



Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS



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PULLOUT
RATIO

A VERY BRIEF INTRODUCTION TO FRACTALS

The beautiful image that you see on the cover is a fractal, a recent entrant into the world of mathematics, dating from the 1980s in the work of the mathematician Benoit Mandelbrot, who himself coined the term 'fractal'. Mandelbrot recognised that most objects found in nature cannot be modelled very well by the regular objects we encounter in Euclidean geometry (triangles, rectangles, circles, spheres...). Here are some famous quotes of his about fractals:

- “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”
- “I conceived, developed and applied in many areas a new geometry of nature, which finds order in chaotic shapes and processes. It grew without a name until 1975, when I coined a new word to denote it, *fractal* geometry, from the Latin word for irregular and broken up, *fractus*.”
- “A fractal is a mathematical set or concrete object that is irregular or fragmented at all scales...”

Essentially, the term denotes a set of points whose dimension can be regarded, in a definite and precise sense, as non-integral. One of the very surprising facts about fractals is that using procedures that are extremely simple to describe, sets of extraordinary geometrical complexity can be generated, having a fractal nature. These arise when the procedures are iterated indefinitely; so recursion is a key component of the algorithm. Two such examples are described in the article by Jonaki Ghosh in this issue of AtRiA (the article deals with the use of GeoGebra in generating fractal shapes).

Many fractals have the property of *self-similarity*: they are made up of scaled-down replicas of themselves. A particularly beautiful example of this is the Sierpinski triangle described in the article referred to above. It will be seen that the set consists of three copies of itself, each at half the scale of the original; and each of those copies consists of three copies of itself, each at half its scale; and so on indefinitely. It is precisely this property which makes it possible to describe such shapes in a very compact manner; yet, their complexity is bewildering.

The topic offers a wealth of opportunities for exploration on one's own.



From the Editor's Desk . . .

AtRiA continues its March, July and November pattern with this, the second issue for 2019. Our cover this time is based on Jonaki Ghosh's TechSpace article on Fractals. Seen and admired from afar, and studied with a mathematician's lens, here is the greatest excitement, generating your own fractals using dynamic geometry software.

In Features, we have the second part of Shailesh Shirali's article on e in the fourth of the series on Mathematical Constants. Shashidhar Jagadeeshan invites us to dwell on an illustration of the beauty of mathematics in the first of his series, we focus this time, on Pentagons. In Vignettes of Calculus, Pythagorean surprise uses very simple calculus to explain very strange results. You are certainly going to have rich takeaways from this section.

ClassRoom, this time, has a wide variety of articles. Two articles on π - one which describes the excitement when π enters class 5 - this has accompanying pedagogical notes, and the other, *A Path to π* for a more mature student. Students Ananda Bhaduri and Rahil Miraj share their understanding on *Circles Inscribed in Segments* and *Inradius-Exradius of a Pythagorean Triangle* respectively. Hara Gopal studies a new kind of triplet and how these can be generated in *CuRe Triplets*, Swati Sircar's TearOut has a ready-made worksheet which helps a student investigate the relationships between the area and perimeter of different figures and in Sense-Making, we look at patterns, counting and generalization. Shikha Takker has thrown light on to the study of errors and misconceptions, *Understanding Learner's Thinking* will certainly add to the mathematics teacher's insight into the working of the student's mind.

Are paper and pencil the only tools which are used to solve problems? A Ramachandran thinks not - his Middle School Problem set are *Do It Yourself* and need paper, cello tape, scissors and an inquisitive mind. Usha Krishnamoorthy reviews two websites on Data Handling and tops her review with how she used these in her classroom.

Remember our code given on the Contents Page. A discreet colour band at the top of each article indicates whether it is best suited for Primary (1-5), Middle School (6-8), High School (9-10) or Pre-University (11-12).

Awaiting your feedback on AtRiA.editor@apu.edu.in or on AtRiUM, our FaceBook page.

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Please Note:

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.



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Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

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This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

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Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

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PullOut

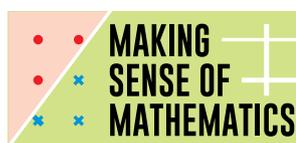
The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

- Padmapriya Shirali
Ratio



Captured Mathematics

Scattered across this issue, we present to you a collection of several interesting photographs which have captured some mathematical concepts.



From Regular Pentagons to the Icosahedron via the Golden Ratio

Part I

SHASHIDHAR
JAGADEESHAN

Introduction

This series of articles will explore an amazing connection between three different objects in mathematics: the regular pentagon, the Golden Ratio and the icosahedron. Obviously, if the Golden Ratio is involved then the Fibonacci sequence can't be far behind and, if the icosahedron is, so is its dual the dodecahedron!

In the first of these articles we will begin with the question of how to construct regular polygons, and restrict our attention to the regular pentagon and some of its properties. The regular pentagon serves as a doorway to a veritable treasure house of interconnected mathematical ideas and it never fails to astonish me.

At the outset I would like to acknowledge that almost all the ideas discussed here can be found in [3]. In this series of articles we expand on some of the ideas and take a few digressions along the way.

Like many things in mathematics, construction of regular polygons comes with a rich history. As a part of school geometry we teach students geometrical constructions. While the standard school geometry box has many instruments like the protractor and set square, the Greeks were particularly interested in those constructions that can be performed using only a straight-edge (a ruler with no markings for measurements) and compass. So for example, many constructions we do in the school syllabus like bisecting an angle and drawing the perpendicular bisector of a given line segment are examples of straight-edge and compass constructions. Apart from some of these easy and obvious constructions it's amazing what one can do with these tools. For example, one can construct a square having the same area as that of a lune, i.e., a plane figure bounded by two circular arcs! For an excellent account and proof of this fact,

Keywords: Regular pentagon, problems of antiquity, ruler and compass construction, golden ratio

I would urge you to read [2]. It is not clear why the Greeks imposed these constraints on themselves. Perhaps it was due to the notion of Plato that the only ‘perfect’ figures were the straight line and the circle, or it was an intellectual game, with very precise rules. It must be noted here that the Greeks themselves did not always restrict themselves to these tools and did not hesitate to use other instruments.

Although the Greeks were extremely successful in many constructions using only a straight-edge and compass, there were some questions that they were not able to resolve. The most famous of these are called **the three problems of antiquity**:

1. *The duplication of a cube*, or the problem of constructing a cube having twice the volume of a given cube.
2. *The trisection of an angle*, or the problem of dividing a given arbitrary angle into three equal parts.
3. *The squaring of a circle*, or the problem of constructing a square whose area is equal to that of a given circle.

Many professional mathematicians, amateurs and cranks have spent several hours trying to solve them and have not been successful, because these problems are impossible to solve! That is right; no matter how smart you are or how hard you work, no matter if you are a Ramanujan or the latest math God, you cannot solve these problems using the straight-edge and compass constraints. Moreover, we can prove that this is so! The notion of impossibility in mathematics is fascinating but to really understand the proofs requires knowledge of some abstract algebra. For those interested please see [4] and [5].

Constructing regular polygons and the regular pentagon in particular

Apart from the three problems of antiquity, the Greeks were also interested in constructing regular polygons. With their ingenuity they were able to construct polygons with 3 sides, 5 sides

and in general, regular polygons with 2^m sides, for $m \geq 2$. They also knew that if they could construct polygons with r sides and s sides, then they could construct one with rs sides, provided r and s are relatively prime (that is, r and s have GCD 1). If we were to list all the regular polygons they could construct with $n < 100$, the list would be regular polygons or n -gons with $n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, 80, 96$. So they had no clue how to construct several regular n -gons, with even small $n = 7, 9, 11, 13, 14, 17, 18, 19!$

One of the reasons Gauss is revered so much is that in 1796, at the age of 19, he demonstrated how to construct a regular polygon of 17 sides using straight-edge and compass! This after nearly 2000 years of stagnation as far as construction of regular polygons was concerned. Gauss showed further that a regular polygon of n sides can be constructed whenever n is prime and is of the form $n = 2^k + 1$ for some integer $k \geq 0$. These primes are actually called *Fermat Primes*. You can check that 17 is one such.

According to the historian Archibald [1], this discovery made a great impression on Gauss.

“But the extraordinary discoveries of Gauss, while yet in his teens, greatly extended this class of polygons and settled for all time the limits of possibilities for such constructions. In this connection the discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses was not only one of which Gauss was vastly proud throughout his life, but also, according to Sartorius von Waltershausen, one which decided him to dedicate his life to the study of mathematics.”

So from our discussion we see that we can construct a regular polygon of n sides provided $n = 2^{i+2}$ for $i \geq 0$ or $n = 2^i p_1 p_2 \cdots p_j$, where the p_i are distinct Fermat primes.

Recall, Fermat primes are primes of the form $n = 2^k + 1$ for some integer $k \geq 0$. The question that naturally comes up is, what if n cannot be

expressed in the above manner? Well! Pierre Wantzel in 1837, proved that if n is not of this form, then it is **impossible** to construct a regular polygon with n sides.

Let us turn our attention to humbler pie! All of you are probably familiar with the construction of an equilateral triangle, a square and a hexagon. The construction of a regular pentagon however is not so straightforward. This is probably because the interior angle of a regular pentagon is 108° , and it is not so easy to construct this angle.

How would we approach such a construction?

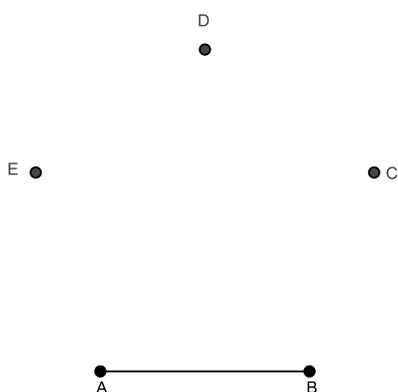


Figure 1

We are given two points A and B and a fixed length $|AB|$ and we need to find points C , D and E , so that $ABCDE$ forms a regular pentagon, that is, all the sides and interior angles are equal. Remember you have only a straight edge and compass. So, we can transfer lengths, draw circles with fixed radii and join two points to form a line. If we think a bit, we will see that if we knew how to construct the diagonal length $|AC|$, we would be done!

We illustrate below:

The point C is the intersection of circle 1 which has centre B and radius $|AB|$ and circle 2 which has centre A and radius the length of the diagonal. We can see now how we can find points D and E using a similar method. Of course we have assumed that the diagonals of a regular pentagon are all equal. We leave this for the reader to prove!

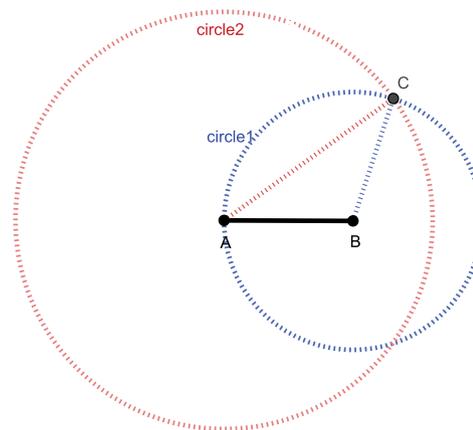


Figure 2

So the problem of constructing a regular pentagon has been reduced to that of constructing the diagonal of a regular pentagon. For the sake of convenience we shall assume that $|AB|$ is of unit length and we shall denote the length of the diagonal by ϕ . So what is ϕ ?

The first step is to figure out all the angles and lengths in the regular pentagon shown in Figure 3.

This follows from the fact that the interior angle of a regular pentagon is 108° and that the triangles $\triangle ABC$, $\triangle ABE$ and $\triangle ECD$ are isosceles. If you stare at Figure 3 long enough you will get it!

We now see that $\triangle ABF \sim \triangle ECD$ giving the following ratio

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

and hence the quadratic equation $\phi^2 - \phi - 1 = 0$.

Using the quadratic formula we find

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

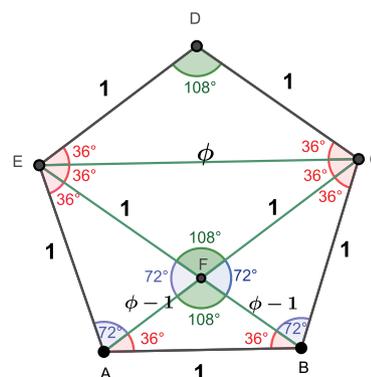


Figure 3

and because $\phi > 0$, we have $\phi = \frac{1+\sqrt{5}}{2}$, the Golden Ratio!

