5 \times y^2$		y^7	
$b^m \times b^n$			

This is the crux of the approach. At first sight, it looks like a great way of helping children explore generalisation in mathematics. Just by working through a few specific examples, the children are able to generalise without the teacher explaining the law! The students discover the rule!

While working as a teacher educator, my focus remained only on the variety of examples the teacher gave the students to examine and the space s/he created for students to look for patterns, hypothesize and arrive at a general form by systematic questioning. I enjoyed watching students come up with hypotheses and I was blissfully unaware of the boundaries that underlie the method.

A few years down the line, as I started thinking about the method, doubts emerged in my mind. Is this really a discovery by the student? Is there not a need for proof after we arrive at generalisation? Is this proof at all? This article brings together my thoughts on the topic.

What is 'inductive thinking'?

'Inductive thinking' is what we routinely do in daily life, without realising it or naming it as such. We meet four people from country X who love spicy food and conclude, "All people from country X love spicy food." Or we meet five people from country Z who enjoy dancing and conclude, "All people from country Z love to dance." In general, we examine a number of particular cases and based on the observations we arrive at a 'generalisation'. If we see something that works several times in a row, we're convinced that it works forever. As the confirming instances pile up, the conclusion gets strengthened.

While the ability to generalise is an inherent and vital ability of the human brain and underpins much of what we do - indeed, the entire scientific enterprise has such thinking at its roots - an uncritical acceptance of its results can be poten-

tially damaging; and this is as true in all spheres, whether mathematics or science or life as a whole. Here are two instances which show the relevance of this comment to mathematics.

Division of a circle into regions

Consider a circle with n points on it. How many regions will the circle be divided into if each pair of points is connected with a chord? (Assume that no two chords are parallel to each other, and no three chords meet in a point.)

By looking at the examples (Figure 1), most of us would be convinced that with 6 points there will be 32 regions. Our guess is that the number of regions when n points are connected is 2^{n-1} ; *but this is only a guess*. We looked at a few examples, found it true for the specific instances and now believe it works for all unexamined cases. There is no logic to explain why we believe it to be that way other than "it is true for the cases we have verified". *We may think we have established something, but we haven't.*

To test the conjecture we need to examine all possible cases. In the above example, if we go ahead and experiment with joining 6 points, we find that there aren't 32 regions! This proves that our conjecture is wrong.

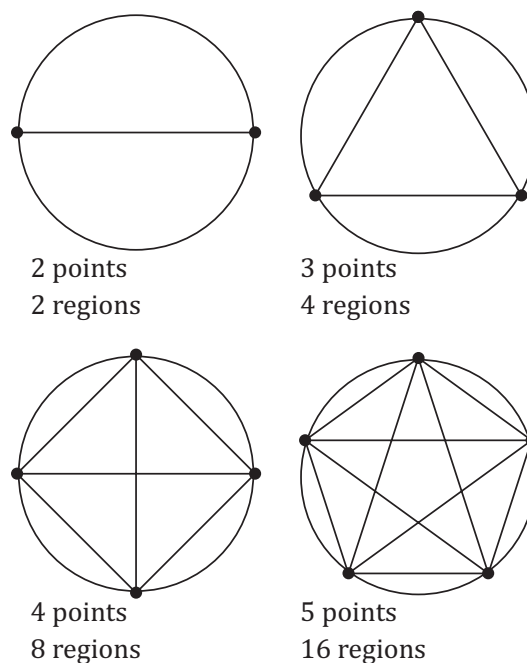


FIGURE 1. Sometimes a sequence can mislead us. . .

A prime generator

Here is another example. One might conjecture that $n^2 - n + 41$ is prime for all natural numbers n , and the evidence is persuasive:

- If $n = 1$ then $n^2 - n + 41 = 41$ is prime.
- If $n = 2$ then $n^2 - n + 41 = 43$ is prime.
- If $n = 3$ then $n^2 - n + 41 = 47$ is prime.

Even if one continues the experiment till $n = 40$ one would not find any evidence that the conjecture is false. But it is easy to see that the statement cannot be true in general, for when $n = 41$, the expression equals 41^2 which is not prime. So by finding a example which does not fit the pattern, we have proved the conjecture wrong.

Notion of a counterexample

Inductive arguments are suggestive: the evidence seems to support the conclusion, but there is no guarantee of the accuracy of that conclusion. If one wishes to prove a statement in mathematics, it is clearly not sufficient to do experiments and make observations; for a conjecture cannot be proved by example. On the other hand, one can *disprove* a conjecture by finding an example that ‘does not fit’ the pattern suggested by the conjecture. *Even one such example is enough to destroy a conjecture.* Such an example, which goes ‘counter’ to the conjecture, is called a *counterexample*. In the prime generator conjecture, $n = 41$ is a counterexample. (The reader may enjoy searching for more such counterexamples for this particular claim.)

But what if the statement is true?

What happens if a statement is actually true and we are searching for a proof? We know that evidence alone is insufficient to prove the statement, and we also know that a counterexample is enough to disprove it. But if the statement is true then one will not find a counterexample at all!

For example, take the claim that $1 + 3 + 5 + \dots + (2n - 1) = n^2$ for all natural numbers n . We will certainly not find a counterexample, because the claim happens to be true.

A more substantive example of this phenomenon is provided by the claim of Fermat (1601–1665): if n is an integer greater than 2, the equation $x^n + y^n = z^n$ has no solution in positive integers. Attempts to find a counterexample would not yield anything, because (as we now know) the statement is true.

Thus, experimentation with a few examples as a method of settling a conjecture does not always work. Then how do we proceed? Is there a way out? *Deductive proof* is called for, and a powerful kind of deductive proof is *mathematical induction*.

Proof by Mathematical Induction

Mathematics distinguishes itself from other disciplines in its structure and its internal consistency. It is built on axioms and postulates, which are self-evident truths and accepted without proof. All theorems, principles and generalisations in mathematics are derived and proved based on these. The large-scale structure of a proof by mathematical induction is simple:

- Prove the theorem for the base case, say $n = 1$.
- Prove that if the theorem is assumed to be true for any value of n , then it must also be true for the next higher value of n . This step is crucial; it must work for any arbitrary n .
- Connect steps 1 and 2: deduce that since the theorem is true for the known case (say $n = 1$), it will be true for the next case ($n = 2$), therefore for the next case ($n = 3$), and so on, for all positive integers n .

Despite its name, ‘mathematical induction’ is another form of deduction. It has similarities to induction in that it generalises to an infinite class from a small sample. And it is possible to execute because of the logical links between the successive unexamined cases.

Induction is one among many approaches that may be used to prove a statement. It would be rare to find a statement for which an alternate proof is not possible. For example, consider the identity $1 + 3 + 5 + \dots + (2n - 1) = n^2$. This has an easy inductive proof based on the identity $n^2 + (2n + 1) = (n + 1)^2$, and this in turn translates into an elegant ‘proof without words’.

But it can also be proved by inverting the order of the summands:

$$\left\{ \begin{array}{ccccccc} 1 & + & 3 & + & 5 & + \dots + & (2n-3) + (2n-1) \\ (2n+1) & + & (2n-3) & + & (2n-5) & + \dots + & 3 + 1 \end{array} \right.$$

and adding the two collections of summands ‘vertically’. Each vertical pair of numbers has sum $2n$, so the two rows together have sum $n \times 2n = 2n^2$.

Hence each row has sum n^2 . Note that this proof is *not* based on the principle of induction.

It is an instructive exercise to compile more such pairs of inductive and non-inductive proofs.

Conclusion

Inductive reasoning involves guessing general patterns from observed data. In science (or in life as a whole), such guesses remain merely conjectures, with varying degrees of probability of correctness. In mathematics, however, certain conjectures can be proved by the technique called ‘mathematical induction’. This technique is not ‘induction’ in the usual sense of the word; rather, it is a method for proving conjectures that have been arrived at by induction or some other route.

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A rare example

A Surprising Fact about Triangles with a 60 degree Angle

Is the converse of a statement always true?

Ever posed this question to a class and then scanned your memory for good examples to clinch your argument? Here is one you could use.

$C \otimes M \alpha C$

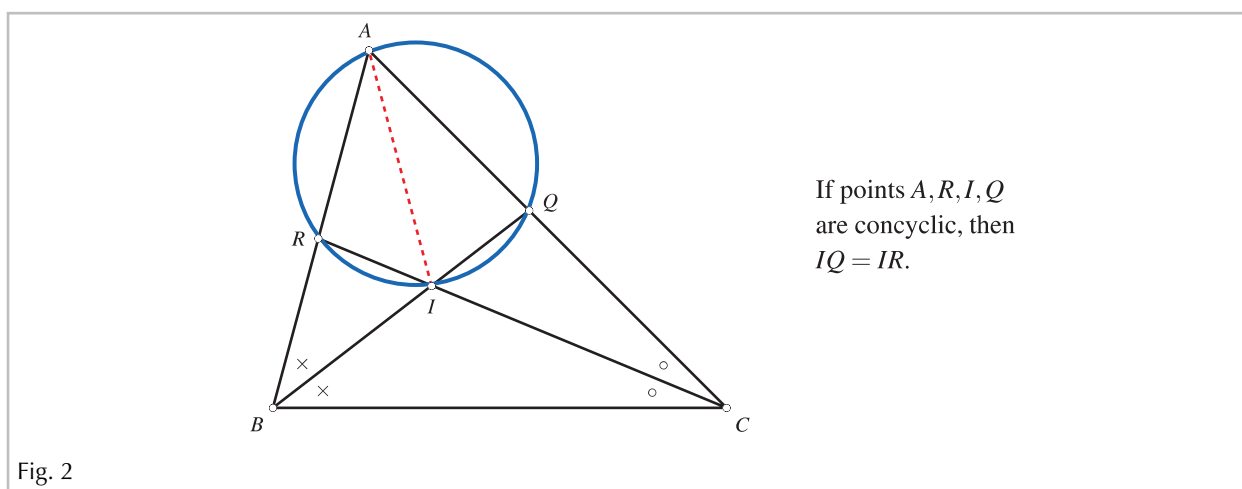
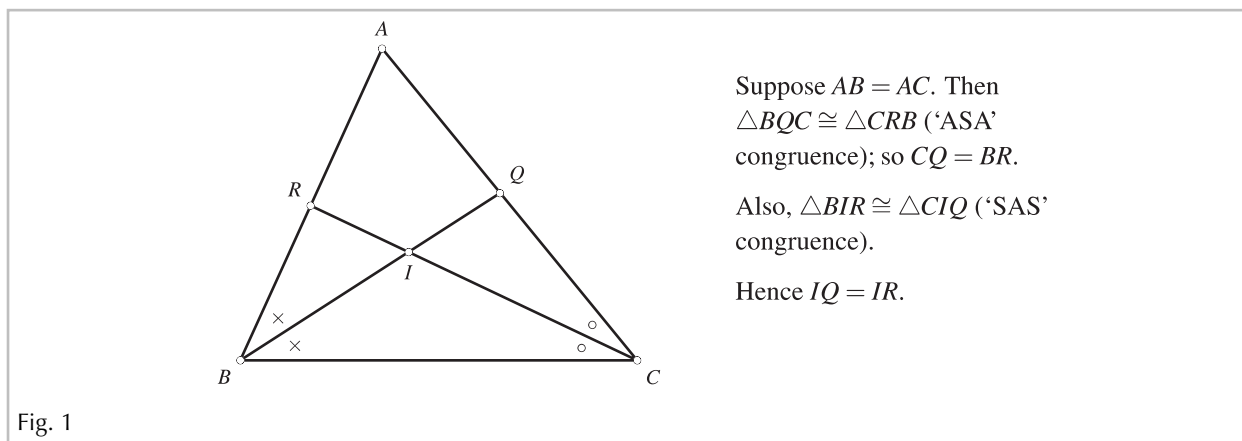
In the study of triangle geometry we get used to various pairs of theorems about isosceles triangles. Here are a few such pairs of statements, all with reference to a triangle ABC . Note their common element: the words ‘*and conversely*’.

(1) “If $AB = AC$ then $\angle B = \angle C$; and conversely.” (That is, if $\angle B = \angle C$ then $AB = AC$.)

(2) “If $AB = AC$ then the medians from B and C have equal length; and conversely.”

(3) “If $AB = AC$ then the altitudes from B and C have equal length; and conversely.”

But occasionally we come across statements that go counter to this pattern; that is, the ‘and conversely’ fails. Here is one such. Given a $\triangle ABC$, let the internal bisectors of $\angle B$ and $\angle C$ meet each other at I (the incentre of the triangle), and let them meet the



opposite sides (AC and AB) at Q and R respectively. For this configuration the following is true and easy to prove: *If $AB = AC$, then $IQ = IR$* (see Figure 1; the proof is given alongside).

Having seen so many statements about isosceles triangles of the form “If p then q ” in which the propositions p and q can exchange places without any loss, we may now guess the following ‘proposition’: *If $IQ = IR$, then $AB = AC$* . But this turns out to be false!

What might a triangle look like in which $IQ = IR$ but $AB \neq AC$? To produce such a triangle we use a standard theorem in circle geometry: *Chords of a circle which subtend equal angles at a point on the circumference of the circle have equal length*.

Suppose that quadrilateral $ARIQ$ is cyclic (Figure 1). Since $\angle IAR = A/2 = \angle IAQ$, chords IR and IQ subtend equal angles at A ; hence they have equal length. Therefore: *If points A, R, I, Q are concyclic, then $IQ = IR$* . (See Figure 2.)

Under what conditions will A, R, I, Q be concyclic? It is known that $\angle BIC = 90^\circ + A/2$. Hence $\angle QIR = 90^\circ + A/2$. Now a quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180° . Hence $ARIQ$ is cyclic if and only if $A + 90^\circ + A/2 = 180^\circ$, which yields $A = 60^\circ$. So: *If $\angle A = 60^\circ$ then $IQ = IR$* . And this holds regardless of the relation between sides AB and AC ! Hence from ‘ $IQ = IR$ ’ we cannot conclude that $AB = AC$. What we can conclude is this: *If $IQ = IR$, then either $AB = AC$, or $\angle A = 60^\circ$, or both*.

This may be rewritten as: *If $IQ = IR$, then either $AB = AC$, or $ARIQ$ is cyclic, or both*. We prove it in this form. We use the ‘sine rule’ which states that in any triangle, the ratio of the side to the sine of the

opposite angle is the same for the three sides ($a/\sin A = b/\sin B = c/\sin C$). We call this common ratio the “side by sine ratio” of the triangle.