Coming back to the question of constructing a regular pentagon, we said it reduces to that of being able to construct ϕ . We have just shown $\phi = \frac{1+\sqrt{5}}{2}$. Is $\frac{1+\sqrt{5}}{2}$ constructible? In school geometry we learn that if x is constructible then so is \sqrt{x} (if you have not learned this then bug your teacher to show you!), therefore $\sqrt{5}$ is constructible and so is $\frac{1+\sqrt{5}}{2}$ and hence the regular pentagon is constructible!

So the Golden Ratio ϕ has already shown up! Regular readers of AtRiA will be familiar with this most famous ratio in mathematics. We refer them to the March 2013, March 2014, November 2016, March 2017, and July 2017 issues.

In Part II of this article we will return to the Golden Ratio for it will lead us to the Fibonacci sequence and more.

A nested sequence of regular pentagons

Extending all sides of the regular pentagon $ABCDE$ we get a five-star figure $FHJGI$ and then a pentagon $FGHIJ$.

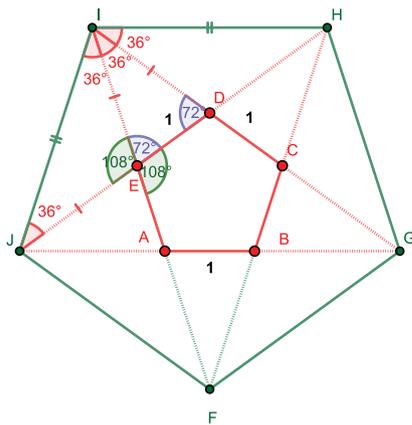


Figure 4

We first establish that the pentagon $FGHIJ$ is a regular pentagon. This is actually not that hard. The first step is to realize that $\angle DEI = 72^\circ = \angle EDI$, because they are both the exterior angles of $\angle AED$ and $\angle CDE$ respectively. This in turn

gives us that $|EI| = |DI|$. By the same argument we can show that $|EI| = |EJ|$ making $\triangle EIJ$ isosceles. Since $\angle AED = \angle JEI = 108^\circ$, and $\triangle EIJ$ is isosceles we have $\angle EIJ = 36^\circ$. By a similar argument we can show that $\angle DIH = 36^\circ$. It is now easy to see that $\angle JIH = 108^\circ$. Further $\triangle JEI \cong \triangle DIH$ by SAS criterion and hence $JI = IH$. We can show similarly that all the sides and angles of the pentagon $FGHIJ$ are equal making it a regular pentagon.

As you might have guessed we can continue this process and get a nested sequence of regular pentagons. The only thing to notice is the orientation is flipped in each iteration.

Since all regular pentagons are similar, the question is what is the scale factor for the sides and diagonals of each new pentagon?

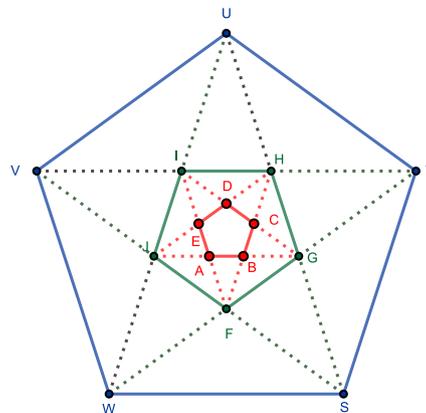


Figure 5

Let us look at Figure 4 again but now with the view of studying the lengths of the sides and diagonals of the pentagons.

From our previous work we know that $\angle BAD = 72^\circ = \angle ABD$ and therefore from the ASA criterion we have $\triangle ABD \cong \triangle EDI$. Hence we see that $|EI| = |DI| = \phi$. By considering $\triangle ABD \cong \triangle CDH$, we get $|DH| = |HC| = \phi$.

Now consider the two triangles $\triangle DIH$ and $\triangle DEC$. They are similar because the corresponding angles are equal. We then have the following ratio:

$$\frac{|HI|}{\phi} = \frac{\phi}{1}$$

and hence $|HI| = \phi^2$. Hence the length of the sides of the new regular pentagon $FGHIJ$ is ϕ^2 . So it looks like each new regular pentagon is scaled by a factor of ϕ^2 .

What about the diagonal of the regular pentagon $FGHIJ$? Since the diagonal of the regular pentagon $ABCDE$ was ϕ the new diagonal should be ϕ^3 . Let us see how we can prove this. There are two ways in which we can see this. The first is to recognize that

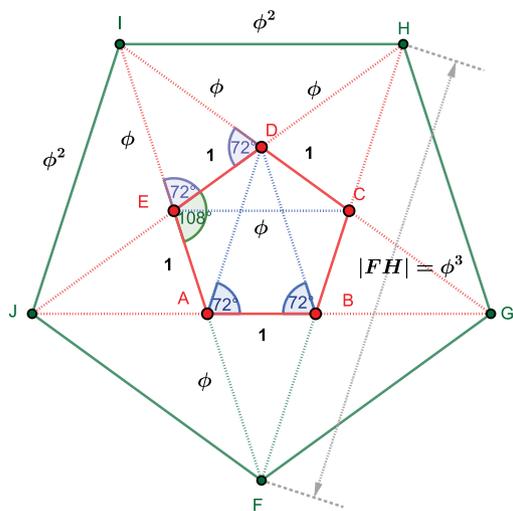


Figure 6

$\triangle IJF \sim \triangle DEC$ and obtain the ratio

$$\frac{|FI|}{\phi^2} = \frac{\phi}{1}$$

yielding $|FI| = \phi^3$. The other way is to use algebra! We know ϕ satisfies the equation $\phi^2 - \phi - 1 = 0$, rearranging we have $\phi^2 = \phi + 1$. Multiplying both sides by ϕ yields $\phi^3 = \phi^2 + \phi$; substituting ϕ^2 with $\phi + 1$ gives $\phi^3 = 2\phi + 1$. Now from $\triangle IHF$ we see $|FI| = |FH| = 2\phi + 1 = \phi^3$!

From the nested sequence of pentagons we have an infinite sequence of side, diagonal, side, diagonal, . . . :

$$1, \phi, \phi^2, \phi^3 \dots$$

In the next part of this article we will see how this sequence, along with a sister sequence, will lead to the famous Fibonacci sequence and as promised, we will see how they all come together in the icosahedron.

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- [5] Raghavan, K.N. *Constructions with ruler and compass: An application of algebra to geometry*. Retrieved from <https://www.imsc.res.in/~knr/130729cmi.pdf>



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The Constants of Mathematics

Part IV

Yet more on the Remarkable Number e

SHAILESH SHIRALI

In this article, we continue our exploration of Euler's constant e .

A maximising number

An often-asked question is the following:

Which number is larger, π^e or e^π ?

There is more to this question than meets the eye. To come upon it, we shall ask a series of similar sounding questions which can be answered quite easily.

- *Which number is larger, $2^{1/2}$ or $3^{1/3}$?*

Since $\text{LCM}(2, 3) = 6$, we answer the question by raising both sides to the 6-th power. We have:

$$\left(2^{1/2}\right)^6 = 2^3 = 8, \quad \left(3^{1/3}\right)^6 = 3^2 = 9.$$

Since $9 > 8$, it follows that $3^{1/3} > 2^{1/2}$.

- *Which number is larger, $3^{1/3}$ or $4^{1/4}$?*

Since $\text{LCM}(3, 4) = 12$, we answer the question by raising both sides to the 12-th power. We have:

$$\left(3^{1/3}\right)^{12} = 3^4 = 81, \quad \left(4^{1/4}\right)^{12} = 4^3 = 64.$$

Since $81 > 64$, it follows that $3^{1/3} > 4^{1/4}$. (Comment. There is an easier way to answer this particular question. See if you can find it!)

Keywords: Derangement, factorial function

- Which number is larger, $4^{1/4}$ or $5^{1/5}$?

Since $\text{LCM}(4, 5) = 20$, we answer the question by raising both sides to the 20-th power. We have:

$$\left(4^{1/4}\right)^{20} = 4^5 = 1024, \quad \left(5^{1/5}\right)^{20} = 5^4 = 625.$$

Since $1024 > 625$, it follows that $4^{1/4} > 5^{1/5}$.

- Which number is larger, $5^{1/5}$ or $6^{1/6}$?

Since $\text{LCM}(5, 6) = 30$, we answer the question by raising both sides to the 30-th power. We have:

$$\left(5^{1/5}\right)^{30} = 5^6 = 15625, \quad \left(6^{1/6}\right)^{30} = 6^5 = 7776.$$

Since $15625 > 7776$, it follows that $5^{1/5} > 6^{1/6}$.

We can continue in this manner. The operations are easy to perform, though the numbers get progressively larger.

After a while, we begin to suspect that *the sequence $\{n^{1/n}\}_{n \geq 3}$ is strictly decreasing*; in other words, that

$$3^{1/3} > 4^{1/4} > 5^{1/5} > 6^{1/6} > 7^{1/7} > 8^{1/8} > \dots \quad (1)$$

If this is true, then *the largest value taken by $n^{1/n}$ for positive integers n is $3^{1/3}$* . The statement **is** true, but we postpone the proof to the end of this section.

Extending the search to half integers. Using the above finding as a starting point, we take up the following exploration. *For which half integer $x > 0$ does $x^{1/x}$ take its largest value?* (A ‘half integer’ is a number of the form $n/2$ where n is an integer.) As earlier, we ask a series of similar sounding questions.

- Which number is larger, $1.5^{1/1.5}$ or $2^{1/2}$?

Since $\text{LCM}(1.5, 2) = 6$, we answer the question by raising both sides to the 6-th power. We have:

$$\left(1.5^{1/1.5}\right)^6 = \left(\frac{3}{2}\right)^4 = \frac{81}{16}, \quad \left(2^{1/2}\right)^6 = 2^3 = 8.$$

Since $8 > 81/16$, it follows that $2^{1/2} > 1.5^{1/1.5}$.

- Which number is larger, $2^{1/2}$ or $2.5^{1/2.5}$?

Since $\text{LCM}(2.5, 2) = 10$, we answer the question by raising both sides to the 10-th power. We have:

$$\left(2^{1/2}\right)^{10} = 2^5 = 32, \quad \left(2.5^{1/2.5}\right)^{10} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}.$$

Since $625/16 > 32$, it follows that $2.5^{1/2.5} > 2^{1/2}$.

- Which number is larger, $2.5^{1/2.5}$ or $3^{1/3}$?

Since $\text{LCM}(2.5, 3) = 15$, we answer the question by raising both sides to the 15-th power. We have:

$$\left(2.5^{1/2.5}\right)^{15} = \left(\frac{5}{2}\right)^6 = \frac{15625}{64}, \quad \left(3^{1/3}\right)^{15} = 3^5 = 243.$$

Since $15625/64 > 243$, it follows that $2.5^{1/2.5} > 3^{1/3}$. (Note the numbers involved; $15625/64 \approx 244.14$, which means that $3^{1/3}$ has only ‘just’ been beaten. A narrow victory for $2.5^{1/2.5}$...)

- Which number is larger, $3^{1/3}$ or $3.5^{1/3.5}$?

Since $\text{LCM}(3, 3.5) = 21$, we answer the question by raising both sides to the 21-st power. We have:

$$\left(3^{1/3}\right)^{21} = 3^7 = 2187, \quad \left(3.5^{1/3.5}\right)^{21} = \left(\frac{7}{2}\right)^6 = \frac{117649}{64}.$$

Since $2187 > 117649/64$, it follows that $3^{1/3} > 3.5^{1/3.5}$.

Continuing, we begin to suspect that *the largest value taken by $n^{1/n}$ for half integers $n > 0$ is $2.5^{1/2.5}$* . This statement too is true, but as earlier we postpone the proof.

Extending the search to quarter integers. We get more ambitious now and extend the exploration to ‘quarter integers’, i.e., numbers of the form $n/4$ where n is an integer. We ask: *For which quarter integer $x > 0$ does $x^{1/x}$ take its largest value?* Thus:

- Which number is larger, $2.25^{1/2.25}$ or $2.5^{1/2.5}$?

Since $\text{LCM}(2.25, 2.5) = 45$, we answer the question by raising both sides to the 45-th power. The computations are messy but they are straightforward and they show that $2.5^{1/2.5} > 2.25^{1/2.25}$.