Examine $\triangle ARI$ and $\triangle AQI$. The side by sine ratio for $\triangle ARI$ is $IR/\sin(A/2)$, and for $\triangle AQI$ it is $IQ/\sin(A/2)$. These two ratios are equal, because $IQ = IR$. So the two triangles have the same side by sine ratio. This means in particular that $AI/\sin \angle AQI = AI/\sin \angle ARI$, and hence that $\sin \angle AQI = \sin \angle ARI$.

Two angles between 0° and 180° have equal sines just when they are equal or supplementary. Hence, either $\angle AQI = \angle ARI$, or $\angle AQI + \angle ARI = 180^\circ$. The first possibility holds when $AB = AC$, and the second possibility when $ARIQ$ is cyclic. Thus our claim is proved.

Acknowledgement

The editors express their grateful thanks to **Mr Sankararaman B** of The Valley School, Bangalore, for bringing this problem to their attention.

number crossword-2

Clues Across

- 3: 2D plus 38
- 5: 5D plus 5
- 7: Internal angle of an equilateral triangle
- 8: Add 10 to 2D times 11
- 9: Number of squares on a chess board
- 10: 24D times 3 with the digits mixed up
- 12: $7A+9A+18A+20A+1D$
- 14: A Pythagorean triple: 313, 25, -----
- 16: 3D minus one digit of 7A
- 18: Last two digits of 15D plus 5
- 19: 10A minus 13D
- 20: Circumference of a circle with radius 1D/3
- 21: 5D jumbled up
- 23: Two thirds of 4D

Clues Down

- 1: Cube root of 9261
- 2: The largest perfect square below 100
- 3: An internal angle of a regular Pentagon
- 4: Largest three digit number
- 5: 14A minus 100
- 6: Largest number formed with 5, 6, 7
- 11: 1D in reverse times 9 plus 13
- 13: A full circle
- 14: 20A times 8 plus 10
- 15: 12A plus 6
- 16: 5A minus 1D
- 17: Three even digits in order
- 22: 24D divided by 3 plus 3
- 24: LCM of 3, 9 and 21

		1				2		
	3		4		5		6	
7			8				9	
	10	11			12	13		
	14		15		16		17	
18			19				20	
	21	22			23	24		



math club

find the next number!

Sequences are an excellent topic for a math club, in part because they offer something for everyone. And since there is no dearth of suitable sequences, we will never run out of material. Perfect for a Math Club!

Here we study a problem which involves drawing lines on a sheet of paper. All we do is draw straight lines using a pencil and count regions! The problem may be posed in an imaginative way: *If we make 10 straight cuts through a flat pizza using a knife, how many pieces can we get?* Here 'pizza' can be replaced by any flat object (a sheet of paper, a pancake, an omelette, a *dosa*). The only rule to be followed is that while the cuts are being made, the pieces are not rearranged in any way (for example, we cannot stack the pieces on top of each other before making the next cut); we simply run the knife through the object 10 times in succession. Ambiguities in meaning can be avoided by dealing with a plane sheet of paper and a pencil: *If we draw 10 straight lines on a sheet of paper, what is the largest number of regions we can get on the paper?* Naturally, the '10' does not have any significance; it can be replaced by any other number. So the mathematical essence of the problem is this: *If we draw n straight lines upon a sheet of paper, what is the largest number of regions we can create on the paper?* Here n represents an arbitrary positive integer. If we let $R(n)$ denote the number of regions created by n lines, we want a formula for $R(n)$. (See Figure 1.)

Why 'largest'? Well, we can 'lose' regions if we are not careful. We could draw two lines parallel to

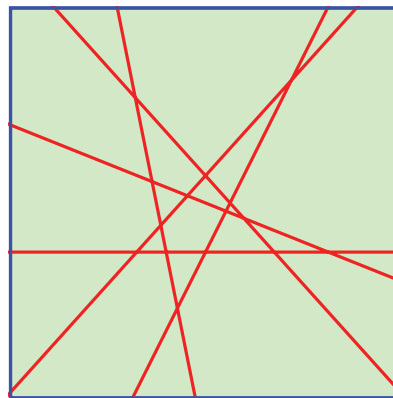


Figure 1 The picture for $n = 6$ is bad enough!

each other and so lose regions; or we could draw several lines through the same point and so lose regions (see Figure 2; in (a) we get only 3 regions when we could have got 4, and in (b) we get 6 regions when we could have got 7).

We shall assume that we do not let either of these situations occur. That is, we shall draw the lines in such a way that (i) every line meets every other line somewhere on the sheet, and (ii) no three lines pass through a single point. Given these two conditions, can we find a formula for the number of regions created by n lines? A related question is the following: *Can it happen that the above two conditions are satisfied, but there are different configurations possible which give different numbers of regions?* This is far from obvious! The first few values of R are easily found: $R(1) = 2$, $R(2) = 4$, $R(3) = 7$, $R(4) = 11$. For $n = 4$ we need to draw some trial configurations

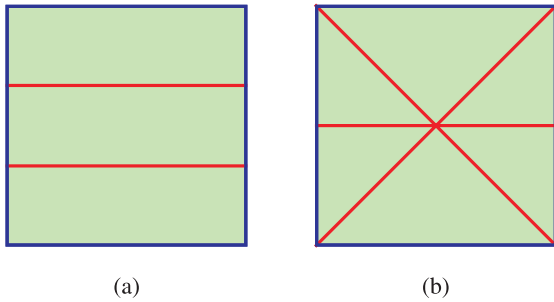


Figure 2 'Losing' regions, in two different ways

before we are convinced that we have the right answer (see Figures 3 and 4).

Examining these values, we quickly spot a pattern (in a Club setting, we will of course coax the children to do the spotting): we see that $4 - 2 = 2$, $7 - 4 = 3$, $11 - 7 = 4$. *The differences between successive entries appear to advance by 1 each time.* If this pattern persists then we expect that $R(5) - R(4) = 5$, and hence that $R(5) = 16$. Is $R(5)$ really equal to $11 + 5 = 16$? Is $R(6)$ really equal to $16 + 6 = 22$? And does the difference pattern continue? Experiment and find out!

We leave the matter open here, and end by posing a few questions for further exploration.

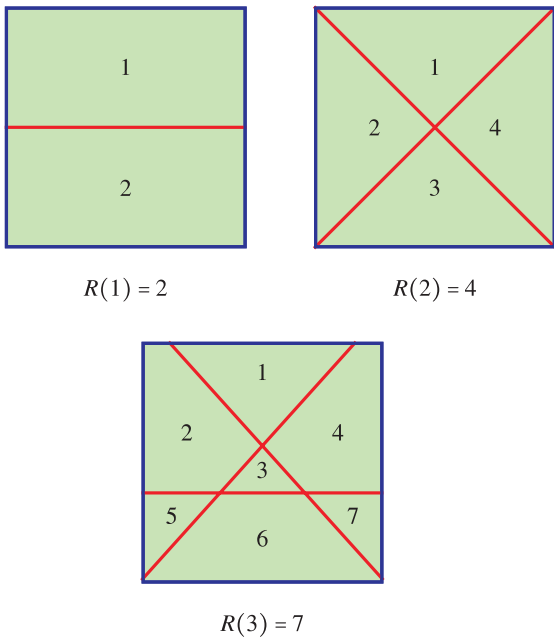


Figure 3 The cases $n = 1$, $n = 2$ and $n = 3$

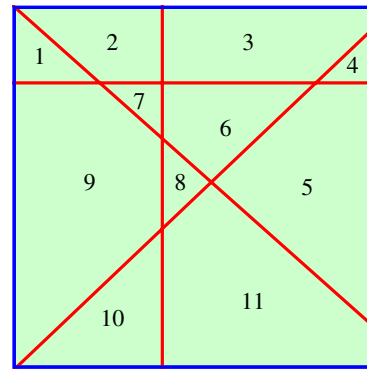


Figure 4 $R(4) = 11$ (Please check that we cannot get more than 11 regions)

Some questions to ponder . . .

- (1) Assuming that the pattern described above continues, what do we expect will be the value of $R(10)$? $R(20)$?
- (2) Assuming that the pattern described above continues, can we find a simple formula for $R(n)$?
- (3) We have repeatedly used the phrase “ . . . assuming that the pattern persists . . . ”. But why *should* the pattern persist? What geometric logic can we give for believing that it will persist? Unless we can give a convincing answer for this, our answer for question #2 will at best be partial and therefore of limited value.
- (4) Could there be some other way of proving the formula we find for $R(n)$?
- (5) The sequence $R(1), R(2), R(3), R(4), \dots$ is defined *geometrically*, but it may have some interesting *arithmetical* properties, which do not necessarily derive from its geometric origins. Try exploring some of these properties.
- (6) In what ways can this exploration be continued? Perhaps with some objects other than straight lines? Or by venturing into three dimensions? Or by studying aspects other than just the number of regions? Find some ways on your own, and continue the study. Happy exploring!

We shall continue our exploration of sequences and related topics in future editions of this column.

Probability taught visually

The Monty Hall Problem

A Spreadsheet Simulation

This problem which has had a history of generating controversy is often used to teach conditional probability. The article explains how repeated trials can be simulated using the random number generator in Excel, and how the result can be experimentally verified and then explained.

JONAKI GHOSH

One of the fundamental concepts in statistics is that of probability. It forms an integral part of most mathematics curricula at high school level. The topic of probability can be enlivened using many interesting problems. The related experiments are, however, time consuming and impractical to conduct in the classroom. Simulation can be an effective tool for modeling such experiments. It enables the student to use random number generators to generate and explore data meaningfully and, as a result, grasp important probability concepts. This section discusses a well known problem known as the Monty Hall problem or the three door problem which is based on conditional probability and lends itself to investigation. Its exploration using a spreadsheet such as MS Excel can lead to an engaging classroom activity. It highlights the fact that handheld calculators enabled by spreadsheet capabilities can enable students to visualize, explore and discover important concepts without necessarily getting into the rigor of mathematical derivations.

Exploring the Monty Hall Problem

Students often find it hard to understand the concept of conditional probability. This is aptly highlighted by the Monty Hall Problem (also referred to as the three door problem) which goes as follows

You are a contestant in a game show where the host (Monty) asks you to choose one among three doors. Behind one of the doors is a car (the prize) and behind the other two are goats. After you select a door, the host doesn't open it; instead, he opens one of the other two doors and reveals a goat (the host, who knows what's behind the doors, never reveals the car). The host then asks you whether you want to stick to your original choice or you would like to switch by choosing the other door. The question is which option is more likely to win you the prize?

This puzzle and its correct solution was published by Marilyn vos Savant in the Sunday *Parade* magazine in the 90s. She claimed that the contestant should switch because the odds of winning the car would be $2/3$, whereas if one sticks to the original door, then the odds of winning would be only $1/3$. This solution created a furor and thousands of readers including professional mathematicians retorted angrily by writing that switching cannot matter, since after one of the goats is revealed, there are two doors left and the probability of obtaining the car is $1/2$.

One wrote: "As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and, in future, be more careful."

Indeed this problem can lead to an interesting classroom discussion. The immediate reaction of most students (like most people) is also that switching cannot matter and that the probability of obtaining the car (after Monty opens one of the doors and reveals a goat) is $1/2$. This is however incorrect. The probability that you will select the door with the car (and therefore win) is $1/3$, and probability that you will lose is $2/3$. If your initial choice was correct then switching would be wrong. However if you chose a wrong door initially (the probability of which is $2/3$) then switching would lead you to win (since the host will reveal the goat behind the other wrong door). Figure 1 shows a tree diagram to explain the problem. Here it is assumed that the player has chosen door 1.

The tree diagram shows that the total probability of winning by switching is $2/3$ whereas the probability of winning by staying with one's original choice is $1/3$. The problem can be analyzed using the concept of conditional probability which will be described towards the end of the article.

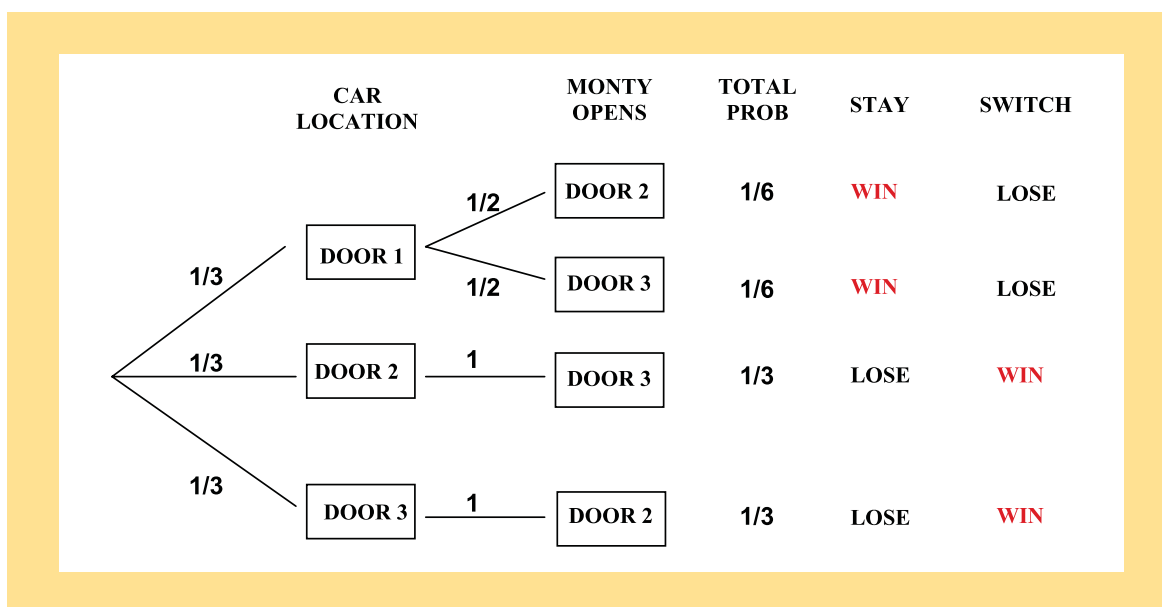


Figure 1 A tree diagram to explain the Monty Hall problem; the player's initial choice is door 1.

Spreadsheet verification in Excel also reveals that the probabilities of winning by switching are $2/3$ and winning by staying is $1/3$. In Figure 2 the experiment has been simulated 100 times with the assumption that the player's initial choice is door 1. The numbers in column A indicate the door behind which the car is placed. This is done by randomly generating the integers 1 through 3, both inclusive. We assume that the player's initial choice is door 1 in all the simulations and enter 1 throughout in column B. To get the entries in column C, we first check whether the entries for A and B are the same (i.e., the player has selected door with the car). If so, we generate a random number between 0 and 1. If the entries of column A and B are both equal to 1, that is, the car is behind door 1 (which is also chosen by the player), then Monty may open either door 2 or door 3 (with probability $1/2$ each) since he cannot reveal the car. This is indicated by the random number between 0 and 1. A random number less than 0.5 indicates that Monty opens door 2 in which case 2 is entered in column C. However, if the random number is 0.5 or greater, then 3 is entered in column C indicating Monty's opening door 3. These entries in column C are filled using a simple macro.

In column D we identify the cases where the player wins by switching. If the entries of

column A and B are different, then YES appears in the corresponding cell. Similarly in column E we identify the cases where the player wins by staying with his original choice. Thus if the entries of columns A and B are the same then YES appears in the corresponding cell. Finally counting the YES's in column D reveals the total number of times the player wins by switching in 100 trials. Interestingly, this number, on an average turns out to be around 67. Thus one may conclude that the probability of winning by staying with one's choice is around 33% and that by switching is 67%.

Simulation of the Monty Hall Problem on Excel

The problem can be simulated on Excel by using the **RAND()**, **INT()** and **If()** functions. The steps of simulation are as follows

Step 1: The first step is to simulate the position of the car behind one of the three doors. This is done by generating 100 integers among 1, 2 and 3 in column A. Enter **=INT(3*RAND()+1)** in cell A2 and drag till cell A101. When creating a column of numbers by dragging, we need to drag the small box in the lower right hand corner of the cell. 100 randomly generated integers will appear in column A. These represent with door behind which the car is placed (See Figure 2).

	A	B	C	D	E	F	G
1	CAR	PLAYER CHOOSES	MONTY OPENS	Switch and win	Stay and win		
2	2	1	3				
3	3	1	2				
4	3	1	2				
5	3	1	2				
6	2	1	3				
7	3	1	2				
8	1	1	2				
9	1	1	2				
10	2	1	3				
11	1	1	3				
12	1	1	3				
13	2	1	3				
14	3	1	2				
15	1	1	2				
16	3	1	2				
17	3	1	2				

Figure 2 Simulating the Monty Hall problem using Excel.

Step 2: In this simulation we are assuming that the player's choice is door 1. Enter 1 in cell B2 and drag till B100 or simply double click on the corner of cell B2.

Step 3: The next step is to simulate the door opened by Monty. This is done by entering `=IF(A2=1, IF(RAND()<0.5, "2", "3"), IF(A2=2, "3", "2"))` in cell C2 followed by a double click on the corner of the cell. This step simulates the door opened by Monty. If the car is behind door 1 and the random number generated by **RAND()** is less than 0.5 then Monty opens door 2, otherwise he opens door 3. On the other hand if the car is behind door 2, then Monty opens door 3 and if the car is behind door 3, Monty opens door 2.

Step 4: In column D we identify the cases in which the player wins by switching and in column E we identify the cases where the player wins by staying with his original choice. In column D cell should show 'YES' if the player's choice is not the same as the position of the car and 'NO' otherwise. This is done by entering `=IF(A2<>B2, "YES", "NO")` in cell D2 followed by a double click in the corner of the same cell. In column E a cell should indicate 'YES' if the player's choice is the same as the position of the case and 'NO' otherwise. This can be achieved by entering `=IF(A2=B2, "YES", "NO")` in E2 followed by a double click in the corner of the cell.

Step 5: Finally we need to count the number of 'YES's in columns D and E to find the number of cases in which the player wins by switching and staying with his choice respectively. This can be obtained by entering `=COUNTIF(D2:D101, "YES")/100` and `=COUNTIF(E2:E101, "YES")/100` respectively in separate cells.

The Monty Hall Problem—An Analysis using Conditional Probability

Let C_i denote the event that the car is behind door i . Clearly $P(C_i) = \frac{1}{3}$ where $i = 1, 2, 3$. Also let M_{ij} denote the event that Monty opens door j when the player chooses door i .

$$P(M_{ij}/C_k) = \begin{cases} 0, & \text{if } i = j \\ 0, & \text{if } j = k \\ 1/2, & \text{if } i \neq k, j \neq k \\ 1, & \text{if } i \neq k, j = k \end{cases}$$

Note that here i denotes the door chosen by the player, j denotes the door opened by Monty and k denotes the door behind which the car is placed.

Now, $P\left(\frac{M_{ij}}{C_k}\right) = 0$ if $i = j$, since Monty will never open the door chosen by the player. Also $P\left(\frac{M_{ij}}{C_k}\right) = 0$ if $j = k$, since Monty will never open the door which has the car behind it.

	A	B	C	D	E	F	G
	CAR	PLAYER CHOOSES	MONTY OPENS	Switch and win	Stay and win		
1	3	1	2	YES	NO		
2	2	1	3	YES	NO		
3	2	1	3	YES	NO		
4	2	1	2	YES	NO		
5	1	1	2	NO	YES		
6	3	1	2	YES	NO		
7	3	1	2	YES	NO		
8	3	1	2	YES	NO		
9	1	1	2	NO	YES		
10	1	1	3	NO	YES		
11	2	1	3	YES	NO		
12	2	1	3	YES	NO		
13	2	1	3	YES	NO		
14	3	1	2	YES	NO		
15	2	1	3	YES	NO		
16	1	1	2	NO	YES		

Figure 3 Simulating the probabilities of winning by switching and staying with one's choice.

If the player chooses the door with the car behind it (that is, $i = k$), then Monty can open any of the other two doors with probability $1/2$ each, thus for $i = k$.

For the case $i \neq j \neq k$, that is, the case when the player chooses a door which does not have the car behind it, then Monty has no option but to open the third door (the one which was not chosen by the player and the one which does not have the car behind it). Hence $P(\frac{M_{ij}}{C_k}) = 1$ for $i \neq j \neq k$.

Now let us compute the probability of the event M_{13} that is the event that Monty opens door 3 when the player chooses door 1. Using conditional probability we have

$$\begin{aligned} P(M_{13}) &= P(M_{13} \cap C_1) + P(M_{13} \cap C_2) \\ &\quad + P(M_{13} \cap C_3) \\ &= P(M_{13}/C_1)P(C_1) + P(M_{13}/C_2) \\ &\quad + P(M_{13}/C_3)P(C_3) \\ &= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3} = \frac{1}{2} \end{aligned}$$

We would now like to compute the probability of the event C_1/M_{13} , that is, the event that the car is actually behind door 1 given that Monty opens door 3 when the player chooses door 1. This would give us the probability of the player winning the car by staying with his choice. Similarly we would like to compute the probability of the event, that is, the event that car is actually behind door 2 given that Monty opens door 3 when the player chooses door 1. This would give us the probability of the player winning the car by switching. The computations are as follows

$$P(C_1/M_{13}) = \frac{P(M_{13} \cap C_1)}{P(M_{13})} = \frac{P(M_{13}/C_1)P(C_1)}{P(M_{13})}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$\begin{aligned} P(C_3/M_{13}) &= \frac{P(M_{13} \cap C_3)}{P(M_{13})} = \frac{P(M_{13}/C_3)P(C_3)}{P(M_{13})} \\ &= \frac{0 \times \frac{1}{3}}{\frac{1}{2}} = 0 \end{aligned}$$

$$\begin{aligned} P(C_2/M_{13}) &= 1 - \{P(C_3/M_{13}) + P(C_1/M_{13})\} \\ &= \frac{2}{3} \end{aligned}$$

Thus $P(\frac{C_1}{M_{13}}) = \frac{1}{3}$ indicates that the probability of the player winning the car by staying with his choice is $1/3$ and $P(\frac{C_2}{M_{13}} = \frac{2}{3})$ indicates that the probability of the player winning the car by switching is $2/3$.

It would be a good exercise to analyse the problem if the number of doors in the game is more than 3. Suppose there were 100 doors, you choose one of them, and Monty then opens 98 of the other doors, would you switch?

Conclusion

The topic of probability has a plethora of interesting problems which can be made accessible to high school students through spreadsheets. The experiments related to these problems may be impractical to conduct manually but simulation can be an effective modeling tool for imitating such experiments. Microsoft Excel proves to be a very handy tool for conducting the explorations and investigations in the classroom. The Monty Hall Problem discussed in this article can be conducted with students of grades 9 and 10 without getting into the mathematical derivations. However in grades 11 and 12 the spreadsheet verification of the problem can be followed by an analysis of the underlying concepts which are rooted in probability theory.

References

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SOLUTION FOR number crossword-1 by D.D. Karopady

	1	2		3	4	5						
	1	9	0		9	7	8					
6	1	7	9	7	1	8	6	5	4			
10	7	2		11	4	6	8	13	2	2		
		14	2	6	5	6	15	8				
16	1	0	8	0		17	2	0	4	0		
		18	4	3	19	4	3	4				
20	8	21	8		22	2	5	6		23	6	7
25	1	2	26	3		5		27	1	6	0	
	28	1	2	0		29	2	1	6			



RAMANUJAN NUMBERS

by $C \otimes M \alpha C$

In the last issue of **At Right Angles** we had noted an observation that Ramanujan had made about the number 1729: *It is the least positive integer that can be written as the sum of two positive cubes in more than one way* (namely, as $10^3 + 9^3$ and as $12^3 + 1^3$), and we asked you to find the next integer, after 1729, with the same property.

A 'brute force' computer assisted search reveals the following such numbers:

$$\begin{aligned} 1729 &= 9^3 + 10^3 = 1^3 + 12^3, \\ 4104 &= 9^3 + 15^3 = 2^3 + 16^3, \\ 20683 &= 19^3 + 24^3 = 10^3 + 27^3, \\ 39312 &= 15^3 + 33^3 = 2^3 + 34^3, \\ 40033 &= 16^3 + 33^3 = 9^3 + 34^3, \\ 64232 &= 26^3 + 36^3 = 17^3 + 39^3. \end{aligned}$$

We can list more such equalities by scaling: $13832 = 18^3 + 20^3 = 2^3 + 24^3$ (from the entry for 1729). But we regard these as uninteresting and do not list them. The numbers with the desired property are seen to be: 1729, 4104, 20683, 39312, 40033, 64232, The next Ramanujan number after 1729 is thus 4104.

Ramanujan's Solution

Instead of a brute force method, can we not look for approaches that are more worthy of being called 'mathematical'?

When we have an equation and we must find integers satisfying it, the equation is referred to as a Diophantine equation (after the Greek mathematician Diophantus). Two well known examples:

(i) the Pythagorean equation $a^2 = b^2 + c^2$, which gives rise to Pythagorean triples; (ii) the Fermat equation $a^n = b^n + c^n$ (with $n > 2$). For taxicab numbers the defining equation is $a^3 + b^3 = c^3 + d^3$.

It turns out that it is possible to solve the equation $a^3 + b^3 = c^3 + d^3$ in a systematic way. The great eighteenth century mathematician Euler did so. So did Srinivasa Ramanujan, during the period when he was still in India, composing his now-famous notebooks. (This was before he went to England, in 1914, at the invitation of G H Hardy.) Here are the formulas he found: if u and v are arbitrary integers, positive or negative, and

$$a = 3u^2 + 5uv - 5v^2, \quad b = 4u^2 - 4uv + 6v^2,$$

$$c = 5u^2 - 5uv - 3v^2, \quad d = 6u^2 - 4uv + 4v^2,$$

then $a^3 + b^3 + c^3 = d^3$, identically. This is nearly the same as our equation, except that c has come on the 'wrong' side. Clearly, this can be fixed by a simple change of sign.

For example, if we put $u = 1$ and $v = -2$ we get $a = -27, b = 36, c = 3, d = 30$, hence:

$$(-27)^3 + 36^3 + 3^3 = 30^3.$$

Since each term in this equality is divisible by 3 we may divide it out without losing anything; we get $(-9)^3 + 12^3 + 1^3 = 10^3$, and therefore by exchanging terms:

$$12^3 + 1^3 = 9^3 + 10^3.$$

Now we see why this identity came so readily to Ramanujan when Hardy mentioned the number 1729; he had found it out many years earlier!

Other (u, v) combinations yield more such nice and non-obvious relations:

- from $(u = 1, v = 2)$, we get $7^3 + 14^3 + 17^3 = 20^3$;
- from $(u = 1, v = -3)$, we get $7^3 + 54^3 + 57^3 = 70^3$;
- from $(u = 2, v = 3)$, we get $3^3 + 36^3 + 37^3 = 46^3$;
- from $(u = 2, v = -3)$, we get $23^3 + 94^3 = 63^3 + 84^3$, and this yields yet one more Ramanujan number: 842751.

It is difficult to say how Ramanujan found these formulas. But that complaint holds for just about everything that Ramanujan found!

Readers who wish to see Euler's derivation of the general integral solution of the equation $a^3 + b^3 + c^3 = d^3$ should consult the book by G H Hardy and E M Wright, *Introduction to the Theory of Numbers*.

Fun Problems

1. A Magic Triangle

We are familiar with the notion of a magic square. Here we consider a related notion: that of a *magic triangle*. In Figure 1 we see a triangle with six circles on its three sides.

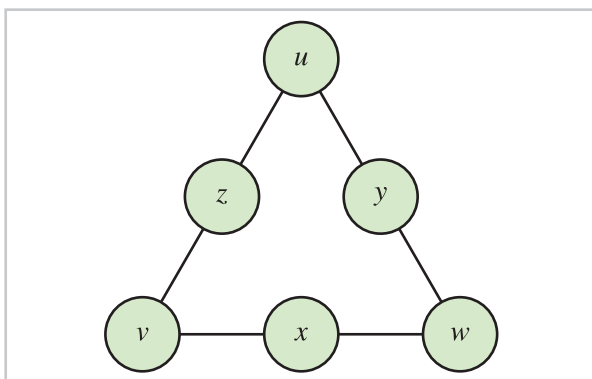


Fig. 1

Using the digit set $\{1, 2, 3, 4, 5, 6\}$ we must put one digit into each circle, using up all the six digits, in such a way that the sum of the numbers on each side is the same. That is, (u, v, w, x, y, z) should be a permutation of $(1, 2, 3, 4, 5, 6)$, and the quantities $u + z + v$, $v + x + w$ and $w + y + u$ must all be the same. Hence the name, ‘magic triangle’. We ask: *In how many different ways can this be done?* We consider two arrangements to be the ‘same’ if one can be obtained from the other by flipping it over along some axis, or by

rotating it about the centre of the triangle through some angle. (Strictly speaking, we should say ‘congruent’ rather than ‘same’.) This may be achieved by agreeing to orient the triangle so that $u < v < w$ (i.e., looking only at the vertices, the smallest vertex number is at the top, and the largest vertex number is at bottom right).