- Which number is larger, $2.5^{1/2.5}$ or $2.75^{1/2.75}$?

Since $\text{LCM}(2.25, 2.75) = 110$, we answer the question by raising both sides to the 110-th power. The computations are *extremely* messy but they do show that $2.75^{1/2.75} > 2.5^{1/2.5}$.

Now we suspect that *the largest value taken by $n^{1/n}$ for quarter integers $n > 0$ is $2.75^{1/2.75}$* . The statement is true, yet again.

Full generalisation. We are now ready to ask the question in its most general form: *For which positive real number x does $x^{1/x}$ assume its largest value?* For this, we make use of differentiation. In particular, we use the fact that the derivative of $\ln x$ is $1/x$. Let $y = x^{1/x}$; here $x > 0$. Then $\ln y = \frac{1}{x} \cdot \ln x$. Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{x \cdot (1/x) - \ln x \cdot 1}{x^2}, \\ \therefore \frac{dy}{dx} &= \frac{y(1 - \ln x)}{x^2}. \end{aligned} \quad (2)$$

From the expression in the last line, it follows that the slope of the curve $y = x^{1/x}$ is positive when $\ln x < 1$, i.e., when $0 < x < e$, and negative when $\ln x > 1$, i.e., when $x > e$. Hence $x^{1/x}$ increases in value as x arises from 0^+ to e and decreases in value thereafter. Therefore: *$x^{1/x}$ assumes its largest value when $x = e$.*

This is an unexpected and most pleasing result; a bonus, in fact!

Justifying some statements made earlier. We are now in a position to justify some of the statements made above on the basis of numerical experimentation. For example, consider the claim that $3^{1/3}$ is the largest value of $n^{1/n}$ for positive integers n . To prove this, we make use of the fact, just proved, that $x^{1/x}$ increases in value as x arises from 0^+ to e and decreases in value thereafter. This statement implies that

$$1^{1/1} < 2^{1/2}, \quad 3^{1/3} > 4^{1/4} > 5^{1/5} > 6^{1/6} > 7^{1/7} > \dots \quad (3)$$

It follows from these inequalities that the only possible candidates for the positive integers n for which $n^{1/n}$ assumes its maximum value are $n = 2$ and $n = 3$. But we already know that $3^{1/3} > 2^{1/2}$. It follows that $3^{1/3}$ is the maximum value of $n^{1/n}$ for positive integers n .

Comment. In as much as e is a fundamental constant of mathematics, it follows from the above that $e^{1/e}$ too may be considered to be a fundamental constant of mathematics. Later in this series of articles, we may be able to dwell on this particular constant.

Solution to the problem posed earlier. We are now in a position to answer the question posed earlier: Which number is larger, π^e or e^π ? Since $e^{1/e}$ is the maximum possible value of $x^{1/x}$ for $x > 0$, it follows that

$$e^{1/e} > \pi^{1/\pi}.$$

It follows immediately from this that

$$e^\pi > \pi^e. \tag{4}$$

Derangements

We close our series of articles featuring e with a discussion of a combinatorial problem: that of counting ‘derangements’.

Consider the string of numbers $(1, 2, 3, \dots, n)$, where n is any positive integer. The n numbers in the string can be arranged in $n! = 1 \times 2 \times 3 \times \dots \times n$ different ways. For example, for $n = 3$, the $3! = 6$ arrangements of $(1, 2, 3)$ are the following: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

Some of these arrangements have the property that *every number in the string is in the ‘wrong’ place*; i.e., it has been shifted relative to its original location. An arrangement that has this feature is known as a **derangement**.

Among the permutations of $(1, 2, 3)$, there are two derangements: $(2, 3, 1)$ and $(3, 1, 2)$. Please verify for yourself that in each of the remaining permutations, there is at least one element in its original location. For example, in $(1, 3, 2)$, element 1 is in its original location, and in $(3, 2, 1)$, element 2 is in its original location; so these two are not derangements.

For any given value of n , how many of the $n!$ permutations are derangements? It is far from obvious how we may compute this number. The number itself is considered to be of sufficient significance that it has been assigned a special symbol, D_n . Mathematicians call it the **derangement number**. We clearly have $D_1 = 0$ and $D_2 = 1$, and we have already found that $D_3 = 2$.

Let us now find the value of D_4 . Since element 1 must not be in its original location, it must be in one of the positions 2, 3, 4. There are thus 3 possible locations for element 1. If we now compute the number of derangements in which element 1 is in position 2, then, by symmetry, D_4 will be 3 times this number. (It stands to reason, surely, that the number of derangements in which element 1 is in position 2 will be equal to the number of derangements in which element 1 is in position 3.) This number is easy to find. In position 1, we must have 2 or 3 or 4. It turns out that each of these possibilities leads to just one possible derangement.

- If element 2 is in position 1, then the derangement must be $(2, 1, 4, 3)$.
- If element 3 is in position 1, then the derangement must be $(3, 1, 4, 2)$.
- If element 4 is in position 1, then the derangement must be $(4, 1, 2, 3)$.

Thus there are 3 derangements in which element 1 is in position 2. It follows from what we said earlier that $D_4 = 3 \times 3 = 9$.

Now let us see how to find the value of D_5 . The possibilities are many more now, so we will have to be more skillful in our reasoning.

Since element 1 must not be in its original location, it must be in one of the positions 2, 3, 4, 5. There are thus 4 possible locations for 1. If we now compute the number of derangements in which element 1 is in position 2, then, by symmetry (as earlier), D_5 will be 4 times this number.

Which element could be in position 1? Clearly it must be one of 2, 3, 4, 5.

If element 2 has come to position 1, then it means that 1 and 2 have swapped positions. To complete the derangement, elements 3, 4, 5 must occupy positions 3, 4, 5; but all in the 'wrong place'. The number of such derangements is clearly D_3 . (To see why, simply rename 3, 4, 5 as 1, 2, 3.)

Next, suppose that element 2 is not in position 1. Look at the array below carefully.

<i>Element</i>		1			
<i>Constraint</i>	Not 2		Not 3	Not 4	Not 5

Observe that elements 2, 3, 4, 5 must be placed in the four empty boxes in the first row, keeping in mind the constraints listed in the second row. *But this is exactly the same as the 4-element derangement problem!* (To see why, rename 2, 3, 4, 5 as 1, 2, 3, 4 respectively.) But this means that the number of possibilities is exactly D_4 .

We see that among the derangements in which element 1 occupies position 2, there are D_3 derangements in which element 2 occupies position 1, and D_4 derangements in which element 2 does not occupy position 1. Therefore, there are $D_3 + D_4$ derangements in all in which element 1 occupies position 2.

Arguing as earlier, there are just as many derangements in which element 1 occupies position 3, and so on. It follows that

$$D_{5*} = 4 \cdot (D_3 + D_4),$$

i.e., $D_5 = 4 \times (2 + 9) = 44$.

A general recursive relation. The argument described above generalises easily. Let n be a positive integer. Consider the derangements of $(1, 2, 3, \dots, n)$. Arguing as earlier, we find that the number of derangements is equal to $(n - 1)$ times the number of derangements in which element 1 occupies position 2.

Next, we subdivide the derangements in which element 1 occupies position 2 into two categories: those in which element 2 occupies position 1, and those in which element 2 occupies some other position. Repeating the argument made earlier, we deduce that there are D_{n-2} derangements of the first kind and D_{n-1} derangements of the second kind. Hence there are $D_{n-2} + D_{n-1}$ derangements in all in which element 1 occupies position 2. It follows that

$$D_n = (n - 1) \cdot (D_{n-2} + D_{n-1}). \tag{5}$$

This kind of relation, in which the terms of a sequence are expressed in terms of the preceding terms by some formula, is termed a **recursion relation**. The most well-known such relation is, of course, that obeyed by the Fibonacci sequence.

The above relation permits us to compute the derangement numbers very easily. We give below the first few terms of the sequence of derangement numbers.

n	1	2	3	4	5	6	7	8	9	10	...
D_n	0	1	2	9	44	265	1854	14833	133496	1334961	...

Connection with e . At this point, the reader must be impatiently asking: All this is very well, but what do the derangement numbers have to do with e ? Why is this topic being discussed in an article about e ? Well, here comes the connection

As noted above, the formula that we have found for D_n is a recursion relation; in order to find any particular derangement number, we would need to compute all the derangement numbers that precede it. But we may want a formula for D_n that computes its value directly in terms of n and not in terms of the preceding derangement numbers. Is there such a formula? There are indeed many such formulas, and perhaps the most surprising of these is the following:

$$D_n = \text{the integer closest to } \frac{n!}{e}. \quad (6)$$

The following array shows this formula in action.

n	1	2	3	4	5	6	7	8	...
D_n	0	1	2	9	44	265	1854	14833	...
$n! / e$	0.37	0.74	2.21	8.83	44.15	264.87	1854.11	14832.9	...

The formula is indeed an astonishing one. Note that it says ‘integer closest to’. This means that it is not a case of uniformly rounding down or uniformly rounding up. You may well wonder how one might ever prove such a formula!

But it isn’t as hopeless as that. Let us see how far we can get by using the recursion relation found above. Define a new sequence $f(n)$ by the relation

$$f(n) = \frac{D_n}{n!}.$$

From this we get $D_n = n! \cdot f(n)$. The relation $D_n = (n-1) \cdot (D_{n-2} + D_{n-1})$ when rewritten in terms of the f -sequence yields the following:

$$\begin{aligned} n!f(n) &= (n-1) \cdot \left((n-1)! \cdot f(n-1) + (n-2)! \cdot f(n-2) \right) \\ &= (n! - (n-1)!) \cdot f(n-1) + (n-1)! \cdot f(n-2), \\ \therefore f(n) &= \left(1 - \frac{1}{n} \right) f(n-1) + \frac{1}{n} \cdot f(n-2), \\ \therefore f(n) - f(n-1) &= -\frac{1}{n} \cdot \left(f(n-1) - f(n-2) \right). \end{aligned} \quad (7)$$

We have discovered something quite remarkable. Let the sequence $u(n)$ be defined as follows:

$$u(n) = f(n) - f(n-1), \quad n \geq 2. \quad (8)$$

Then we have proved the following:

$$u(n) = -\frac{1}{n} \cdot u(n-1). \quad (9)$$

This relation enables us to find a formula for $u(n)$ in terms of n . From this, we will find a formula for $f(n)$ in terms of n .

We first compute the initial values of the f -sequence, and from these the initial values of the u -sequence. The computations in the array below are self-explanatory

n	1	2	3	4	5	6	7	8	...
D_n	0	1	2	9	44	265	1854	14833	...
$f(n)$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{11}{30}$	$\frac{53}{144}$	$\frac{103}{280}$	$\frac{2119}{5760}$...
$u(n)$		$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$	$-\frac{1}{5040}$	$\frac{1}{40320}$...

It is easy to guess from the last line that

$$u(n) = \frac{(-1)^n}{n!}, \quad (10)$$

and this may be proved using relation (9) and the principle of induction. We shall leave the details of the proof to the reader. (The proof is straightforward.)

From (10) and (8), we can prove (again, using induction) that

$$f(n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}. \quad (11)$$

As earlier, we shall leave the details of the proof to the reader.

Since $f(n) = D_n/n!$, relation (11) yields a formula for the n -th derangement number:

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right). \quad (12)$$

Now we begin to see the connection with e . We are familiar with the exponential series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

from which we obtain (by putting $x = -1$) the following infinite series for $1/e$:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots. \quad (13)$$

It follows from (12) and (13) that $D_n \approx n! / e$ for large values of n . We could stop here, but we wish to prove the stronger statement that D_n is equal to the integer closest to $n! / e$ for all positive integers n . More work is needed to establish this.

The infinite series in (13) is an example of an *alternating series whose terms decrease in absolute value and converge to 0*. (The terms alternate in sign, hence the name. A series with these features always converges. See [2].) For such a series, it is known that the limiting sum always lies between two consecutive partial sums. To illustrate what this means, consider the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots,$$

whose limiting sum is $1/(1 + 1/2) = 2/3$. The partial sums of the above series are:

$$1, \quad \frac{1}{2}, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{11}{16}, \quad \frac{21}{32}, \quad \frac{43}{64}, \quad \cdots$$

Observe that $2/3$ lies between each pair of consecutive partial sums.