We answer the question of ‘how many ways’ and in the process we uncover some pretty relationships which we shall describe as “theorems about magic triangles”. The first one is: *In a magic triangle as defined here, the sum of the numbers at the vertices is a multiple of 3. So is the sum of the numbers at the ‘middles’ of the three sides.*

Let $C = u + v + w$ be the sum of the three corner numbers, and let $M = x + y + z$ be the sum of the three middle numbers. (The claim made is then: *C and M are multiples of 3.*)

Now, clearly:

$$C + M = 21, \quad (1)$$

since between them, $\{u, v, w\}$ and $\{x, y, z\}$ exhaust all of the numbers $\{1, 2, 3, 4, 5, 6\}$, and the sum of these six numbers is 21. Let s denote the common sum of the three numbers on each side.

Then we have:

$$\begin{cases} u + z + v = s, \\ v + x + w = s, \\ w + y + u = s, \end{cases} \quad (2)$$

hence by addition, $2(u+v+w) + (x+y+z) = 3s$, i.e.,

$$2C + M = 3s. \quad (3)$$

From (3) and (1) we get, by subtraction:

$$\begin{cases} C = 3s - 21 = 3(s - 7), \\ M = 21 - C = 3(14 - s), \end{cases} \quad (4)$$

implying that both C and M are multiples of 3, just as we had claimed.

We have proved our first theorem about magic triangles!

Again, since C and M are each sums of three distinct numbers from the set $\{1, 2, 3, 4, 5, 6\}$, each of them is at least $1 + 2 + 3 = 6$, and at most $4 + 5 + 6 = 15$. It follows that the pair (C, M) is one of the following: $(6, 15)$, $(9, 12)$, $(12, 9)$, $(15, 6)$. The corresponding values of s are 9, 10, 11, 12. Hence we have the following possibilities:

	Possibility #1	Possibility #2	Possibility #3	Possibility #4
C	6	9	12	15
M	15	12	9	6
s	9	10	11	12

It is clear that these can be the only possibilities. However this by itself does not imply that all these possibilities can be realized, that such configurations really do exist; it may happen that some other condition 'comes in the way'. We can find out only by trial and error. Doing so, we find that each combination listed can indeed be realized. For proof we simply give the magic triangles (Figure 2).

We see that *there are precisely four different magic triangles*. We have discovered our second theorem — and proved it at the same time.

Note the curious symmetry between triangles I and IV (each one is a sort of 'turned out' version of the other), and between triangles II and III. One could call it a kind of 'duality'.

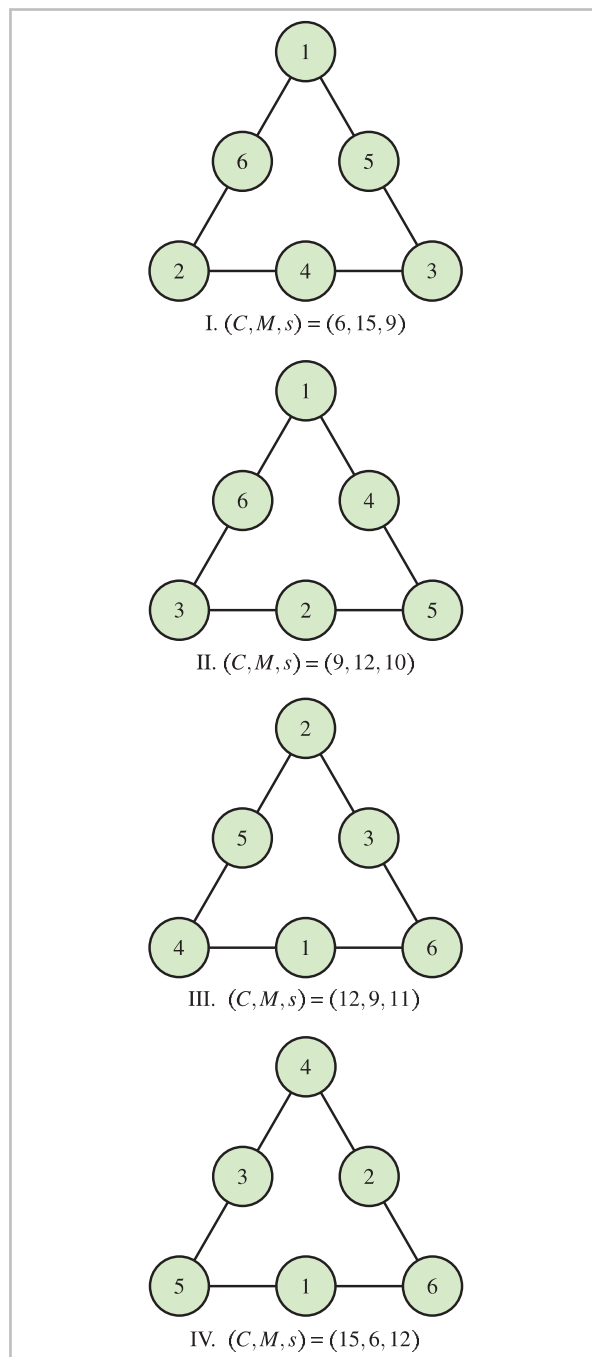


Fig. 2 The four possibilities

On examining the configurations we find another theorem, which we had not anticipated: *In each magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.* (The difference is 3 in configurations I and IV, and 1 in configurations II and III.)

Perhaps you will uncover more such theorems?

2. Problems for Solution

Notation. We introduced cryptarithms in the last issue of *At Right Angles*. We pose a few more problems of this genre here. But we make a change in the notation: *To denote a two-digit number with tens digit A and units digit B we use the notation \overline{AB} and not AB .* The line covering AB serves to tell us that this is the intended meaning. Without the line we would not be able to distinguish between the two-digit number with tens digit A and units digit B and the ordinary product AB .

Problem I-2-F.1 Find an algebraic proof of the property stated above: *In a magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.* That is, $u - x = v - y = w - z$ (with reference to Figure 1).

Problem I-2-F.2 Explore the analogous problem in which the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are placed

along the sides of a triangle, one at each vertex and two on the interiors of each side, so that the sum of the numbers on each side is the same. What different theorems can be found for this configuration? Conduct a complete exploration.

Problem I-2-F.3 Show that the cryptarithm

$$\overline{AT} + \overline{RIGHT} = \overline{ANGLE}$$

has no solutions! (The problem has nine unknowns: A, T, R, I, G, H, N, L, E.)

Problem I-2-F.4 Solve the following cryptarithm:

$$\overline{CATS} \times 8 = \overline{DOGS}.$$

Problem I-2-F.5 Solve this cryptarithm:

$$\overline{ABCDEF} \times 5 = \overline{FABCDE}.$$

3. Solutions of Problems from Issue-I-1

Problem I-1-F.1 *To solve the cryptarithm $\overline{ABCD} \times 4 = \overline{DCBA}$.*

- Let $x = \overline{ABCD}$ and $y = \overline{DCBA}$. Both x and y are four digit numbers. Since A is the leading digit of x we take A to be non-zero.
- Since $4x$ is a four digit number, $A = 1$ or 2 . As A is the ones digit of $4x$, it must be even. Hence $A = 2$.
- Since D is the leading digit of $4x$, it follows that $D = 8$ or 9 . Since the ones digit of $4x$ is 2 (and not 6), it follows that $D = 8$.
- Noting the ‘carry’ of 3 from the multiplication $D \times 4$, and the fact that there is no ‘carry’ from the multiplication $B \times 4$, it follows that

$$4(10B + C) + 3 = 10C + B,$$

giving $2C = 13B + 1$. The relation implies that B is odd.

- Since B and C are digits, it must be that $B = 1$ and therefore $C = 7$.

- Hence $x = 2178$ and $y = 8712$. It is easily checked that $y = 4x$.
- So the given cryptarithm, $\overline{ABCD} \times 4 = \overline{DCBA}$ has *precisely one solution*.

Comment. After getting $A = 2$ and $D = 8$ there are other ways of getting B and C . We have only described one such way, and we encourage you to find other ways.

Problem I-1-F.2 *To solve the cryptarithm $(\overline{TWO})^2 = \overline{THREE}$.*

- Looking at their leading digits (both are T), we realize that $T = 1$.
- Since $\overline{TWO}^2 < 20000$, it follows that $\overline{TWO} < 142$.
- As the ones digit of a perfect square is one of $0, 1, 4, 5, 6, 9$, it follows that E is one of these digits. Hence \overline{EE} is one of the numbers $00, 11, 44, 55, 66, 99$.

- Now we recall the test for divisibility by 4: *A number n is divisible by 4 if and only if the number formed by the last two digits of n is itself divisible by 4.* We can make a stronger statement. Let n' denote the number formed by the last two digits of n . Then: *n and n' leave the same remainder under division by 4.* (To see why, observe that all powers of 10 higher than 10^2 are multiples of 4.)
- Next, recalling that all squares leave remainder 0 or 1 under division by 4, we see that \overline{EE} leaves remainder 0 or 1 under division by 4. Hence we eliminate the possibilities 11, 55, 66 and 99 for \overline{EE} . So $E = 0$ or 4.
- If $E = 0$ then $O = 0$ too (two different letters cannot represent the same digit), and we disallow this. Hence $E = 4$, and $O = 2$ or 8. So the number formed by the last two digits of \overline{TWO} is either $10W + 2$ or $10W + 8$.
- Suppose that $O = 2$. Since $(10W + 2)^2 = 100W^2 + 40W + 4$, we get $W = 1$ or 6 (as the last two digits of $(\overline{TWO})^2$ are 44). But 6 is too large (remember that $\overline{TWO} < 142$), and 1 has been 'used up'. So $O \neq 2$.
- Hence $\overline{TWO} \in \{108, 128, 138\}$.
- Testing these three possibilities we find that only the last one 'works' and we get the answer: $138^2 = 19044$. Check it out for yourself!

Comment. After getting $O = 2$ or 8 there are other ways possible. As earlier, we encourage you to find these ways on your own.

Problem I-1-F.3 To count the total number of 1s in the string 1, 2, 3, ..., 99, 100, 101, ..., 999998, 999999.

The neatest way of solving this is to pad each number in the given string with a suitable numbers of 0s from the left side, so that every number in the string has six digits. We also start the list with 000000. So the list is: 000000, 000001, 000002, ..., 999998, 999999.

There are now 10^6 numbers in the list, each with six digits, so the total number of digits in the entire string is simply 6×10^6 . The crucial observation to make is that the number of 1s, 2s, 3s, ..., 8s, 9s in the new string is exactly the same as in the original string; only the number of 0s has changed.

Now by symmetry it should be clear that each of the digits 0, 1, 2, ..., 9 occurs with the same frequency in the new string. Hence there are 6×10^5 occurrences of each of the ten digits in the new string.

So the number of 1s in the original string is $6 \times 10^5 = 60,000$. (The number of 2s, 3s, ..., 9s is exactly the same. Only the number of 0s is different.)

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

1. Problems for Solution

Problem I-2-M.1 Using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once each can you make a set of numbers which when added and subtracted in some order yields 100 as the answer?

(For example, you could make the collection {132, 58, 40, 69, 70} and try the 'sum' $132 - 58 + 40 - 69 + 70$. But that does not work!)

Problem I-2-M.2 Let all the natural numbers be listed, except the multiples of 3:

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16,
17, 19, 20, 22, . . .

Find a simple formula for the n -th term of the above sequence, in terms of n .

Problem I-2-M.3 Let all the natural numbers be listed, except the perfect squares:

2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15,
17, 18, 19, . . .

Find a simple formula for the n -th term of the above sequence, in terms of n .

Problem I-2-M.4 Amar, Akbar and Antony are three friends. The average age of any two of them is the age of the third person. Show that the total of the three friends is divisible by 3.

Problem I-2-M.5 A set of consecutive natural numbers starting with 1 is written on a sheet of paper. One of the numbers is erased. The average of the remaining numbers is $5\frac{2}{9}$. What is the number erased?

Problem I-2-M.6 The average of a certain number of consecutive odd numbers is A . If the next odd number after the largest one is included in the list, then the average goes up to B . What is the value of $B - A$?

Problem I-2-M.7 101 marbles numbered from 1 to 101 are divided between two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers of marbles in A increases by $\frac{1}{4}$; the average of all the numbers of marbles in B also increases by $\frac{1}{4}$. Find the number of marbles originally present in basket A. (From the 1999 Dutch Math Olympiad.)

2. Solutions to problems from Issue-I-1

Solution to problem I-1-M.1

- (a) *Right triangles in which the longer leg and hypotenuse are consecutive natural numbers:* (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41).
- (b) *Right triangles in which the legs are consecutive natural numbers:* (3, 4, 5), (20, 21, 29).
- (c) *Rectangles with integer sides and a diagonal 25:* Sides 7 and 24; sides 15 and 20.
- (d) *Two PPTs free from prime numbers:* (16, 63, 65), (33, 56, 65).

Solution to problem I-1-M.2 Let (a, b, c) be a PT. Is it possible that among a, b, c :

- (i) *All three are even?* Possible; e.g., (6, 8, 10).
- (ii) *Exactly two of them are even?* Not possible; if two of a, b, c are even, the third one must be even.
- (iii) *Exactly one of them is even?* Possible; e.g., (3, 4, 5).
- (iv) *None of them is even?* Not possible; if two of a, b, c are odd, then the third one must be even, necessarily.

Solution to problem I-1-M.3 Let (a, b, c) be a PT. Is it possible that among a, b, c :

- (i) *All three are multiples of 3?* Possible; e.g., (9, 12, 15).
- (ii) *Exactly two of them are multiples of 3?* Not possible; if two of a, b, c are multiples of 3, the third one too must be a multiple of 3.
- (iii) *Exactly one of them is a multiple of 3?* Possible; e.g., (3, 4, 5).
- (iv) *None of them is a multiple of 3?* Not possible. For the proof please see the solution to Problem I-1-S-2 elsewhere in this issue, where we prove more.

Solution to problem I-1-M.4 To list PPTs in which one of the numbers in the PPT is 60.

So we must find all pairs (m, n) of coprime integers, with opposite parity, such that one of $m^2 - n^2, 2mn, m^2 + n^2$ is 60.

Since m, n have opposite parity, $m^2 - n^2$ and $m^2 + n^2$ are odd; so we cannot have $m^2 - n^2 = 60$ or $m^2 + n^2 = 60$. The only possibility is $2mn = 60$, i.e., $mn = 30$. In what ways can we express 30 as a product of two coprime positive integers? (They will automatically be of opposite parity, since 30 is divisible by 2 but not by 4.) Here are the ways: $30 \times 1, 15 \times 2, 10 \times 3, 6 \times 5$. Hence (m, n) can be (30, 1), (15, 2), (10, 3) or (6, 5). This yields four possible PPTs: (899, 60, 901), (221, 60, 229), (91, 60, 109), (11, 60, 61).

Solution to problem I-1-M.5 Take two fractions with product 2. Add 2 to each one, and multiply each by the LCM of the denominators. You get two natural numbers which are the legs of an integer sided right triangle.

Let the two fractions be a/b and $2b/a$ where a, b are coprime. Adding 2 to each we get $(a + 2b)/b$ and $2(a + b)/a$. Multiplying by the LCM of their denominators which is ab , we get $a(a + 2b)$ and $2b(a + b)$. To verify the property we must show that $a^2(a + 2b)^2 + (2b)^2(a + b)^2$ is a perfect square. You should verify that this expression simplifies to $(a^2 + 2ab + 2b^2)^2$.

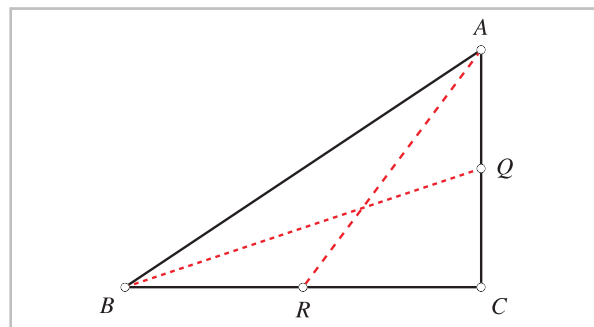


Fig. 1 Problem I-1-M-6

Solution to problem I-1-M.6 The medians of a right triangle drawn from the vertices of the acute angles have lengths 5 and $\sqrt{40}$. What is the length of the hypotenuse?

The situation is depicted in Figure 1, with $AR = 5$ and $BQ = \sqrt{40}$. Let the sides of $\triangle ABC$ be a, b, c .

Apply the Pythagorean theorem to $\triangle ARC$ and $\triangle BQC$:

$$AR^2 = b^2 + \left(\frac{a}{2}\right)^2, \quad BQ^2 = a^2 + \left(\frac{b}{2}\right)^2,$$

$$\therefore AR^2 + BQ^2 = \frac{5(a^2 + b^2)}{4},$$

so $5(a^2 + b^2)/4 = 25 + 40 = 65$. Hence $a^2 + b^2 = 52$. It follows that $c = \sqrt{52}$.

In general, if the medians drawn from the acute angles of a right triangle have lengths u and v , then the hypotenuse c is given by $5c^2 = 4(u^2 + v^2)$.

Solution to problem I-1-M.7 *ABCD is a square of side 1; P and Q are the midpoints of sides AB and BC; PC and DQ meet at R. What can be said about $\triangle PRD$?*

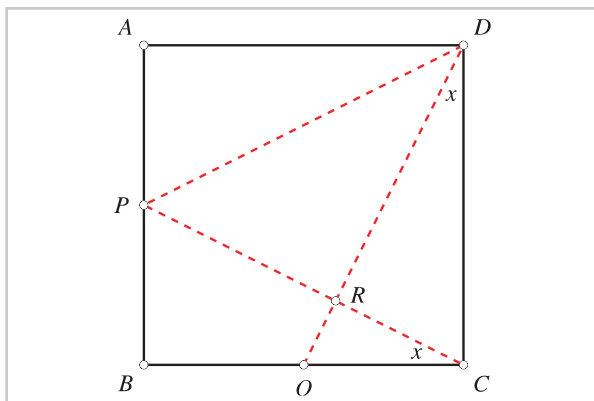


Fig. 2 Problem I-1-M.7

The situation is depicted in Figure 2. From $\triangle PBC \cong \triangle QCD$ it follows that the two angles marked x are equal, and so $\angle DRC$ is a right angle. So $\triangle PRD$ is a right triangle. To find its sides we use the principle of similarity. Observe that $\triangle CRQ \sim \triangle CBP$. Hence $CR/CQ = CB/CP$. Next, note that $CP^2 = 1^2 + (1/2)^2 = 5/4$, hence $CP = \sqrt{5}/2$. Also, $CQ = 1/2$. From this we get $CR/(1/2) = 1/(\sqrt{5}/2)$, hence $CR = 1/\sqrt{5}$. In the same way we get $DR = 2/\sqrt{5}$. We now get the length of PR :

$$PR = \frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}} = \frac{3}{2\sqrt{5}}.$$

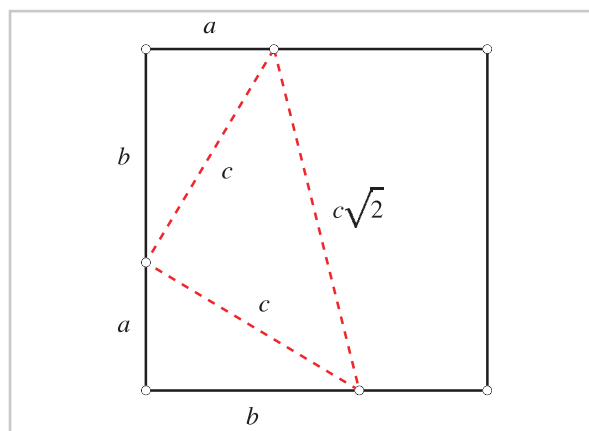


Fig. 3 Problem I-1-M.9

So the sides of the right triangle PRQ are in the following ratios:

$$PR : DR : PD = \frac{3}{2\sqrt{5}} : \frac{2}{\sqrt{5}} : \frac{\sqrt{5}}{2} = 3 : 4 : 5.$$

The right triangle PRD is a 3 : 4 : 5 triangle.
A pleasant surprise!

Solution to problem I-1-M.8 Find all right triangles with integer sides such that their perimeter and area are numerically equal.

Note the phrase ‘numerically equal’; *area can never equal perimeter*, as the two are dimensionally distinct. An example of a triangle satisfying the condition is the one with sides (6, 8, 10); its area is $(6 \times 8)/2 = 24$ square units, and its perimeter is $6 + 8 + 10 = 24$ units. We solve the problem in full in the ‘Senior Problems’ section.

Solution to problem I-1-M. 9 *If a, b are the legs of a right triangle, show that*

$$\sqrt{a^2 + b^2} < a + b \leq \sqrt{2(a^2 + b^2)}.$$

For proof see Figure 3, in which $c = \sqrt{a^2 + b^2}$. The relation $c < a + b$ follows from the triangle inequality (“Any two sides of a triangle are together greater than the third one”), and $a + b \leq \sqrt{2(a^2 + b^2)}$ follows from the fact that the least distance between a pair of opposite sides of the square is $a + b$, and $\sqrt{2(a^2 + b^2)} = c\sqrt{2}$ is the length of *one* possible segment connecting the same pair of sides.

Problems for the Senior School

Problem Editor : PRITHWIJIT DE & SHAILESH SHIRALI

'Area equals Perimeter'

In the 'Problems for the Middle School' in Issue-I-1 we asked for a listing of all integer sided right triangles with the property that the area *numerically* equals the perimeter. We study this problem in some detail here. Let a, b, c be the sides of the triangle, c being the hypotenuse. Then we have the following relations:

$$\text{area} = \frac{ab}{2}, \text{ perimeter} = a + b + c, \quad c^2 = a^2 + b^2.$$

Hence we have:

$$a + b + \sqrt{a^2 + b^2} = \frac{ab}{2},$$

$$\therefore \sqrt{a^2 + b^2} = \frac{ab}{2} - a - b.$$

From the last relation we get, by squaring:

$$a^2 + b^2 = \left(\frac{ab}{2} - a - b \right)^2$$

$$= \frac{a^2 b^2}{4} - a^2 b - a b^2 + 2ab + a^2 + b^2.$$

Simplifying, we get the relation $a^2 b^2 - 4a^2 b - 4a b^2 + 8ab = 0$. The expression on the left has ab as a factor, and as this is not zero, we get:

$$ab - 4a - 4b + 8 = 0.$$

We must find all pairs (a, b) of positive integers satisfying this equation. Note that we have two unknowns but one equation; so the equation may have many pairs of solutions. It is an example of a *indeterminate Diophantine equation*. To solve it we use factorization. The expression $ab - 4a - 4b + 8$ does not factorize but if the '8' were replaced by '16' we would get a nice factorization. And that gives us the idea for a solution:

$$ab - 4a - 4b + 8 = (ab - 4a - 4b + 16) - 8$$

$$= (a - 4)(b - 4) - 8.$$

Since $ab - 4a - 4b + 8 = 0$ we get

$$(a - 4)(b - 4) = 8.$$

Hence $\{a - 4, b - 4\}$ is one of the following: $\{8, 1\}$, $\{4, 2\}$. It follows that $(a, b) = (12, 5)$ or $(8, 6)$. Hence there are just two integer sided right triangles with the property that the area and perimeter are numerically equal. These are the triangles with sides $(5, 12, 13)$, with area and perimeter numerically equal to 30; and $(6, 8, 10)$, with area and perimeter numerically equal to 24.

Problems for Solution

Problem I-2-S.1 A series of numbers beginning with 2012 is in AP (arithmetic progression) as well as GP (geometric progression). Find the sum of the first 100 terms of the series.

Problem I-2-S.2 Find the sum of all four digit numbers with the following property: *the sum of the first two digits equals the sum of the last two digits*. Also compute the number of such numbers.

Problem I-2-S.3 Show that no term of the sequence 11, 111, 1111, 11111, ... is the square of an integer.

Problem I-2-S.4 The radius r and the height h of a right-circular cone with closed base are both an integer number of centimetres, and the volume of the cone in cubic centimetres is equal

to the total surface area of the cone in square centimetres. Find the values of r and h .

Problem I-2-S.5 Given a $\triangle ABC$ and a point O within it, lines AO , BO and CO are drawn intersecting the sides BC , CA and AB at points P , Q and R , respectively. Prove that

$$\frac{AR}{RB} + \frac{AQ}{QC} = \frac{AO}{OP}.$$

Problem I-2-S.6 The triangular numbers 1, 3, 6, 10, 15, 21, 28, ... are numbers of the form $n(n+1)/2$ for positive integers n . The square numbers 1, 4, 9, 16, 25, 36, 49, ... are numbers of the form n^2 for positive integers n . Show that every triangular number greater than 1 is the sum of a square number and two triangular numbers.

Solutions of Problems in Issue-I-1

The following facts are needed. Let n be any integer; then: n^2 leaves remainder 0 or 1 modulo 3; n^2 leaves remainder 0 or 1 modulo 4; n^2 leaves remainder 0, 1 or 4 modulo 5; and n^2 leaves remainder 0, 1 or 4 modulo 8.

Solution to problem I-1-S.1 For a PPT (a, b, c) , one of a, b is odd and the other is even, and the even number is a multiple of 4.

Certainly, a and b cannot both be even. Suppose that both a and b are odd. Then $a^2 \equiv 1 \pmod{4}$ and also $b^2 \equiv 1 \pmod{4}$, which yields $a^2 + b^2 \equiv 2 \pmod{4}$, i.e., $c^2 \equiv 2 \pmod{4}$.

But no such square exists; all squares are 0 (mod 4) or 1 (mod 4). Hence it cannot happen that both a, b are odd. So one of $\{a, b\}$ is even, and the other one is odd.

Suppose that a is even and b is odd. Then c is odd, and we have $b^2 \equiv 1 \pmod{8}$ and also $c^2 \equiv 1 \pmod{8}$, implying that $c^2 - b^2$ is a multiple of 8. Hence a^2 is a multiple of 8. This implies that a is an even multiple of 2, i.e., a is a multiple of 4.

Solution to problem I-1-S.2 For a PPT (a, b, c) , the product abc is a multiple of 60.

We have already shown that if (a, b, c) is a PPT then one of $\{a, b\}$ is a multiple of 4, and hence that ab is a multiple of 4. If we can show that one of $\{a, b, c\}$ is a multiple of 3, and one of $\{a, b, c\}$ is a multiple of 5, then the problem will be solved since abc will then be divisible by $3 \times 4 \times 5 = 60$.

Suppose that both a and b are non-multiples of 3. Then $a^2 \equiv 1 \pmod{3}$ and also $b^2 \equiv 1 \pmod{3}$, and so $a^2 + b^2 \equiv 2 \pmod{3}$. Therefore $c^2 \equiv 2 \pmod{3}$. But there is no square of this form.

Hence it cannot be that both of a, b are non-multiples of 3. So one of a, b is a multiple of 3.

Suppose that a, b, c are non-multiples of 5. Then a^2, b^2, c^2 are all 1 or 4 (mod 5). But none of the possibilities 'fits'; it is impossible to have $a^2 + b^2 \equiv c^2 \pmod{5}$ within these possibilities. Hence it cannot be that a, b, c are all non-multiples of 5. So one of a, b, c is a multiple of 5.

Hence $\{a, b, c\}$ contains a multiple of 3, a multiple of 4 and a multiple of 5. Since 3, 4, 5 are coprime, it follows that abc is a multiple of 60.

Solution to problem I-1-S.3 Any right-angled triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x -axis and whose three vertices have integer coordinates.

Suppose the given triangle has integer sides a, b, c where c is the hypotenuse. We construct a triangle with sides ac, bc, c^2 , the side of length c^2 having end-points $B'(0,0)$ and $A'(c^2,0)$. Since its sides bear the ratios $a : b : c$, it is similar to the given triangle and hence is a right triangle (Figure 1). Its height is $h = (a c \times b c) / c^2 = ab$. It is easily verified that the coordinates of the third vertex are integers; for, if D' is the foot of the perpendicular from C' to side $A'B'$, then $B'D' = \sqrt{a^2 c^2 - a^2 b^2} = a^2$ and $C'D' = b^2$.

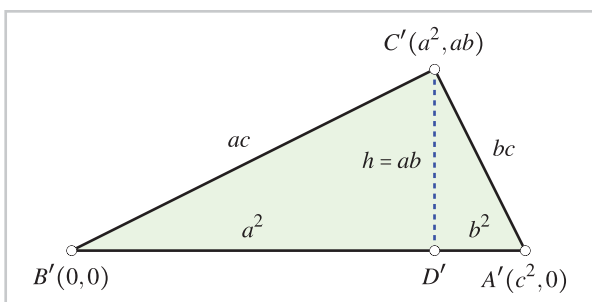


Fig. 1

Solution to problem I-1-S.4 Let a, b and c be the sides of a right-angled triangle. Let θ be the smallest angle of this triangle. Show that if $1/a, 1/b$ and $1/c$ too are the sides of a right-angled triangle, then $\sin \theta = (\sqrt{5} - 1)/2$.