To prove that $D_n =$ the integer closest to $n! / e$, we reason as follows. We have:

$$\frac{n!}{e} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots \right),$$

which is true by relation (13). This may be rewritten, using (12), as

$$\frac{n!}{e} = D_n + a_n, \quad (14)$$

where

$$a_n = \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \frac{(-1)^{n+3}}{(n+1)(n+2)(n+3)} + \cdots. \quad (15)$$

The series in (15) is an alternating series whose terms decrease in absolute value and tend to 0. This implies that the limiting sum a_n is less than $1/(n+1)$ in absolute value. Since $n \geq 1$, it follows that $|a_n| < 1/2$ for all values of n . Since

$$D_n = \frac{n!}{e} + \text{a quantity which lies strictly between } -\frac{1}{2} \text{ and } \frac{1}{2} \quad (16)$$

and D_n is an integer, the desired conclusion follows: that D_n is equal to the integer closest to $n! / e$.

A few 'crazy' results

To bring this three-part series of articles on e to a close, we quote (without proof) a few fascinating results concerning e . Some of them are so strange that we can only call them crazy.

Arithmetic mean, geometric mean. We all know that the arithmetic mean (AM) of any collection of positive numbers is greater than or equal to the geometric mean (GM) of that collection. Let us apply this well-known statement to the collection of numbers $1, 2, 3, \dots, n$, where n is any positive integer. Let A_n and G_n denote the AM and the GM respectively of this collection. For example, $A_4 = 5/2$ and $G_4 = 24^{1/4}$. How do A_n and G_n compare with each other when n is extremely large? Here is the remarkable finding:

$$\lim_{n \rightarrow \infty} \frac{A_n}{G_n} = \frac{e}{2}.$$

That's crazy, isn't it?

Two infinite products for e . To conclude, we exhibit two infinite products for e :

$$e = \frac{2}{1} \left(\frac{4}{3} \right)^{1/2} \left(\frac{6 \cdot 8}{5 \cdot 7} \right)^{1/4} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15} \right)^{1/8} \cdots,$$

$$e = 2 \left(\frac{2}{1} \right)^{1/2} \left(\frac{2 \cdot 4}{3 \cdot 3} \right)^{1/4} \left(\frac{4 \cdot 6 \cdot 6 \cdot 8}{5 \cdot 5 \cdot 7 \cdot 7} \right)^{1/8} \cdots.$$

Just as crazy! The first of these was proved by Catalan. For the second product, please see [1].

References

1. N. Pippenger, An Infinite Product for e , *American Mathematical Monthly* 87(5) (1980), 391.
2. Wikipedia, Alternating series test, https://en.wikipedia.org/wiki/Alternating_series_test



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The tree branches in 3 at each node,
counting from the bottom, it is possible
to see 3, 9, 27... branches.



Powers of 3: Tree at Azim Premji University, PES College Campus

Photo & Ideation: *Swati Sircar*

A Counter-intuitive PYTHAGOREAN SURPRISE

SHAILESH SHIRALI

The following is sometimes presented as a phenomenon in mathematics that goes counter to our intuition. Imagine a tightly stretched rope from one end of a field to another, tied down at its ends. For simplicity, we take the length of the field to be 100 m; so the length of the rope is 100 m. Now replace this rope by one that is slightly longer, say by 20 cm. There is some slack in the rope, so we should be able to lift the midpoint of the rope to some height (Figure 1). Imagine pinching the rope at its midpoint and raising the rope till it is taut. Question: *To what height can you raise the midpoint?* Try to estimate the answer without doing any computations; what does your intuition tell you?



Figure 1

The answer may be obtained by an application of the Pythagorean theorem. After going through the steps, a surprise awaits us. Let h be the height of the midpoint; then we have:

$$h^2 + 50^2 = 50.1^2,$$
$$\therefore h = \sqrt{50.1^2 - 50^2} = \sqrt{10.01},$$

which gives $h \approx 3.16$. So the height of the midpoint is roughly 3.16 m (which means that the rise in the midpoint is more than 15 times the increase in length of the rope). That is high enough for a tall person riding a large horse to go underneath the rope quite comfortably, without having to duck! (Was your guess anywhere close to that?)

Keywords: Pythagoras, intuition

How can this be? Is there some way of explaining this very high ratio, some way of understanding this counter-intuitive phenomenon? Here is one perspective which may help us understand the situation better.

Comment. In passing, we ask what it means to ‘explain’ a phenomenon in mathematics. One may try to explain a natural phenomenon by appealing to basic principles of physics or chemistry or biology (for example, we may try to explain some aspect of crystals using quantum mechanics; or we may try to explain some aspect of animal behaviour using the theory of evolution and natural selection), but what parallel does this have in mathematics? Some may argue that the calculation shown above is explanation enough! In one sense this is so; but the counter-intuitive nature of the answer is not to be denied, and what we are looking for are the *features* of this setup which underlie the counter-intuitive behaviour. If we are able to identify these features, then we may be able to predict other situations where a similar behaviour occurs.

Explanation. We consider the following function $f(h)$, defined for $h \geq 0$: if the length of the rope is increased from 100 to $100 + 2h$ and the rope is pulled upwards at its midpoint till it is taut, then the height of the midpoint is $f(h)$. We clearly have:

$$f(h) = \sqrt{(50 + h)^2 - 50^2},$$

i.e.,

$$f(h) = \sqrt{100h + h^2}, \tag{1}$$

Its graph is shown in Figure 2.

Now observe the following:

$$\frac{f(h)}{h} = \frac{\sqrt{100h + h^2}}{h} = \frac{\sqrt{100 + h}}{\sqrt{h}}$$

$$\approx \frac{10}{\sqrt{h}} \text{ for } h \text{ close to } 0. \tag{2}$$

This shows that for values of h very close to 0, the value of $f(h)/h$ will be very large.

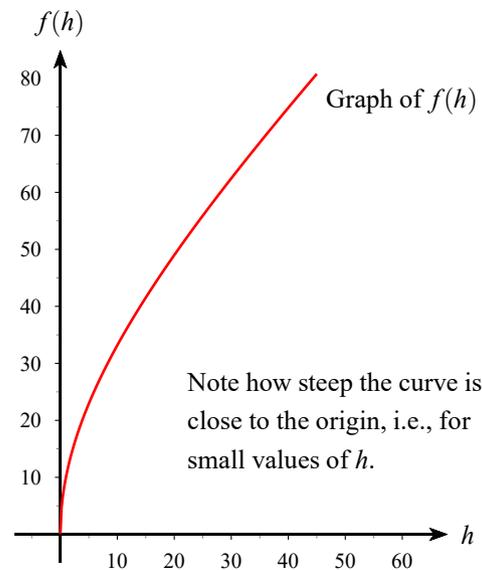


Figure 2. Graph of $f(h) = \sqrt{(50 + h)^2 - 50^2}$ for $0 \leq h \leq 50$

In our situation, we had $h = 0.1$, so $f(h) / h \approx 10 / \sqrt{0.1} \approx 31.6$. So the rise in height of the midpoint of the rope is more than 15 times the increase in length of the rope. (Recall the definition of h ; the increase in length of the rope is $2h$, not h .) What we have just found is consistent with what we computed earlier.

If the rope is increased in length from 100 m to 100.02 m (i.e., by 2 cm), then we have $h = 0.01$ and therefore $f(h) / h \approx 10 / \sqrt{0.01} = 100$; the rise in height of the midpoint is now 50 times the increase in length of the rope.

A viewpoint from the Mean Value Theorem. The ‘explanation’ given above may seem adequate, but some readers will appreciate the following additional perspective.

We have noted visually how steep the curve is close to the origin, i.e., for small values of h . This can be seen algebraically by computing the derivative of f :

$$f'(h) = \frac{1}{2\sqrt{100h + h^2}} \times (100 + 2h)$$

$$= \frac{50 + h}{\sqrt{100h + h^2}} \tag{3}$$

From (3) it is easy to see what happens as h tends to 0 (from the positive side):

$$\text{As } h \rightarrow 0^+, f'(h) \rightarrow \infty$$

This tells us that for values of h close to 0, the slope $f'(h)$ assumes large values. The graph affirms this observation.

Next, note that by Lagrange's Mean Value Theorem, for any $h > 0$, there exists a number t , $0 < t < 1$, such that

$$\frac{f(h) - f(0)}{h - 0} = f'(0 + th). \quad (4)$$

Here $f(0) = 0$. Hence for any $h > 0$, there exists a number t , $0 < t < 1$, such that

$$\frac{f(h)}{h} = f'(th).$$

This means that for $0 < h \ll 1$, the value of $f(h)/h$ will be large. This is consistent with the observation made earlier. For the specific numbers used: $h = 0.1$, $f(h) \approx 3.16$, and

$$\frac{f(0.1)}{0.1} \approx 31.6. \quad (5)$$

For the sake of completeness, let us find the value of t for which relation (4) holds, with $h = 0.1$.

We find, repeatedly using the approximation

$$\frac{f(h)}{h} \approx \frac{10}{\sqrt{h}} \text{ for } h \approx 0:$$

$$f'(0.05) \approx 22.38, \quad f'(0.03) \approx 28.88,$$

$$f'(0.025) \approx 31.63, \quad f'(0.02) \approx 35.37,$$

showing that for $h = 0.1$, (4) holds with $t \approx 1/4$.

This value lies between 0 and 1, as it is meant to. So our findings are consistent with the claim made by Lagrange's theorem.

Closing remark. We had stated at the start that in looking for an 'explanation' of this phenomenon, what we are looking for are the features that make the phenomenon possible. Now we are in a position to answer this question. *Essentially, the phenomenon happens when we have a differentiable function whose derivative at 0 is infinite or extremely large.* Armed with this insight, we should be able to find other functions that behave similarly.

It is worth noting that this insight comes as a result of using derivatives and invoking the mean value theorem. The use of such heavy machinery may not have seemed warranted at the start. Note, however, that it has yielded an insight which we may not have got if we had stuck to a pre-calculus approach.



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Student Corner – Featuring articles written by students.

Circles Inscribed in Segments

ANANDA BHADURI

As I was going through Evan Chen's *Euclidean Geometry for Mathematical Olympiads*, I came across this remarkable problem.

We are given that ω is a circle with centre O . AB is a chord of ω . If a circle is tangent to ω at P (internally) and tangent to AB at Q , prove that P, Q and the midpoint of the arc \widehat{AB} not containing P are collinear.

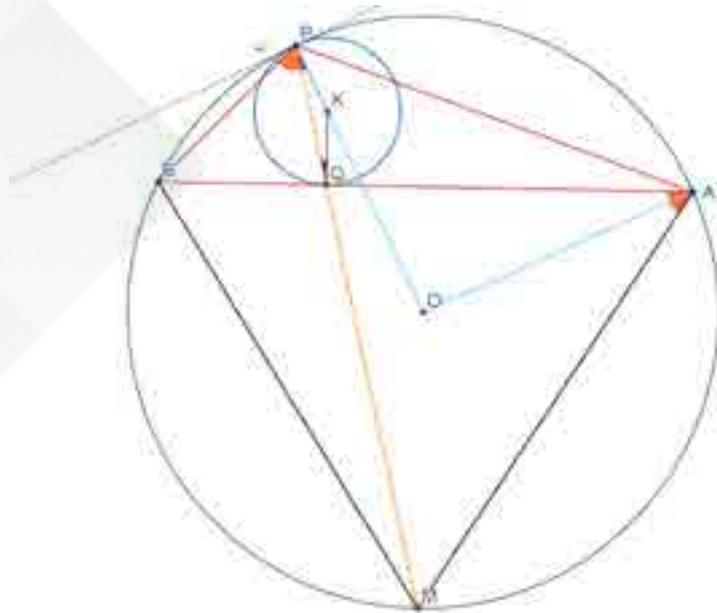


Figure 1

I start solving this problem by reasoning backwards. Once I have reduced it into a much more convenient form, I will present a proof.

Keywords: Circles, angles, chord, tangent, power of a point, orthogonal circles

Let PQ meet the arc \widehat{AB} not containing P at M. We need to prove that M is the midpoint of the arc AB not containing P. That is, we need to prove that

$$MA = MB.$$

This reduces to proving that $\angle MAB = \angle MBA$. Now, we use the fact that angles in the same segment are equal. Since $\angle MAB = \angle MPB$ and $\angle MBA = \angle MPA$, it is sufficient to prove that

$$\Leftrightarrow \angle MPB = \angle MPA \quad \dots\dots\dots(1)$$

We will prove (1) below

Proof:

Observe that O, P and K are collinear, where K is the centre of the inscribed circle, O is the centre of the outer circle and P is the point of contact.

Since the circle inscribed in the segment is tangent to AB at Q,

$$\angle KQA = \angle KQB = 90^\circ \quad \dots\dots\dots(2)$$

We compute $\angle MPA$ and show that it must be equal to $\angle MPB$.

$\angle MPA$ is the sum of the two angles $\angle MPO$ and $\angle OPA$.

$$\begin{aligned} \angle MPO & \text{ is nothing but } \angle KPQ \\ & = \angle KQP && \text{(KP = KQ)} \\ & = 90^\circ - \angle PQB && \text{(from (2))} \\ & = 90^\circ - \angle AQM \\ \angle OPA & = \frac{180^\circ - \angle POA}{2} && \text{(from the isosceles triangle POA)} \\ & = 90^\circ - \angle PMA && (\angle POA = 2 \angle PMA) \end{aligned}$$

Adding these two values,

$$\begin{aligned} \angle MPA & = \angle MPO + \angle OPA \\ & = 180^\circ - (\angle AQM + \angle PMA) \\ & = \angle MAB \\ & = \angle MPB && \text{(angles in the same segment)} \end{aligned}$$

And this proves (1).

Now, we come to our first corollary:

The length of the tangent from M to any circle inscribed in segment AB is equal to MA.

We begin by using the fact that triangles MPA and MAQ are similar. This is because $\angle MPA = \angle MAB = \angle MAQ$ (as proved earlier) and $\angle PMA = \angle QMA$.

Similarity yields the following ratio:

$$\frac{MQ}{MA} = \frac{MA}{MP}$$

This leads to:

$$MP \cdot MQ = MA^2$$

Observe that $MP \cdot MQ$ is the **power of point M** with respect to the inscribed circle. (The definitions of words written in bold can be found in the appendix.) But the power of a point of a point outside the circle is equal to the square of the length of the tangent from the point to the said circle.

If the length of the tangent from M to the inscribed circle is t . Then,

$$MA^2 = MP \cdot MQ = t^2$$

What does this result imply? *The length of the tangent from M to the circle is fixed by the segment AB and does not depend on the position of the circle within the segment.*

Now for our second corollary:

Let γ be the circle passing through A and B with centre M and radius MA or MB. Let γ intersect the inscribed circle at X and Y and intersect PQ at I. Then,

- i. *MX and MY are tangent to the inscribed circle.*
- ii. *I is the incentre of triangle PAB.*

The first part is the first corollary in disguise. In fact, the inscribed circle and γ are **orthogonal circles**.

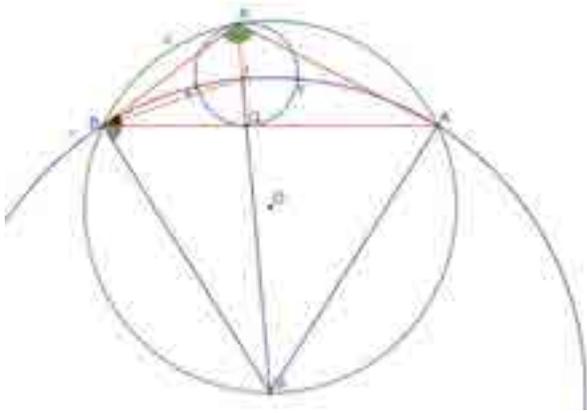


Figure 2

What about the second part? Since I lies on PM (which is the bisector of $\angle APB$), it is sufficient to prove that IB is the bisector of $\angle PBA$.

Since $MB = MI$,

$$\angle MBI = \angle MIB \quad (2)$$

$$\begin{aligned} \angle MBI &= \angle MBA + \angle ABI \\ &= \angle MPA + \angle ABI \\ &\quad \text{(angles in the same segment)} \\ &= \frac{1}{2}\angle APB + \angle ABI \quad \text{(MP bisects } \angle APB) \end{aligned}$$

$$\begin{aligned} \angle MIB &= \angle IPB + \angle IBP \\ &\quad \text{(exterior angle of a triangle)} \\ &= \angle MPB + \angle IBP \\ &= \frac{1}{2}\angle APB + \angle IBP \end{aligned}$$

Substituting these values in (2),

$$\frac{1}{2}\angle APB + \angle ABI = \frac{1}{2}\angle APB + \angle IBP$$

$$\Rightarrow \angle ABI = \angle IBP$$

Hence, IB bisects $\angle PBA$.

I would like to present a problem from the IMO 1992 shortlist which was proposed by Shailsh Shirali. This problem appeared in the article on Mathematical Olympiads in India in the November edition.

Problem 1

Two circles G_1 and G_2 are inscribed in a segment of circle G and touch each other externally at a point W . Let A be a point of intersection of a common internal tangent to G_1 and G_2 with the arc of the segment and let B and C be the endpoints of the chord. Prove that W is the in-centre of triangle ABC .

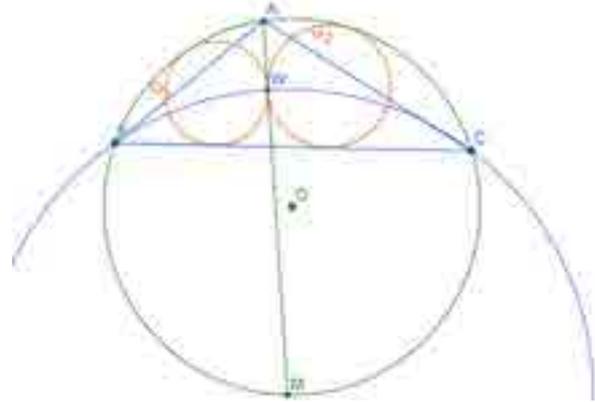


Figure 3

(Note from the editor: For those unfamiliar with the term ‘internal tangent,’ an internal tangent to two circles is a common tangent to the two circles such that the two circles lie on opposite sides of the tangent line. In the case of an external tangent, the two circles lie on the same side of the tangent line. Wikipedia uses the terms ‘inner tangent’ and ‘outer tangent’ but these are non-standard terms.)

We define point M to be the intersection of the common internal tangent of G_1 and G_2 and the circle G . The figure suggests that M is the midpoint of the arc \widehat{BC} not containing A. To prove this, we will require this lemma:

The tangents from a point P to G_1 and G_2 are equal if and only if it lies on the common internal tangent to G_1 and G_2 .

If P is on the common internal tangent, the tangent from P to both the circles is equal to the length PW. Therefore, we need to prove that the tangents from a point not on the common internal tangent to G_1 and G_2 cannot be equal.

Here is a brief sketch of the proof. The reader is expected to fill in the details.

- Let there be a point P' not on the common internal tangent such that the tangents from P' to both the circles are equal.
- Let P'W meet G₁ and G₂ at K and L respectively. Prove that K and L are distinct.
- Consider the powers of point P' (see appendix 1) with respect to both the circles. Prove that they must be unequal.
- Since the power of a point with respect to a circle is the square of the length of the tangent, we get a contradiction.

How can we use this lemma? According to the first corollary, the tangent from the midpoint of the arc \widehat{BC} not containing A to circles G₁ and G₂ must be equal. Hence, the midpoint lies on the common internal tangent and is point M.

This also implies that the tangent from M to both the circles is equal to MW. But it is also equal to MA (from the first corollary).

Therefore, we can construct a circle centred at M with radius MW which passes through A and B. Now the figure of the problem resembles that of the second corollary. We can use the second part of the second corollary to complete the proof.

Appendix 1: Power of a point

There are many ways to define the power of a point. For a point P and a circle with centre O and radius r , the power of point P is the quantity $OP^2 - r^2$. If P is outside the circle, the power of point P is the length of the tangent from P to the circle squared. If a line through P intersects the circle at A and B, then the power of point P is also $PA \cdot PB$ (Note that the power of a point inside a circle is negative according to the first definition. But the third definition suggests that it is positive. The power of a point inside a circle is indeed negative and $PA \cdot PB$ is the *absolute value* of the power of a point.)

Here, we only prove that the three definitions are equal for a point outside a circle.

Suppose PT is the tangent from P to the circle.

It is obvious from the Pythagoras theorem that $PT^2 = OP^2 - r^2$

We have to prove that $PT^2 = PA \cdot PB$.

Let TC be a diameter of the circle.

This implies that $\angle TAC$ is a right angle.

$$\Rightarrow \angle TCA = 90^\circ - \angle ATC$$

But $\angle CTP$ is also a right angle,

$$\Rightarrow \angle ATP = 90^\circ - \angle ATC$$

$$\Rightarrow \angle ATP = \angle TCA = \angle TBA$$

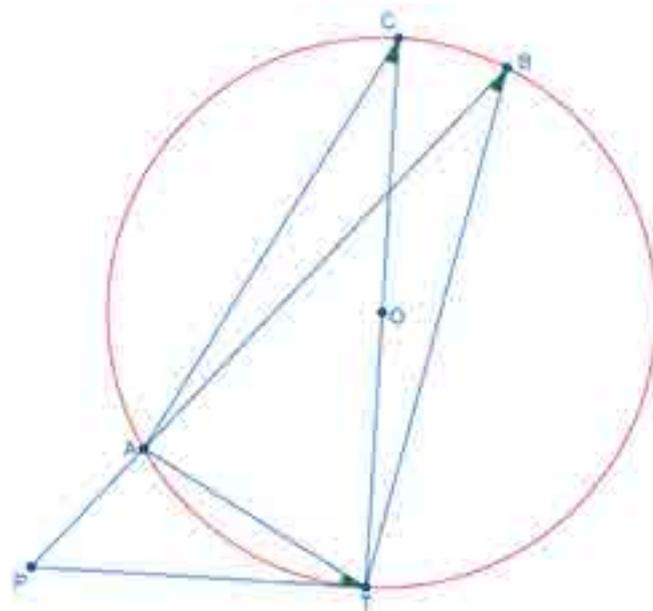


Figure 4

Also,

$$\angle APT = \angle BPT$$

The last two equations imply that triangles PAT and PBT are similar.

We get this ratio from similarity:

$$\frac{PA}{PT} = \frac{PT}{PB}$$

This gives us the required result.

Appendix 2: Orthogonal circles

Let Ω and ω be two circles intersecting with centres O_1 and O_2 respectively which intersect at A and B . If O_1A and O_1B are tangents to ω , the two circles are orthogonal. In fact, this implies that O_2A and O_2B are tangents to Ω . Orthogonal circles have lots of nice properties. We encourage the reader to find them. For example, one interesting feature is that the points O_1 , O_2 , A , B lie on a circle whose centre is the midpoint of segment O_1O_2 . The proof is trivial and is left to the reader.

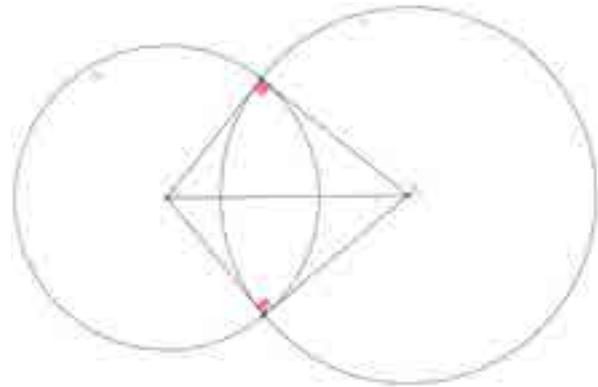


Figure 5

We conclude this appendix with a question: What is the power of point O_1 with respect to ω ? What about that of O_2 with respect to Ω ?

Acknowledgments

I thank my brother Maitreya for his review comments on the draft versions of this article and his suggestion to include the problem from the IMO shortlist of 1992.

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Radii of In-circle and Ex-circles of a Right-Angled Triangle

RAHIL MIRAJ

In this article, I provide a relation connecting the lengths of the tangents from the vertices of a right-angled triangle to its incircle and ex-circles, in terms of its inradius and ex-radii. I give a geometric proof as well as an analytic proof.

A standard result which will be used repeatedly is the following: *Given a circle and a point outside it, the lengths of the two tangents that can be drawn from the point to the circle have equal length.* A list of more such results and formulas of relevance is provided at the end of the article.