Proof. Suppose that θ lies opposite side a ; then $a < b \leq c$, hence $1/a > 1/b \geq 1/c$, and the hypotenuse of the triangle with sides $1/a, 1/b, 1/c$ is $1/a$, so $1/b^2 + 1/c^2 = 1/a^2$. Multiplying by a^2 and using $\sin \theta = a/c$, $\tan \theta = a/b$ we get:

$$\tan^2 \theta + \sin^2 \theta = 1, \quad \therefore \tan^2 \theta = \cos^2 \theta,$$

$$\therefore \sin^2 \theta = \cos^4 \theta$$

hence $(\sin \theta - \cos^2 \theta)(\sin \theta + \cos^2 \theta) = 0$. Since $\sin \theta + \cos^2 \theta > 0$, the quantity $\sin \theta - \cos^2 \theta$ must vanish. Hence $\sin^2 \theta + \sin \theta - 1 = 0$, and $\sin \theta = (\sqrt{5} - 1)/2$.

Solution to problem I-1-S.5 Find all Pythagorean triples (a, b, c) in which: (i) one of a, b, c equals 2011; (ii) one of a, b, c equals 2012.

(i) Suppose $a = 2011$; then $c^2 - b^2 = 2011^2$, hence $(c - b)(c + b) = 2011^2$. Since 2011 is prime (did you know that?), the only way of

accomplishing this is by setting $c - b = 1$ and $c + b = 2011^2$. Solving these we get $b = 2022060$ and $c = 2022061$. So the triple is $(2011, 2022060, 2022061)$. Of course we can swap the first two entries. Suppose $c = 2011$; then $a^2 + b^2 = 2011^2$. To solve this we shall appeal to a result which we only state for now: *Let p be a prime number of the form $3 \pmod{4}$. Suppose that x and y are integers such that $x^2 + y^2$ is a multiple of p . Then both x and y are multiples of p .*

The result may be easily verified for small p of the stated form, e.g., $p = 3, 7, 11$. To prove the result in full generality requires knowledge of the 'Little Theorem of Fermat'. The result applies since 2011 is a prime number of the form $3 \pmod{4}$. It implies that both a and b are multiples of 2011. Let $a_1 = a/2011$ and $b_1 = b/2011$; then $a_1^2 + b_1^2 = 1$, which clearly has no solution in positive integers. Hence the equation $a^2 + b^2 = 2011^2$ too has no solution in positive integers.

So there are just two PTs in which one of the numbers is 2011: the one listed above and one of its permutations.

(ii) Suppose $a = 2012$; then $c^2 - b^2 = 2012^2$. By writing $2012^2 = 2^4 \times 503^2$ and noting that $c - b$ and $c + b$ have the same parity, we see that $(c - b, c + b)$ is one of the following: $(2, 2^3 \times 503^2)$, $(2^2, 2^2 \times 503^2)$, $(2^3, 2 \times 503^2)$. Solving three sets of equations we find that (a, b, c) can be any of the following:

$$(2012, 1012035, 1012037),$$

$$(2012, 506016, 506020),$$

$$(2012, 253005, 253013).$$

As earlier, we can also swap the first two entries.

Suppose $c = 2012$; then $a^2 + b^2 = 2^2 \times 503 \equiv 0 \pmod{503}$. As 503 is a prime number of the form $3 \pmod{4}$ the result stated above applies; so both a and b are multiples of 503. Let $a_1 = a/503$ and $b_1 = b/503$; then $a_1^2 + b_1^2 = 2^2$, which clearly has no solution in positive integers. Hence the equation $a^2 + b^2 = 2012^2$ too has no solution in positive integers.

Solution to problem I-1-S.6 Find all PPTs (a, b, c) in which a, b, c are in GP.

Suppose (a,b,c) is a PPT in which a,b,c are in GP. Then $b^2 = ac$, and also $b^2 = c^2 - a^2$. Hence $c^2 - a^2 = ac$. Let $x = c/a$. Dividing $c^2 - a^2 = ac$ by a^2 we get $x^2 - 1 = x$, hence $x^2 - x - 1 = 0$, and $x = (\sqrt{5} + 1)/2$. This number is irrational, whereas c/a , a ratio of two positive integers, must be rational. So no such PPT exists.

Solution to problem I-1-S.7 In any triangle, show that the sum of the squares of the medians equals $\frac{3}{4}$ of the sum of the squares of the sides. See below for the solution.

Solution to problem I-1-S.8 The figure shows a $\triangle ABC$ in which P, Q, R are points of trisection of the sides, with $BP = \frac{1}{3}BC$, $CQ = \frac{1}{3}CA$, $AR = \frac{1}{3}AB$. Show that the fraction $(AP^2 + BQ^2 + CR^2)/(BC^2 + CA^2 + AB^2)$ has the same value for every triangle.

We solve a generalized version of this problem which includes Problem I-1-S-7. Let $\triangle ABC$ be given, and let P, Q, R be points on BC, CA, AB respectively such that with $BP/BC = CQ/CA = AR/AB = 1/n$ where n is a given number (Figure 2). We wish to find the value of $(AP^2 + BQ^2 + CR^2)/(BC^2 + CA^2 + AB^2)$. Observe that $n = 2$ yields Problem I-1-S-7, and $n = 3$ yields Problem I-1-S-8.

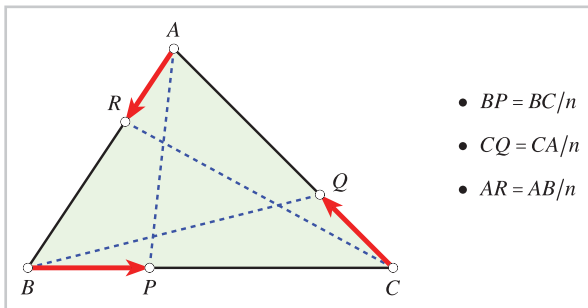


Fig. 2

We solve this using vectors. Let $\vec{u}, \vec{v}, \vec{w}$ represent the vectors $\vec{BP}, \vec{CQ}, \vec{AR}$ respectively. Then $\vec{BC} = n\vec{u}$, $\vec{CA} = n\vec{v}$, $\vec{AB} = n\vec{w}$. Since $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$ (the zero vector), it follows that $n(\vec{u} + \vec{v} + \vec{w}) = \vec{0}$ and hence that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. By dotting each side of this relation with itself we get:

$$u^2 + v^2 + w^2 = -2(\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}),$$

where $u^2 = \vec{u} \cdot \vec{u}$, etc. (Here u denotes the length of \vec{u} , etc.) Now $\vec{AP} = \vec{AB} + \vec{BP} = n\vec{w} + \vec{u}$, and similarly for \vec{BQ} and \vec{CR} . Hence:

$$\begin{aligned} AP^2 + BQ^2 + CR^2 &= (n\vec{w} + \vec{u})^2 + (n\vec{u} + \vec{v})^2 + (n\vec{v} + \vec{w})^2 \\ &= (n^2 + 1)(u^2 + v^2 + w^2) + 2n(\vec{u} \cdot \vec{v} \\ &\quad + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}) \\ &= (n^2 - n + 1)(u^2 + v^2 + w^2), \end{aligned}$$

and since $BC^2 + CA^2 + AB^2 = n^2(u^2 + v^2 + w^2)$, we get:

$$\frac{AP^2 + BQ^2 + CR^2}{BC^2 + CA^2 + AB^2} = \frac{n^2 - n + 1}{n^2}.$$

This is the required answer. Note that the ratio does not depend on the shape of the triangle. It depends only on n .

If $n = 2$ (Problem I-1-S-7) the ratio simplifies to $3/4$; if $n = 3$ (Problem I-1-S-8) the ratio simplifies to $7/9$. So we have solved two problems in one stroke, and also obtained a more general result in the process.

As a side remark we note that the fraction $(n^2 - n + 1)/n^2$ achieves its *least value* of $3/4$ when $n = 2$. Try proving this for yourself.

NRICH

<http://nrich.maths.org/public/>

WEBSITE REVIEW

by **SNEHA TITUS**

From the website:

"The NRICH Project aims to enrich the mathematical experiences of all learners. To support this aim, members of the NRICH team work in a wide range of capacities, including providing professional development for teachers wishing to embed rich mathematical tasks into everyday classroom practice. On our website you will find thousands of our free mathematics enrichment materials (problems, articles and games) for teachers and learners from ages 5 to 19 years. All the resources are designed to develop subject knowledge, problem-solving and mathematical thinking skills. The website is updated with new material on the first day of every month."

The blurb above speaks the truth – I've been using this website for over 3 years now and have found it to be a rich resource of many things mathematical. The home page (to be redesigned this month – so I will speak of the page I have been used to) makes mathophiles itch to get their scratch pads and pencils out – the math problems of the month are featured in prominence here. There are problems at Stages 1-5 and the website invites students to send in their solutions for publication. This I feel is one of the best features

of the NRICH page. What better incentive for students to hone their mathematical communication skills than the promise of having their work put up online! And the solutions I have seen have been chosen not just for correct answers but for innovative thinking, logical presentation and good approaches to problem solving.

On the right of the home page, you find resources for teachers – a comprehensive array of articles, games and interactive resources. You can access previous issues where, reassuringly, problems come with solution sheets. Also, on the right, are the stem NRICH pages. The latter is a treasure trove of useful information with a plethora of articles ranging from different ways to braid your hair (Rapunzel would have been quite content in her tower with access to this page!), to studying epidemics ('Why do epidemics take off? Why don't they just carry on for ever once they've started?' These simple models will help us to understand what's going on, and how science can help us to prevent epidemics happening in the first place) and best of all, ideas for Stem Clubs which will certainly be of use to overburdened mathematics teachers who would

love to try new things but do not have the time to think them through. I found several student friendly activities on the Bridges of Königsberg as well as a smooth transition to networks and traversability which helped me show students how it was possible to think past a problem which had no solution. (Note: You can read the web pages in Gujarati, Hindi, Tamil, Telugu or even Urdu, if you are so inclined!)

Searching on NRICH is easy and you can search by level or topic or activity (worksheets, games, articles and so on). My personal favourite is a game found at <http://nrich.maths.org/6402> which I stumbled upon while doing a search on 'division'. It's a great way to practise the tables but it goes beyond that because it actually encourages students to find the shortest possible route to guessing a number by a series of divisions. According to a friend who is a harried father of four, this game has occupied his children happily, on otherwise interminable car journeys. And his ten year old has figured out the algorithm after playing for a while.

Need I say more?

exploring

CONSTRUCTIVISM

“What did you do at school?” is a routine question that most mothers ask their children when they return home. My daughter Priya was in Class One and when I asked her about her school day, she led me to the tamarind tree in our courtyard, picked a little leaf and said “I learned this.” Elaborating further as she pointed to every pair of leaflets, she said “Look, this is two ones are two, two twos are four, ...” “Is this what teacher used in class today?” I enquired. “No, no. Teacher wrote this on the blackboard and made us say ‘two ones are two’. But mamma, I had seen this ‘two ones are two’ leaf while playing.” I loved the new name that the tamarind leaf had got. A ‘two ones are two’ leaf! Mathematics abounds all around. My daughter taught me this is through many more examples. Here is another interesting one.

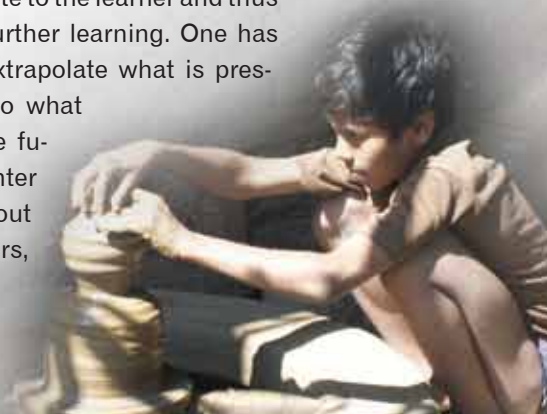
Priya was about three when she was helping me arrange eggs in the refrigerator. As we placed the eggs (in twos) in the egg-rack, she commented “5 is not a partner number, 6 is a partner number, 3 is not a partner number, and 4 is a partner number.” Amused that my daughter was hinting at odd and even numbers, I asked her to explain. “With 3 eggs, we can place 2 eggs next to each other, but 1 is left behind. When there are 4 eggs, all get partners, none are left behind.” I told her that ‘not a partner number’ is the same as odd and ‘partner number’ is the same as even, but she seemed happier with her terminology; it made more sense than ‘odd’ and ‘even’. Later, when she learned about the same in school, I reminded her of this incident. I had read about constructivism and designed constructivist activities, but it was this experience that gave me a chance to explore constructivism. Based on this and my experiences as a teacher, I share a few thoughts about constructivism.

➤ Constructivism values individual thinking strategies: In Mathematics, there can be no one fixed method to solve a given question. Sadly, teachers insist on a particular method, answer keys supplied to examiners allot marks for a set pattern of steps, and we end up with stereotyped answers. “Why can’t I solve it using my method?” is an often heard query. Following a set of steps may be beneficial as it brings in some kind of standardization and facilitates the teacher’s task, but in insisting on ‘following a fixed method’ we fail to nurture individual thinking strategies, we fail to allow creativity and this is the first stumbling block to constructivist thinking.

➤ Constructivism involves sensory input: Mathematics teaching is often considered challenging as much of the content is abstract. We may not have a plausible hands-on activity for every concept in Mathematics, but wherever possible a multimodal approach

(using both cognitive and psycho-motor domains) should be used. To learn more about multimodal learning in Mathematics, I recommend Rashmi Kathuria’s work which can be accessed on <http://mykhmsmathclass.blogspot.in/>, <http://mathematicslearning.blogspot.in/>, <http://mathematicsprojects.blogspot.in/>.