The following nomenclature should be noted. Other than the incircle of a triangle, three other circles can be drawn that touch the sidelines of a triangle. These are called the **ex-circles** of the triangle. The ex-circle opposite vertex A is known as the ' A ex-circle', and likewise for the two other ex-circles. The radius r_a of the A ex-circle is called the ' A ex-radius', and similarly for the radii of the two other ex-circles.

Consider a right triangle ABC , with the right angle at vertex C (Figure 1). Let the lengths of the tangents from the vertices to the incircle be x, y, z ($AQ = AR = x$; $BP = BR = y$; $CP = CQ = z$). Then $AB = c = x + y$, $CA = b = z + x$, $BC = a = y + z$.

We claim the following relations for the radii of the incircle and the three ex-circles:

- Radius of incircle, $r = z$.
- Radius of A ex-circle, $r_a = y$.
- Radius of B ex-circle, $r_b = x$.
- Radius of C ex-circle, $r_c = x + y + z$.

Geometric Proof. Let ABC be a triangle with $\angle C = 90^\circ$ (Figure 1). Let I be its incentre and let I_A, I_B, I_C respectively be the three ex-centres. Let P, Q, R be the points of contact of the tangents to the incircle from the vertices of $\triangle ABC$. Let D, E, F, G, S, T be the points of contact of the ex-circles with the sidelines of the triangle, as shown in Figure 1. Let $AQ = x$; $BP = y$ and $CP = z$; then we also have $AR = x$; $BR = y$ and $CQ = z$. Hence $AB = c = x + y$; $BC = a = y + z$ and $AC = b = z + x$.

Since $\angle C = 90^\circ$, it follows that $IQCP$ is a square; therefore $r = z$.

For the same reason ($\angle C = 90^\circ$), both $I_B DCE$ and $I_C FCG$ are squares.

Keywords: Incircle, ex-circle, tangent, Pythagorean triangle, Pythagorean triple

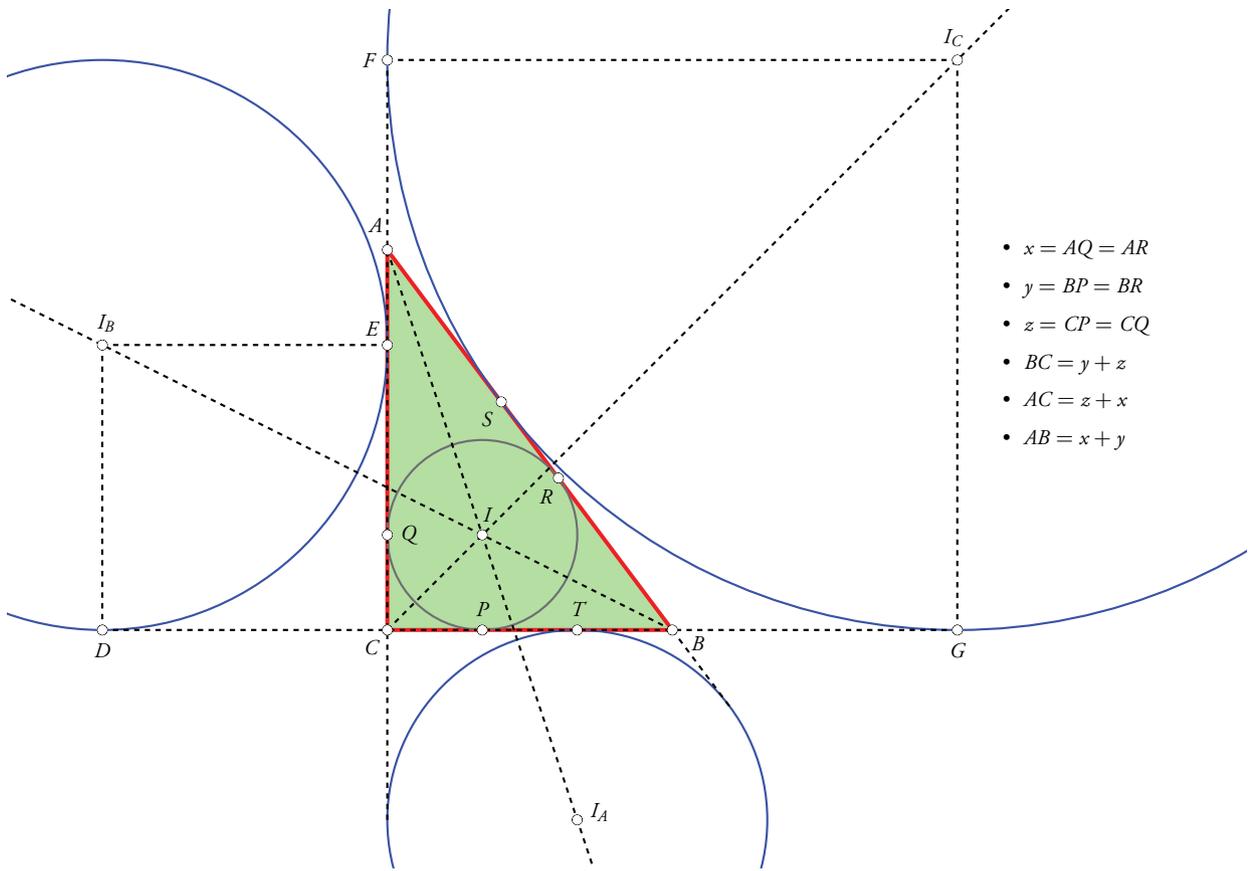


Figure 1. Triangle ABC with its incircle and ex-circles

Now we shall show that $AE = CQ$. This is a well-known property, but we include the proof here. Starting with $CQ = z$, we obtain the following in succession:

$$AQ = b - z, \quad AR = b - z, \quad BR = c - (b - z) = c - b + z, \quad BP = c - b + z.$$

Since we also have $CP = z$, it follows that $z + (c - b + z) = a$, which yields $2z = a + b - c$. So we have $2CQ = a + b - c$.

Next: the length of the tangent from B to the B ex-circle is $BC + CE = a + b - AE$. This length is also given by $BA + AE = c + AE$. Hence we have $c + AE = a + b - AE$, giving $2AE = a + b - c$. This shows that $AE = CQ$. We similarly have $CP = BT$ and $AS = BR$.

From the relation $AE = CQ$ we obtain:

$$r_b = EC = AC - AE = AC - CQ = AQ,$$

i.e., $r_b = x$. In the same way, we prove that $r_a = y$.

Finally, we have:

$$r_c = FC = FA + AC = AS + (x + z) = BR + (x + z),$$

i.e., $r_c = x + y + z$. All the claims made earlier have now been proved

Analytic Proof. We use the following known fact about right-angled triangles (to see why this is true, please see the boxed item below): if $\triangle ABC$ is right-angled at C and its sides are a, b, c , then there exists a positive real number $t < 1$ such that

$$a : b : c = 2t : 1 - t^2 : 1 + t^2.$$

Hence there exists a positive constant k such that

$$a = 2kt, \quad b = k(1 - t^2), \quad c = k(1 + t^2).$$

Now recall what we had proved above: $2z = a + b - c$. In the same way we may show that $2x = b + c - a$ and $2y = c + a - b$ (but we ask you to verify these relations for yourself). Let s be the semi-perimeter and Δ the area of $\triangle ABC$. We now have:

$$b + c - a = 2k(1 - t), \quad c + a - b = 2kt(1 + t), \quad a + b - c = 2kt(1 - t).$$

Hence we have,

$$x = k(1 - t), \quad y = kt(1 + t), \quad z = kt(1 - t),$$

and so:

$$x + y + z = k(1 + t),$$

i.e., $s = k(1 + t)$; and

$$\Delta = \frac{1}{2}ab = k^2t(1 - t^2).$$

We also know that $\Delta = rs$. Hence:

$$r = \frac{\Delta}{s} = kt(1 - t) = z,$$

as claimed. Another such relation is $\Delta = r_a(s - a)$. This yields:

$$r_a = \frac{\Delta}{s - a} = kt(1 + t) = y.$$

Similarly, $\Delta = r_b(s - b)$, so

$$r_b = \frac{\Delta}{s - b} = k(1 - t) = x,$$

and $\Delta = r_c(s - c)$, so

$$r_c = \frac{\Delta}{s - c} = k(1 + t) = x + y + z = s.$$

References

1. Evan Chan, *Euclidean Geometry in Mathematical Olympiads*, MAA press, (2016)

Acknowledgments. I thank my father Prof. Dr. Farook Rahaman for several illuminating discussions.

Justification of claim made about right-angled triangles

We prove the following claim about right-angled triangles: if $\triangle ABC$ is right-angled at C and its sides are a, b, c , then there exists a positive real number $t < 1$ such that

$$a : b : c = 2t : 1 - t^2 : 1 + t^2.$$

Proof. As a triangle is right-angled at C , it follows that $a^2 + b^2 = c^2$, and therefore that $a^2 = c^2 - b^2 = (c - b)(c + b)$. Hence:

$$\frac{a}{c + b} = \frac{c - b}{a} = t, \text{ say.}$$

It is obvious that t is positive, and $t < 1$ follows from the triangle inequality ($a < c + b$). The above two equalities now yield:

$$c + b = \frac{a}{t},$$

$$c - b = at.$$

From these two equalities, we get:

$$2c = \frac{a}{t} + at, \quad 2b = \frac{a}{t} - at.$$

Hence:

$$\frac{c}{a} = \frac{1 + t^2}{2t}, \quad \frac{b}{a} = \frac{1 - t^2}{2t}.$$

The stated claim now follows.

Box 1



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A path to π

A RAMACHANDRAN

Consider the following situation. A regular polygon of n sides is placed symmetrically inside another (larger) regular polygon of n sides, i.e., with corresponding sides parallel and with a constant distance between them. We now have a 'path of uniform width' running around the inner polygon. Now the following question is asked: By how much does the perimeter of the outer polygon exceed that of the inner?

To answer the above question look at Figure 1. AB and BC are adjacent sides of the inner polygon, while DE and EF are adjacent sides of the outer polygon. EB extended would pass through the common centre O of the polygons. This line would also bisect $\angle ABC$ and $\angle DEF$. From B draw BH perpendicular to DE with H on DE . Now $\angle ABC$ is an interior angle of the regular polygon and its measure would be

$$(2n - 4) \times \frac{90^\circ}{n} = 180^\circ - \frac{360^\circ}{n},$$

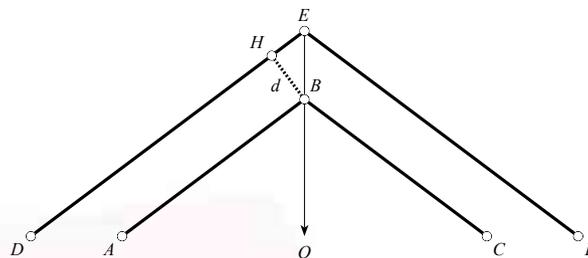


Figure 1

n being the number of sides of the polygon. $\angle ABO$ then equals $90^\circ - 180^\circ/n$, and $\angle EBH$ equals $180^\circ/n$.

Now, $\tan \angle EBH = EH / HB = EH / d$; here d is the path width. So $EH = d \tan(180^\circ/n)$. Side DE exceeds corresponding side AB by twice this amount. So the perimeter of the outer polygon exceeds that of the inner by $2nd \tan(180^\circ/n)$.

The above expression takes different values as n varies. As n increases the regular polygon approaches a circle, and the

Keywords: Regular polygon, nested polygons, path, angle, tan, pi, investigation

situation becomes one of two concentric circles. In such a case it is well known that the outer perimeter (or circumference of the circle) exceeds the inner by $2\pi d$, where d is the width of the annular path.

Hence one could say that the value of $2nd \tan(180^\circ/n)$ approaches the value $2\pi d$ as n

increases. Cancelling $2d$, we could say that $n \cdot \tan(180^\circ/n)$ approaches the value π as n increases.

It will be an interesting exercise for a high school student to compute the values of this expression for increasing values of n and check out how rapidly it converges to π .



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

NOTE FROM THE EDITOR

The following two inequalities are well-known (here x is measured in radians):

$$\text{If } 0 < x < \frac{\pi}{2}, \text{ then } \sin x < x < \tan x.$$

These inequalities imply the following two limits:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

They also imply the following:

- $(\sin x)/x$ approaches its limiting value of 1 from below;
- $(\tan x)/x$ approaches its limiting value of 1 from above.

Now put $x = \pi/n$; then ' $x \rightarrow 0$ ' is equivalent to ' $n \rightarrow \infty$.' Hence we have the following results:

$$\lim_{n \rightarrow \infty} n \cdot \sin \frac{\pi}{n} = \pi, \quad \lim_{n \rightarrow \infty} n \cdot \tan \frac{\pi}{n} = \pi.$$

We also have the following:

- $n \cdot \sin \pi/n$ approaches its limiting value of π from below;
- $n \cdot \tan \pi/n$ approaches its limiting value of π from above.

It would be of interest to tabulate values of $n \cdot \sin \pi/n$ as well as $n \cdot \tan \pi/n$ against increasing values of n and to verify these statements. Students are invited to carry out these tasks.

π Enters the 5th Class

**SABITHA
RAGHUNATH &
SNEHA BHANSALI**

I write this to tell myself that it was not a dream ...

This past year, I taught a bunch of fifth standard kids in Sahyadri School KFI (Krishnamurti Foundation India), who, like all others of their age, were high-energy kids; they were willing to explore but found it difficult to sit down in one place. I had a great relationship with them. The air in the classroom was of love, trust and wonder!

We did all we could with newspaper cutouts of squares and rectangles to explore the relation (if any) between area and perimeter. (We also used biscuits as part of this exploration, but more about that on another day.) An exciting 40 minutes of measuring area and perimeter of different squares and rectangles ended with “Hurry up! Let’s go to the Assembly, quick,” when one voice was heard, “but what about Circles, Akka?”

That was a trigger good enough for me to get into action mode.

I cut out about 50 colorful circles of different sizes from old magazine sheets using circular plates, bottle caps, and such objects available (there was no compass used); borrowed thread from the Art room; took a few half-meter scales from the physics lab; and was in the class with an objective to measure the Perimeter of the Circle.

The steps followed during this exploration were:

1. Each kid took two circles, the size and color being their choice.
2. Using the 50 cm scale and jute thread, each child measured the boundary – the “Perimeter” of the circle.
3. I introduced the word “Circumference.” (Even at this point, there was no clue as to what was in store for us – to either the teacher or the student.)

A question is raised, “Akka, what else should we measure?”

Keywords: Perimeter, circle, circumference, diameter, ratio, pi, investigation, pedagogy

A pin drops in my head ...

“Can we find the length of the longest line in a circle?”

“How Akka?”

Let us try ...

One innovative mind, “Fold the circle and we will get it, Akka!”

“I know Akka, that’s a diameter ...”

My own innovation kicks in and I make a 23×5 table on the black board with all their (a fifth standard class of 23 children) names written, just like what you can see here.

S. No.	Name	Boundary of the circle (A)	Longest line (B)	$A \div B$

4. “Children, please measure the length of the longest line within the circle...”
5. “Divide the length of the boundary of the circle by the length of the longest line.”
6. “Come to the black-board and write down your measurements.”
7. “Do the division and write down the quotient in the appropriate column against your name, beside the measurements you have already written.”
8. “Please write the same table in your notebook.”

The table created interesting observations; there were many kids who wanted to express their thoughts:

1. Why do we all have ‘3.’ something?
2. Why do I have a recurring decimal?
3. Akka, did you measure the circles first and then cut them?
4. Is my division correct?
5. Can we cut circles on our own?

6. Check my division, Akka, it just doesn’t seem to end...

And so on ...

That was the Eureka moment.

A 10-year-old had discovered the value of π for herself...

In the next class, we spoke about π to an extent and the mistakes that happen during the measurement of Perimeter. That was one WOW class...!

The wonder of Mathematics was the highlight of those two classes. I am not sure how many among those 23 children would have got the concept completely, but I am very certain that they had a glimpse of the mystery of Mathematics. In the years to come, they would see more of it...

My own learnings from this were the following:

1. Preparation for a class is an absolute necessity.
2. Time to do an activity is a range. (Each child measured the boundary of the circle, the longest line and then stuck the circles in his or her notebook. I noticed that each child’s requirement is different; each child’s attitude is different. Some children wish to do the task as perfectly as possible and take more time; others wish to race through the task; yet others take time to begin – they are the slow starters; and yet others are hesitant and uncertain, as they are not sure they have the skill to proceed.)
3. Every answer is correct. (I used jute string to measure the perimeter, so there was the factor of stretchability and therefore some variations in the lengths measured. The emphasis was to discover the approximation up to the first decimal place.)
4. Set aside enough time to ensure that every single child has completed the task. This is important.
5. Students who have finished the task can help those who have not.

6. Your preparation is never enough.
7. Be prepared to be surprised.
8. There is room for error...

Please note that in the following picture, one of the kids has chosen to write the columns in her own order. It was an enriching class for quite a few of us.



	Perimeter of the circle	Longest line	A-B	Name
1	25 cm	7.5 cm	3:3	MACHI
2	34 cm	4.2 cm	3	ANISHKA
3	23.4 cm	7.2 cm	3:2	JIVA
4	52.5 cm	17 cm	3:0.8	HIVA
5	56.0 cm	16.4 cm	3:3	SAMRUDDHI
6	10.5 cm	3.3 cm	3:3	SHREYA
7	22 cm	11.30 cm	1.2:3	AMIDHA
8	45 cm	4.7 cm	3:2	ANIKA
9	10 cm	3.1 cm	3:2	RIMI
10	23 cm	7.2 cm	3:1	SARVASHA
11	51 cm	15.5 cm	3:0.9	YUVA
12	10.5 cm	3.1 cm	3:3	NANAKI
13	35 cm	17.3 cm	3:0.0	DEVANSHI
14	8.5 cm	3.1 cm	3:4.6	SRIDHAN
15	35.2 cm	14.5 cm	2:9.1	TISHAN
16	31.5 cm	9.2 cm	3:4.1	CHITANYA

Perimeter of circle = 23.1 cm
 the circle
 Longest = 7.2 cm
 line
 A-B = 3:2

	Circumference PERIMETER OF THE CIRCLE	Longest line	A-B	Name	Perimeter of a circle is (circumference) Longest line in a circle (DIAMETER)
17	24 cm	7.1 cm	3	DHARVYA	
18	35 cm	11.6 cm	3:0.4	ARYA	
19	26.6 cm	7.4 cm	3	SARVASHA	
20	53 cm	15 cm		SARVASHA	
21	10 cm	3 cm	3:3	MAYUR	A = 3 = π
22	45 cm	14.7 cm	3:2	NIKHIL	3

Pedagogical Endnotes

It is worth noting that this exploration which culminated in an estimate of π integrated many different concepts and skills in early mathematics, such as measurement, perimeter, circumference, division of decimal numbers, presentation of data in tabular form, pattern finding, stating/articulating a conjecture, thinking about a converse and so on. In short, it encompassed a good many facets of what it means to “do mathematics.” It is worth reflecting

on the many strands that emanate from this seemingly simple exploration.

We list below some pedagogical implications of the actions done by the children. The column on the left lists statements and instructions from the above article, while the corresponding entries on the right make some pedagogical remarks concerning the statements.

Pedagogical aspects of instructions given by the teacher

Statement number	Statement from the text, above	Pedagogical remarks
4	“Children, please measure the length of the longest line within the circle...”	This instruction requires children to fold the circular shape exactly, use a measuring scale accurately, and record the length of the diameter using decimals.
5	“Divide the length of the boundary of the circle by the length of the longest line.”	The children had just learnt division of decimals. Perhaps more work, time and energy were required to learn this concept and feel comfortable with that. It was thus a good opportunity to revisit the concept while integrating it into the current task of exploring the ratio of two specific numbers.
6	“Come to the black-board and write down your measurements.”	Noticing that the least count is 0.1 cm or 1 mm, the entries in the table would not have more than one digit after the decimal point.
7	“Do the division and write down the quotient in the appropriate column against your name, beside the measurements you have already written.”	It may not be obvious to many children that there can be multiple answers to such an exercise, and none of these answers is ‘wrong.’ Such learning comes only after the children have had sufficient exposure of ‘doing mathematics.’
8	“Please do make the same table in your notebook.”	This served as a quick introduction to collecting and recording the data.

Pedagogical aspects of observations made by the children and questions posed by them

Statement number	Statement from the text, above	Pedagogical remarks
1	“How come we all have 3. something?”	Is there a pattern in the data and if so, how do we know that the pattern actually exists?
2	“Why do I have a recurring decimal?”	Making connections across the different concepts that seem to be unrelated at first glance.
3	“Akka, did you measure the circles first and then cut them?”	What comes first and what comes as a follow-up property: Do the circles follow this particular pattern, or were they ‘constructed’ first in order to obtain this pattern?

Statement number	Statement from the text, above	Pedagogical remarks
4	“Is my division correct?”	How do I make sure my answer is correct? It involves moving from procedural learning towards conceptual learning.
5	“Can we cut the circles on our own?”	Will the pattern be true for any circles, irrespective of who constructs them?
6	“Check my division, Akka; it just doesn't seem to end...”	Is my answer wrong because I have not encountered such a thing before?



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Proof without Words:

An area of a quadrilateral inscribed in a parallelogram

Given: parallelogram ABCD, points K,I,N,G on its sides and $AI = DG$. Prove that the area of KING is equal to half the area of ABCD (Fig.1). Proof (Fig.2).

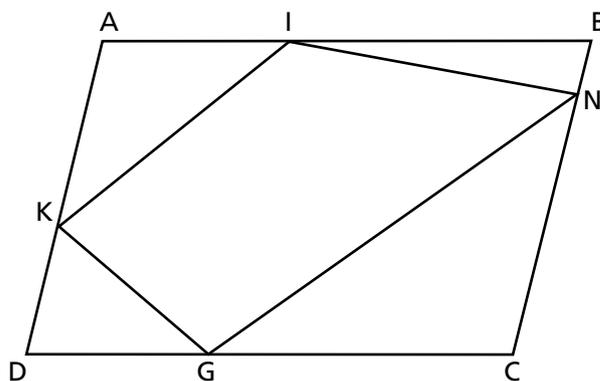


Figure 1

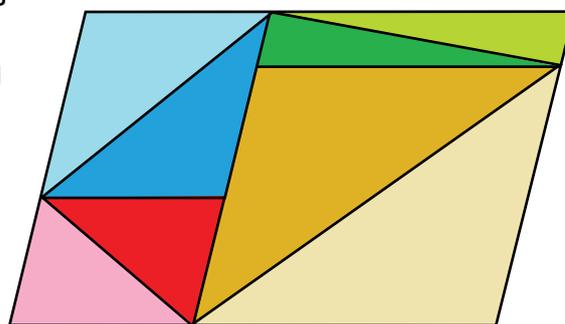


Figure 2

Understanding Learners' Thinking through an Analysis of Errors

SHIKHA TAKKER

In the ClassRoom section of the November 2018 issue of At Right Angles, Prof. Hridyakant Dewan wrote about the interpretation of errors in arithmetic. The paper lists errors made by students while doing arithmetic. The author asserts that algorithms (he calls them “quick fixes”) given by teachers contribute to students’ errors and diverts them from conceptual understanding. He mentions that these errors could be a result of over generalisations made by students. However, he also states that these are “transmitted to students as short-cuts to get the required answer”. In the end, he makes an appeal that teachers plan tasks which help students in gaining conceptual understanding.

This paper is a response to the arguments made by Prof. Dewan on the need for recognising and dealing with errors in mathematics. Extending his sentiment of analysing errors as gateways to students’ thinking, but seeking a more refined and nuanced understanding to dealing with errors, I argue that

- (a) It is important not to homogenise the errors made by students. Instead, errors might point to a student’s attempt to make sense of ‘new information’ based on their ‘existing knowledge and resources’. The process of analysing errors helps in understanding the student’s thought processes and narrowing down the misconceptions from the concept to the sub-concept(s) that s/he might be struggling with. I will use evidences from my research on mathematics classrooms as well as the data from the research literature to substantiate this point.
- (b) In the process of analysing errors, it is important to identify their mathematical source and locate it in the field of a topic. While this would help in further ‘zooming in’¹ to understand the student’s thinking, ‘zooming out’ to understand how an error

Keywords: Errors, misconceptions, analysis, careless mistakes, cognition, zooming in

might affect a student's further learning is equally significant.

- (c) When expecting teachers to plan tasks that enable conceptual understanding, it is important that they are provided with the necessary support to unpack the mathematics underlying students' responses (errors, explanations, alternative methods). One of the ways of supporting teachers is to take a participatory approach to understand teachers' struggles and collaboratively conduct pedagogic experiments *enroute* to a reformed pedagogy.