➤ Constructivism uses dovetailing, scaffolding and extrapolation: Mathematics involves connections. An analytical teacher takes into account the previous content that needs to be dovetailed into the present content being explored. One needs to provide the minimum support that is adequate to the learner and thus provide leverage to further learning. One has to help the learner extrapolate what is presently being learned to what will be learned in the future. When my daughter learned formally about odd and even numbers,



I reminded her of the 'eggs and partner numbers' incident. Next I took a number of small circles and we arranged them in pairs. So if we took seven pairs, we had 14 circles. If we had 20 circles, we had ten pairs. The next step was to try and arrange circles in different combinations, not just pairs. For example, 16 could be arranged as a pair pattern (2 x 8) but going beyond pairs, we could arrange 16 as (4 x 4); 15 could be arranged as a 3 x 5 pattern; 18 could be arranged as either 3 x 6 or 2 x 9 pattern. This was dovetailing what Priya knew about even numbers into factors of a number, which was something she did not yet know. I had to provide help for one example. The remaining examples were like a game. This step of providing minimal support is scaffolding. Constructivism also takes into account extrapolation. This activity of arranging circles was now made challenging by giving 31 circles or 19 circles to arrange in a pattern. This helped introduce the concept of prime numbers. Meaningful connection between what is known and what needs to be known is the crux of constructivism.

➞ Constructivism encourages queries: A healthy learning environment welcomes questions. I once had a question from my daughter: "When we add, subtract and multiply, we begin with the unit's place. Why do we follow a reverse order when we divide?" Such questions indicate that the learner is looking for meaning, and this is the corner stone of constructivist learning.


➞ Constructivism is contagious: Learners who indulge in constructivist learning apply it to all forms of learning. They tend to use it for all subjects. They

tend to experiment, to interact with content. They look out for alternative ways to arrive at knowledge gaining. Most important, they see application of what they learn to real life. When Priya learned that metals expand on heating, she had this experience to share with me. She said "When we leave for school each morning, the two panels of the iron gate of our housing complex slide open easily. But when we come home in the afternoon, they are hot and have expanded. So the iron panels are touching each other and we have to apply force or sometimes kick the gate to open it." I had experienced the same phenomenon but my adult mind had not made the connection. Allowing learners to see the application of what they learn and encouraging them to quote examples beyond the textbook should be a prime focus in constructivist learning.

➞ Learn from and with your learners: All teachers need to learn from and with their learners. Learners could be forming connections based on misconceptions, and this will mean learning something erroneous. My daughter is now thirteen and learning about tests of congruence of triangles. Recently she told me that when she was small and had seen figures of triangles, she thought that segments with one stroke across them were smaller than those with two strokes! Thankfully this misconception was corrected. Thus constructivism has a lot to do with the ideas that the learner forms about content and here vigilance on part of the mentor is required. Else such misconceptions affect further learning. Teachers need to be vigilant about how learners learn, how they think and what they think.

My experience with my daughter has taught me how learners think. Once a teacher is in sync with how the learner thinks, the strategies used to stimulate learning can be aligned to the learner's thinking strategies. Constructivism fosters a 'learning to learn' attitude, an asset in today's era. As educators let us learn how students learn, so that learning is enriching and enjoyable. And before I end, thank you, Priya, for being my teacher!

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Generating PPTs

Letter Received from a Reader

We have received an interesting letter from **Mr Aravind Badiger**, in which he describes another way of generating Primitive Pythagorean Triples (PPTs). Let x be a positive rational number, and let $a = x$, $b = (x^2 - 1)/2$, $c = (x^2 + 1)/2$; then a, b, c are rational numbers such that $a^2 + b^2 = c^2$ and $c - b = 1$. To ensure that a, b, c are in ascending order we must have $x > \sqrt{2} + 1 \approx 2.414$. By choosing x appropriately and after multiplying by an appropriate constant to clear fractions we get a family of PPTs in which $c - b$ has a constant difference; i.e., the difference between the largest two numbers in the triple is a constant. Here are some examples.

Let x take the odd integral values $3, 5, 7, \dots$; we get the PPTs $(3, 4, 5), (5, 12, 13), (7, 24, 25), \dots$ in which $c - b$ has constant value 1.

Let x take the even integral values $4, 6, 8, \dots$; after doubling to clear fractions we get the PPTs $(8, 15, 17), (12, 35, 37), (16, 63, 65), (20, 99, 101), \dots$ in which $c - b$ has constant value 2.

Let x take the fractional values $5/2, 7/2, 9/2, 11/2, 13/2, \dots$; after clearing fractions we get the PPTs $(20, 21, 29), (28, 45, 53), (36, 77, 85), (44, 117, 125), (52, 165, 173), \dots$ in which $c - b$ has constant value 8.

Let x take the fractional values $11/3, 13/3, 17/3, 19/3, \dots$ (fractions of the type odd number/3); after clearing fractions we get the PPTs $(33, 56, 65), (39, 80, 89), (51, 140, 149), (57, 176, 185), \dots$ in which $c - b$ has constant value 9.

Let x take the fractional values $8/3, 10/3, 14/3, 16/3, \dots$ (even number/3); we get the PPTs $(48, 55, 73), (60, 91, 109), (84, 187, 205), (96, 247, 265), \dots$ in which $c - b$ has constant value 18.

If we put $x = m/n$ we get $(a, b, c) = (m/n, (m^2 - n^2)/2n^2, (m^2 + n^2)/2n^2)$, and after scaling by $2n^2$ we recover the familiar formula for generating PPTs.

The interesting feature of this method is that it permits us to group PPTs into families in which the two largest numbers of the triple have a constant difference. And as a bonus, it makes it clear that *this difference is always of the form k^2 or $2k^2$* (where k is some integer). This is an interesting finding in itself.

The Closing Bracket ...

Professor William Thurston (1946–2012), who passed away in August of this year, was one of that rare breed: a high level research mathematician who made worthy contributions to school level mathematics education. His work in geometry had an enormous influence on mathematics at the research level (it won him a Fields Medal, in 1982); and his paper titled “Mathematical Education” was studied very closely by the team (appointed by NCERT) that prepared the Mathematics Position Paper of the National Curriculum Framework 2005.

The following poignant question once appeared on the ‘Math Overflow’ site: “What can one contribute to mathematics? I find that mathematics is made by people like Gauss; while it may be possible to learn their work and understand it, nothing new is created by doing this. It seems plausible that, with all the clever people working so hard on mathematics, there is nothing left for someone such as myself to do. Perhaps my value would be to act more like cannon fodder?” Thurston’s deeply thoughtful and humane response to this has great value for us and is worthy of careful study. Here are some extracts:

It’s not mathematics you need to contribute to. It’s deeper than that: how might you contribute to humanity and to the well-being of the world by pursuing mathematics? Such a question is not possible to answer in a purely intellectual way, because the effects of our actions go far beyond our understanding. We are deeply social and instinctual animals, so much that our well-being depends on many things we do that are hard to explain in an intellectual

way. That is why you do well to follow your heart . . . Bare reason is likely to lead you astray. None of us are wise enough to figure it out intellectually. . . The product of mathematics is clarity and understanding. Not theorems, by themselves. Is there any real reason that even such results as Fermat’s Last Theorem, or the Poincaré conjecture, really matter? Their real importance is not in their specific statements, but their role in presenting challenges that led to developments that increased our understanding. . . The world does not suffer from an oversupply of clarity and understanding. How and whether specific mathematics might lead to improving the world is usually impossible to tease out, but mathematics collectively is extremely important. . . I think of mathematics as having a large component of psychology, because of its strong dependence on human minds. Dehumanized mathematics would be more like computer code. Mathematical ideas, even simple ideas, are often hard to transplant from mind to mind. . . Because of this, mathematical understanding does not expand in a monotone direction. . . The real satisfaction from mathematics is in learning from others and sharing with others. All of us have clear understanding of a few things and murky concepts of many more. There is no way to run out of ideas in need of clarification. The question of who is the first person to set foot on some square meter of land is really secondary. Revolutionary change does matter, but revolutions are few, and they are not self-sustaining — they depend very heavily on the community of mathematicians.

May that act as an inspiration to the community of math writers like ourselves!

– SHAILESH SHIRALI

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Call for Articles

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :

AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University

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TEACHING DECIMAL FRACTIONS USING A PAPER KIT

PADMAPRIYA SHIRALI

A PAPER KIT
APPROACH

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A Resource for School Mathematics

One often sees mistakes of the following kinds in children's work: $0.2 < 0.199$; $0.4 \times 10 = 0.40$; $0.05 = 0.5$; they read aloud 0.32 as 'point thirty two'. One sees a lack of conceptual understanding even in the case of children who are able to solve problems correctly by applying the rule of aligning the decimal points (addition/subtraction problems) and counting decimal places (multiplication problems). Solving the problems correctly does not necessarily imply that concepts have been fully understood. Curiously, children who are logical may have a larger number of these misconceptions!

Why do these kinds of mistakes happen? One possible reason is that children apply the conceptual understanding they have gained about whole numbers to decimal numbers. They try to fit new knowledge into existing schema; in this case, the understanding they have built of whole numbers. Here are some examples of this phenomenon.

- In whole numbers a 'longer' number (a number with more digits) is bigger than a 'shorter' number (one with fewer digits); e.g., $1045 > 950$. But this doesn't hold in decimal numbers; e.g., $0.2 > 0.199$.
- In whole numbers, when you multiply a number by 10 you place a zero at the end of the number. Example: $40 \times 10 = 400$. But when we do $.4 \times 10$ we get 4, not .40.
- In whole numbers, a zero placed in *front* of a number has no value and can be dropped. But we cannot drop the zero in 0.05.

Another reason is that they do not have a proper understanding of place value which presupposes multiplicative thinking. That is, they do not understand that the value of a digit gets ten times larger when its position shifts one step towards the left.

Yet another reason could be that they have not internalized the understanding that the decimal and fractional part of a number is less than a whole.

So, how should we introduce decimals? Some teachers use money, others use measurement. But each of these turns out to have some limitations. (We discuss these at the end of this article.)

Let us now list the prerequisite skills that children should have before we teach decimals.

- Basic concepts of fractions:
 - a. part sizes are same and equal
 - b. number of parts that make up a whole
 - c. what numerator and denominator denote
 - d. idea of equivalent fractions.
- Place values: Multiplicative relationship between succeeding places (ten is 10 times a unit, 100 is 10 times a ten, etc.)

GUIDING PRINCIPLES TO FOLLOW IN INTRODUCING DECIMALS AND FRACTIONS

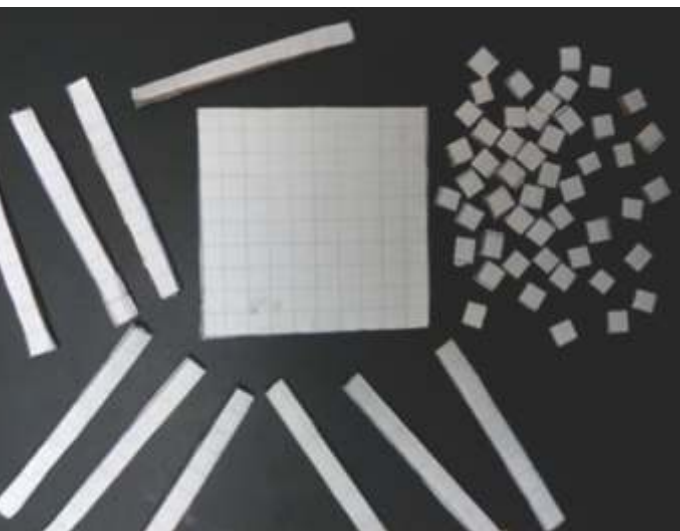
The first step should be through usage of a concrete model, followed by a pictorial representation and lastly the introduction of the abstract symbol. Experience, understanding and verbalization are the sequence to be followed.

Children need to understand that decimal numbers are part of the number system. Fractions and decimal numbers are not internalized by children in the same way as natural numbers (to which they are exposed from childhood). It is necessary for the teacher to contextualize and bring in as many examples as possible of the usage of these numbers in daily life.

ACTIVITY ONE

Purpose:
Preparation of the decimal kit.

Materials required:
Square dot paper or square grid paper (notebooks with Square Grid paper are available in the market and are used generally in primary sections) minimum of 3 sheets.



Help the children to prepare a decimal kit. This action itself helps in gaining understanding of tenth and hundredth and the relationship of these to the whole. It is better to prepare a kit than procure a ready-made one. Teacher should also prepare a kit consisting of several whole grids (to represent units), tenths and hundredths for demonstration purposes.

Paste these sheets on firm card paper. Then ask the children to:

- Outline a 10 x 10 grid which will be considered as the whole, and cut it out.
- Outline one more 10 x 10 grid, divide it into 10 equal parts, and cut out the 10 parts.
- Outline one more 10 x 10 grid, divide it into 100 equal parts, and cut out the 100 parts.

(If teachers find this activity time consuming they can give it as home work. Children will be thrilled to get homework which requires them to draw and cut.)

The process of this kit making should help each child in visualizing that a whole is 10 times as large as a tenth, and a tenth is 10 times as large as a hundredth.

ACTIVITY TWO

Purpose:
To teach tenth and hundredth and discuss their relationship with the whole.

Materials required:
Decimal kit

(This is at the oral level. Symbolic representation is introduced after the students become fully familiar with the language, the words tenth and hundredth.)

Use the kit to pose various questions:

Ask them to pick up one strip and tell them that it is called a tenth. Point out that each strip is one-tenth of the whole. Show pictures of grids with different numbers of tenths shaded, and get them to read the number of tenths shaded. Ask them “How many tenths make a whole?” Let them count and realize that 10 tenths make a whole.

Connection between the name ‘tenth’ and the process of the whole being made into ten equal parts should be established clearly.

Now ask them to pick up a small square and tell them that it is called a hundredth. Point out that each small square is one-hundredth of the whole. Lead them to realize that 100 hundredths make a whole. Show pictures of grids with different numbers of hundredths shaded, and get them to read the number of hundredths shaded.

Connection between the name ‘hundredth’ and the process of the whole being made into hundred equal parts should be established clearly.

Now ask how many hundredths make a tenth, how many hundredths make 2 tenth, etc.

Encourage them to use their materials initially while answering the questions.

The words ‘tenth’ and ‘hundredth’ are new for children and teachers need to stress the endings of the words so that children do not confuse them with the words ‘ten’ and ‘hundred’ which they are already familiar with. Point out that ten is 10 units, and that a tenth is less than a whole, it is one part of a whole which has been made into 10 parts. Similarly, point out that hundred is 100 units, whereas a hundredth is one part of a whole which has been made into 100 equal parts.

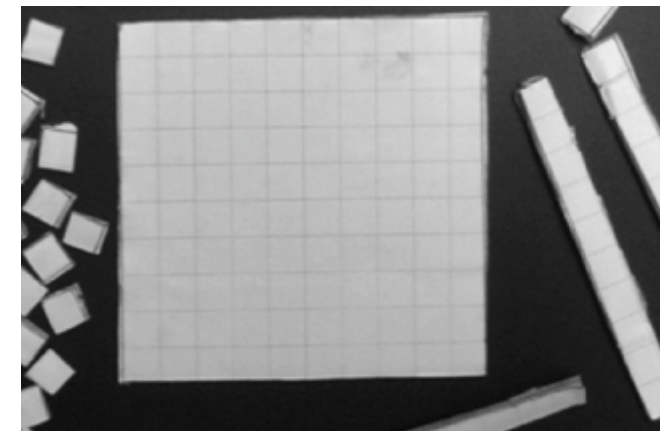
The teacher must emphasize the fact that a tenth and a hundredth are less than a whole. The proper understanding of these relationships paves the way for conceptual understanding of decimals.

You can pose other questions which require summation like:

- How many hundredths are there in 4 tenths and 5 hundredth?
- How many tenths are needed to make 2 wholes? How many tenths are needed to make 3 wholes?
- How many hundredths will make 3 tenth? How many hundredths will make 7 tenth?
- By how much more is 8 hundredth bigger than 2 hundredth?
- How much more is 1 tenth than 1 hundredth?
- How much less is 9 hundredth than 1 tenth?
- How many times a hundredth is 1 tenth?

You may find the children stumbling on questions which differentiate between ‘how much more’ and ‘how many times’ but these are areas we need to consciously strengthen. One requires additive thinking, whereas the other requires multiplicative thinking.

Even at the end of primary school children are not fully conversant with multiplicative thinking and often resort to additive thinking.

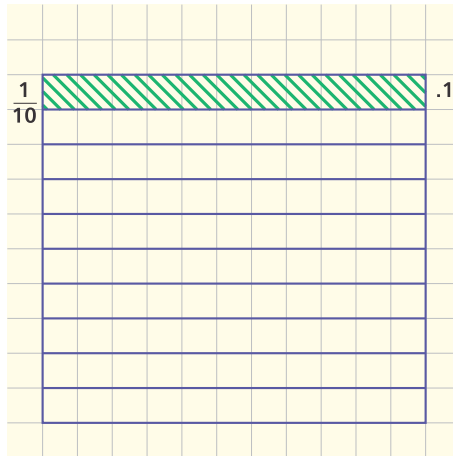


ACTIVITY THREE

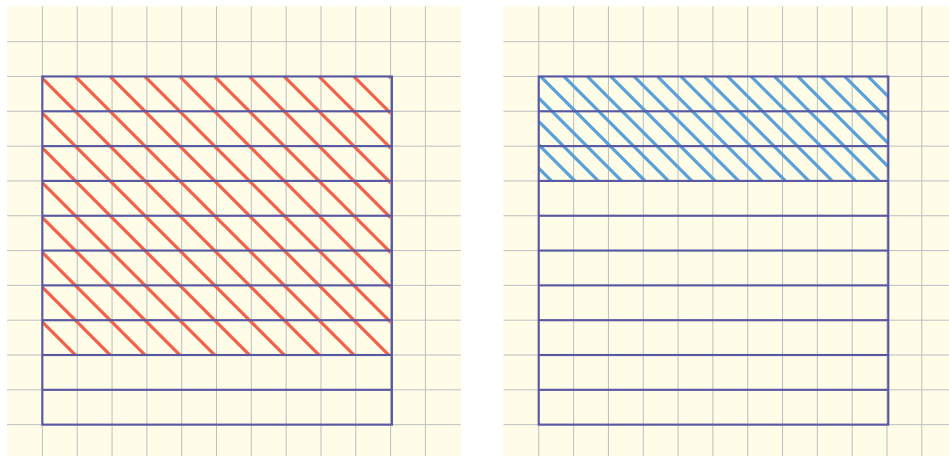
Purpose:
To establish the relationship between fraction and decimal (tenth), to use the symbol and the name.

Materials required:
Square grid paper, colour pencils, Number line

Ask children to outline a 10 x 10 square grid and divide it into 10 equal parts. Ask them to shade one tenth. Let them record the information as shown in the picture.



Give them practice in shading different tenths and recording the information.

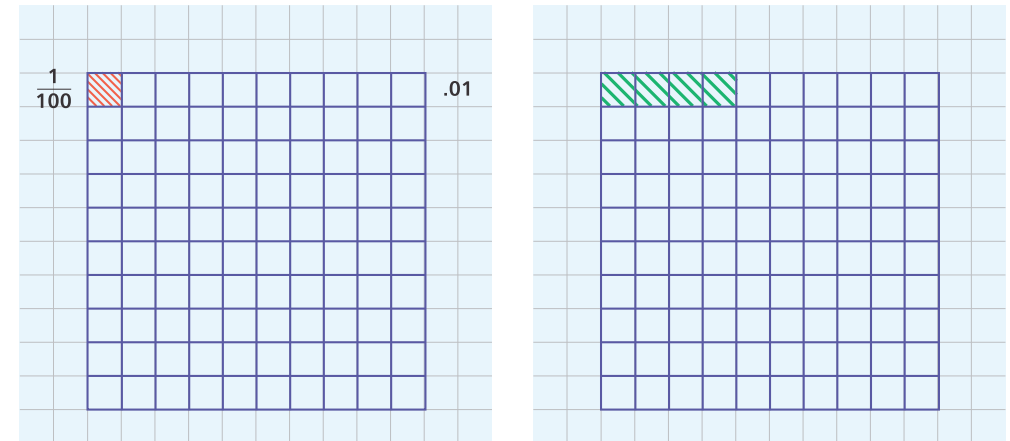


ACTIVITY FOUR

Purpose:
To establish the relationship between fraction and decimal (hundredth), to use the symbol and the name.

Materials required:
Square grid paper, colour pencils

Ask children to outline a 10 x 10 square grid and divide it into 100 equal parts. Ask them to shade one hundredth. Let them record the information as shown in the picture.



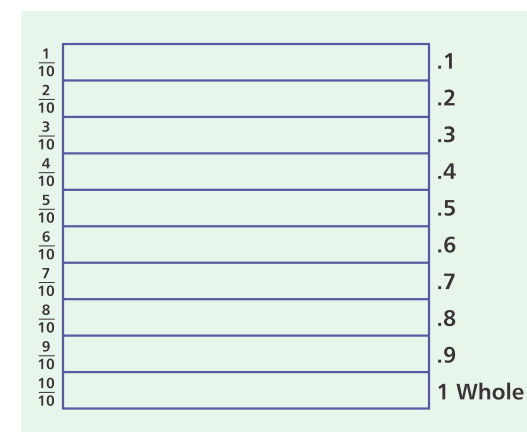
Give them practice in shading different hundredths and recording the information.

ACTIVITY FIVE

Purpose:
To prepare a single grid that consolidates the relationship between fractions and decimals for future reference.

Materials required:
Square grid paper, colour pencils

Ask children to outline a 10 x 10 square grid and divide it into 10 equal parts.



Let them record the information as shown in the picture.

ACTIVITY SIX

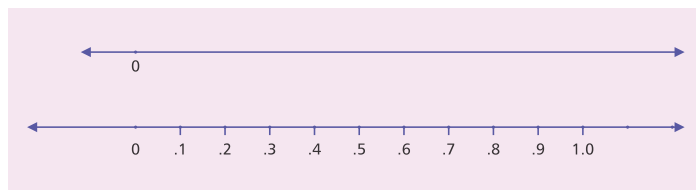
Purpose:

To illustrate that decimal numbers are less than a whole and lie between 0 and 1.

To build estimation skills

Materials required:

Two long number lines as shown in the picture, made from paper or cloth, one showing 1 whole made into 10 equal parts and another without any markings; coloured bulletin board pins.



Initially show the number line which has been divided into parts and point out the different decimal numbers.

Ask children to point different tenths like .1, .5, .9, etc., on the number line.

Similarly you can also make and use a hundredth number line.

Ask children to point different hundredths like .25, .36, .78, etc., on the number line.

ESTIMATION GAME

Turn the number line strip over so that the number line with markings cannot be seen by children.

Now pose the question “Where will .2 be?” Let the child try to estimate where it will lie and mark that part with a pin. Pose more questions of that kind and let different children guess the positions of their numbers.

Now turn up the folded part and children can see for themselves how close their estimates were to the actual. Whoever comes closest to their guess is the winner.

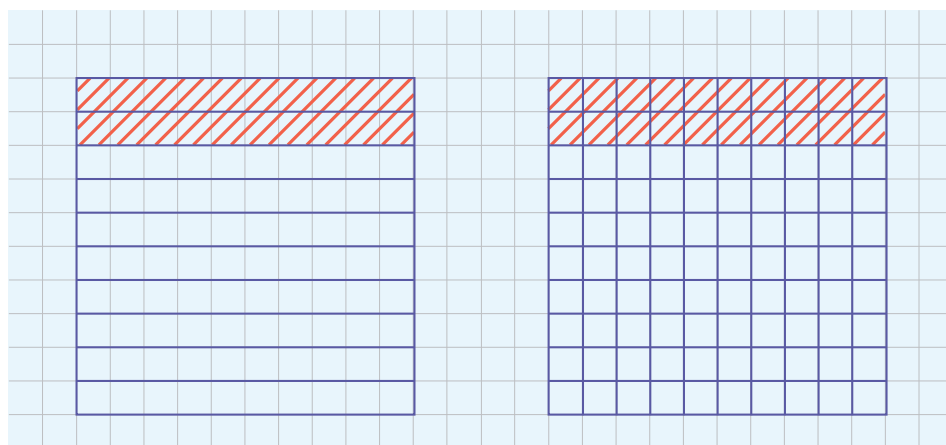
ACTIVITY SEVEN

Purpose:

To compare decimal numbers, to establish the equivalence of numbers like .2 and .20.

Materials required:

Square grid paper, colour pencils

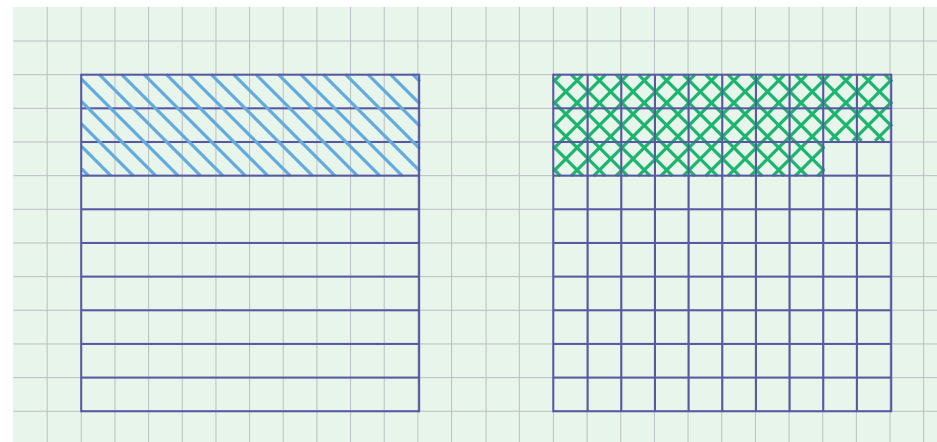


Ask children to shade 2 tenths in the left side grid (of the book) and write the decimal number underneath.

Ask them to shade 2 hundredths in the adjacent grid (right side of the book) and write the decimal number underneath. Let them compare and state the result.

Pose the question “Which is greater, .3 or .28?”

After they respond ask them to shade 3 tenths on one side and 28 hundredths on the other side and verify if their answer was right.



Pose another such question and ask them to justify the answer. Note if they are able to use language correctly. Also check if they are first comparing the tenths and then the hundredths.

Elicit the rule for comparison of decimal numbers from them. Give many exercises which require them to draw and compare to help in the visualization process.

EXTENSION: Now give questions which have mixed decimal numbers.

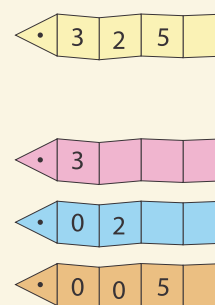
ACTIVITY EIGHT

Purpose:

To show mixed numbers Involving whole and decimal parts.

Materials required:

Square grid sheets, Number cards.



Place the corresponding number cards of a decimal number to show the breakup of these numbers.

Demonstrate a mixed decimal number (involving whole and decimal part) using whole grid, tenths and hundredths as shown in the picture. Record it on a place value chart which can also be made on the black board.

Teach the children how to read the number in different ways, for example: 2.44.

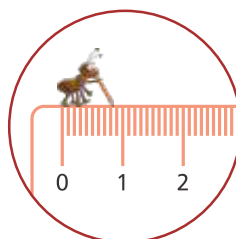
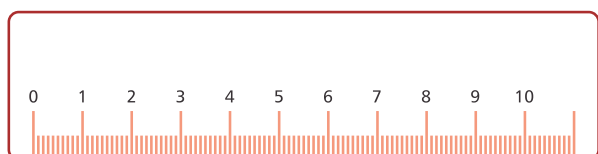
ACTIVITY TEN

Practical activities:

Do plenty of measurement activities and price related activities to give practice in using decimals.

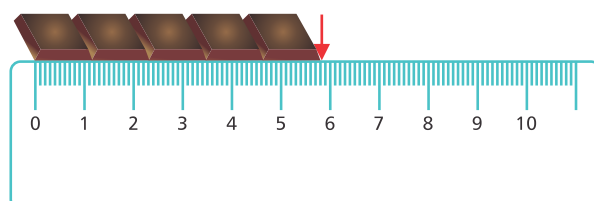
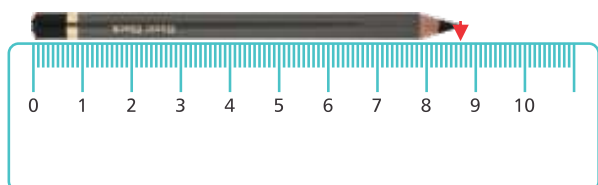
Materials required:

Scale, small objects, money



Measurement provides a natural context to introduce decimals, through the idea of accuracy (i.e., if a length is between 9cm and 10cm, and one is interested in finding the exact measurement, maybe to make a photo frame, one would have to further divide the gap; and 10 seems a convenient number because of our base 10 number system). However measurement doesn't give a conceptual understanding of place value.

Usage of money has also a similar difficulty because the relationship of 100 paise to 1 rupee cannot be represented concretely and still requires abstract understanding. i.e., a 10 paise coin is one tenth of a rupee and could be written as 0.1, and one paise is one hundredth of a rupee and can be written as 0.01. We can discuss that 0.1 and 0.10 both represent the same amount and the misconception that 0.10 is point ten paise can be addressed. (Here the multiplicative property of place value can be explained in more detail.)



Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre, in Rishi Valley School (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com