As we discuss each of these arguments, we will also understand what we mean by 'student errors' or 'misconceptions'.

De-homogenisation of Errors

In my early years of teaching elementary school mathematics, I used to classify all the students' mistakes as 'careless mistakes'. While interacting with colleagues then, and with teachers across different states and at different grade levels later, as a researcher, I realised that this is a rather common thought. Aligned with this thought is the belief that, in mathematics there are either correct or incorrect responses. The belief rests on the understanding that the incorrect responses emerge from students' careless mistakes. But not all students' errors are of the same kind. Let us take a set of errors to discuss this further (refer Fig. 1).

<p>(a) Add 256 and 319. Ans: 2 6 5 + 3 1 9 — 5 8 4</p>	<p>(b) What is the smallest multiple of 7? Ans: 14</p>
---	---

Figure 1: Examples of Errors or Careless Mistakes

What do we notice in these two responses? In the first response (a), the student has added 265 instead of 256 although the addition is correct. This response is fairly common, and often,

teachers are in the dilemma of whether to give "full marks" for such a response. The dilemma arises from the fact that this is an incorrect response to the question asked although the student knows the concept that is being tested. In the second response (b), it is not clear whether the student does not understand "smallest multiple" or has overlooked the word "smallest" and written the first multiple of 7 that came to her mind. The student definitely understands that 14 is a multiple of 7. What, do you think, is the source of such errors? These errors could result from an incomplete reading of the problem, overlooking some part(s) of the given information, misreading the numbers, and so on. Ryan & Williams (2007) mention that such errors might arise from incorrect remembering of the facts, cognitive overload² or jumping to conclusions. Adding to this, we all know, that these errors might arise from the anxiety of problem solving during examinations or due to performance pressure. Like adults, students make these errors (or mistakes or slips) while solving a problem. Clearly, such errors do not have a connection to the age of the learner. In other words, they are agnostic to the developmental level of the learner. Such errors, often classified as "careless mistakes", do not seem to provide sufficient evidence for the lack of students' understanding or incorrect ways of thinking (or misconceptions). The reason is that such errors compel us to ask whether the student would have responded in a similar way if s/he was paying complete attention to the problem at hand, and was not pressured to perform correctly.

Now, let us see a different set of errors (refer Fig. 2). See if you recognise these errors.

Students make error (a) when they do not understand how to carry-over in an addition problem. Kamii & Dominick (1997) have noted that students make such errors in addition when they do not know where to place the 'carried

1 Magdalene Lampert (2001) uses the phrase 'zooming in' to analyse specific aspects of her teaching such as individual student's responses to a problem.

2 Cognitive overload is the mental demand that a learning task poses on the learner.

$\begin{array}{r} 127 \\ + 534 \\ \hline 6511 \end{array}$	$\begin{array}{r} 727 \\ - 534 \\ \hline 213 \end{array}$	$\underline{2} = 3 + 5$	$0.5 \times 10 = \underline{0.50}$	$(a + b)^2 = a^2 + b^2$
(a) Grade 3, Addition of Whole Numbers	(b) Grades 3-4, Subtraction of Whole Numbers	(c) Grades 2-5, Addition / Subtraction Sentences	(d) Grade 6, Decimal Numbers	(e) Grades 5-6, Early Algebra

Figure 2: Errors made by students

over' part of the sum. Similar to error (a), there is subtraction error (b). Here, the students subtract the smaller digit from the bigger digit irrespective of their positions in the number. Error (c) was found to be common among a wide range of students. It is common among students who are beginning to learn algebra or sometimes even later (Falkner, Levi & Carpenter, 1999). Error (d) emerged among Grades 5 and 6 students in my classroom research. In Error (e), students do not understand how to expand the given algebraic identity.

Teachers and researchers of mathematics in India and elsewhere have noted these errors. The research suggests that these errors appear at a specific developmental stage in the learning of a topic. These systematic errors often point to deep-rooted thinking or what we call as 'misconceptions'³ in students. Such misconceptions are persistent unless a deliberate pedagogical intervention is made to address them (Sarwadi & Shahrill, 2004). Given our understanding of students' errors, let us do a task (refer Fig. 3).

Task 1: Make a list of errors that you have seen or made while learning or teaching mathematics. See if you are able to classify them into "careless mistakes" and "systematic errors".

Figure 3: Task on classifying errors

Zooming In

As teachers or educators, we often assume that all errors are a result of lack of attention by the student or lack of practice (in the particular case of mathematics). A preliminary analysis of errors might help us understand that these could indeed be some logical extensions made by students in an attempt to make sense of the new information. Let us study the errors listed in Fig. 2 a little more carefully. The errors (a) and (b) are a result of a student's difficulty in regrouping of the higher place values. In response (a), the student finds it difficult to add the 1 ten with the other tens in the addends. The student treats the digits separately and adds 1 and 5, 2 and 3, 7 and 4 separately. This indicates that the digits of a number are separated and only the relation between the addends is focused upon. As a teacher, we can guess that a student with this kind of thinking will be correctly able to solve addition problems, which do not require a carry-over. In carry-over addition problems, the understanding of re-grouping, after adding the digits with the same place value, becomes important and needs to be accounted for. Now, let us consider response (b). Here the student subtracts 2 from 3, without realising that the minuend needs to be subtracted from the subtrahend. Similar to (a), the digits are treated separately and the next higher place value is not considered while doing the subtraction. Which problems, do you think, would this student be

3 A misconception implies that the learner's conception of a particular idea or topic, of a rule or algorithm, is in conflict with its accepted meaning and understanding in mathematics (Barmby et al., 2009 cited in Goswami, 2018)

able to solve correctly? The operations of addition and subtraction require us to see the relation between the place values of a number and across place value in different numbers. Missing either of them can lead to specific difficulties among students. In the Teacher Manual, published by NCERT (2010), there is a detailed discussion on how these errors are linked to students' difficulty in following the standard algorithms (p. 34).

While (c) seems to be an addition problem, we have found that a wide range of students from Grades 2-5 make this error. Can you guess what might be the student's thinking behind this response? Here the student seems to ignore the equals sign after 2 and (mis) reads the equation as $2 + 3 = 5$. The reason could perhaps be the way in which typical addition or subtraction problems are posed. In a typical numerical equation, when written horizontally, the operators appear to the left of the equation and the final (one number) answer appears on the right side of the equals sign. Students who consider the missing blank to be 2 in this case tend to give a similar response to the problem of the kind $a + b = \underline{\quad} + d$. Here the students filled the blank by writing the sum of a and b . For example, for the numerical equation, $6 + 7 = \underline{\quad} + 8$, some students filled the blank with 13, ignoring the '+ 8' that follows. It is interesting to note that these students might be able to solve these problems correctly when given in a standard format, such as $a + b = \underline{\quad}$. Students with this kind of thinking find it difficult to treat 'equal to' as a 'balance', where two expressions on either side are equal. Developing a more relational understanding of the 'equal to', helps students in developing early algebraic thinking (for details refer to Takker, Kanhere, Naik & Subramaniam, 2013).

Response (d) was found among students of Grade 5 and 6, when they were asked to multiply a decimal number with 10 and its powers. We will discuss this response in a little more detail in the next section. The last response (e) is the expansion of the algebraic identity, which is treated by focusing on the brackets more than squaring the term as a whole. In other words,

students seem to be extending the understanding of opening the brackets, as in $(a + b) c = (ac + bc)$, to the squaring of the term $(a + b)^2$.

Zooming In and Zooming Out

We can make some interpretations about these errors based on our knowledge of the research literature in the field and from our experience of working with students. This knowledge about students' ways of thinking gets developed from our attempts to listen to the students' reasoning. Based on your knowledge of students' thinking, can you predict why students would have responded to Error d (refer to Fig. 2) in this particular way?

Probing students and helping them articulate their thinking is an important means to understand why students respond to a question in a particular way. Let us see what we understand from the reasons given by students for Error (d) (refer Fig. 4).

Sumit	Five times ten is fifty. So we put the fifty, first. There is a [decimal] point here [points to the point before 5 in 0.5] so it is here [points to the point in 0.50].
Garima	0.5 times 1 is 0.5 and 10 means adding a zero at the end, so 0.50.
Roshni	Five tens are fifty. And then zero point is the same.
Jolly	$5 \times 10 = 5 + 0 = 50$ $0.5 \times 10 = 0.50$

Figure 4: Reasons for Error (d)

What did you notice about these students' explanations? Is there any similarity in these explanations? Or do you think that they are all different explanations?

First, we note that although all these students gave 0.50 as the answer, their reasons for arriving at this answer and their ways of thinking are different. Sumit and Roshni use the multiplication table of 5. Sumit seems to think that the position of the decimal point before 5 in 0.5 should be retained in the answer. Roshni keeps the position of zero and point intact. Both

use their knowledge of multiplication table of 5 and then decide where to place the decimal point. Garima begins by multiplying the decimal number 0.5 with the whole number 1 and then follows the rule for multiplication with the power of ten by ‘adding’ the required number of zeroes. Jolly multiplies 5 and 10, but evidently translates ‘adding a zero’ as ‘multiplying by ten’. He then concludes the multiplication.

Second, none of the students thought that the answer is .50 [point five zero]; they have kept the zero at the first and the last place intact. So when these students were asked whether point five zero [.50] is a correct answer, their responses varied. While Sumit did not reach a conclusion and was unsure, Garima, Roshni and Jolly were sure that .50 was not the answer. Put differently, they doubted the equivalence of 0.50 and .50. Of course, the next question would be whether they think that point five [.5] or zero point five [0.5] would be the correct answer. What is your guess?

What do these explanations tell us about students’ thinking or their learning? We see that the students treated the decimal number 0.5 like 5 and then placed the decimal point when writing the final product. They seemed to be aware of how to multiply a whole number with 10; in this case five times ten is fifty. They also know the convention of placing the decimal point at some place in the product after performing the operation (done by treating decimal numbers as whole numbers). While their prior knowledge helps them in making these decisions correctly, they are unable to identify the correct place for the decimal point in the product. So then we ask - where could this error be stemming from?

While learning whole numbers, students are taught that multiplication with powers of ten means “adding the zeroes⁴”, that is, appending the same number of zeroes as the power of 10, after the product. That is, 5 times 100 is broken

into 5 times 1, that is 5, and then to take care of the two zeroes with 100, they are “added” in the answer, which gives, 500. So it is clear that the students have used this understanding when treating the decimal number 0.5 like a whole number 5. Could the ‘placement of the decimal point’ understanding be also linked with their knowledge of whole numbers? While the explanation of ‘adding the zeroes’ when multiplying with powers of 10 is common, no teacher would ever tell the students to keep the position of the decimal point and the zero before the decimal point, in the product, as it was in the multiplicand. Clearly, the students seem to extend their understanding from whole numbers to decimal numbers in deciding the position of zero and the decimal point. They are trying to use their prior knowledge of multiplying a whole number with powers of ten to find the product of a decimal number with powers of ten. This kind of reasoning is not necessarily taught, but is an extension made by students to make sense of the new knowledge. This instance of teaching is a case in point to suggest that as teachers and educators we need to be aware of such extensions made by the students. Even though such extensions may not be a direct consequence of the way decimal number multiplication is taught, we find students using their knowledge of whole numbers while working with fractions, integers and algebraic identities.

What can teachers do?

As teachers, we first need to be able to identify and distinguish between the errors made by students. Fortunately, teachers are not alone in this search. The research literature on students’ errors and thinking helps us in spotting such errors and understanding what might be the possible misconceptions underlying such responses. Further, these errors can be treated as opportunities for discussion in classrooms. We

4 “Adding the zeroes” is a common phrase used for accounting the zeroes in the product when multiplying with powers or multiples of 10. The phrase does not precisely explain the act of appending zeroes, but is often used correctly.

can create problems (or tasks), which explicitly address the roots of these misconceptions. As Ryan & Williams (2007) state, “they [errors] offer a window into the conceptual structures that children are building and hence can be suggestive of appropriate intervention.”

How do we design tasks to help students overcome such misconceptions? What could be the objective(s) of such tasks? Would the objective be to (a) correct the students’ mistake by telling them the correct answer and giving more problems for practice? or (b) design a task, which addresses the reason why the specific error emerged, and then challenge it by providing an adequate mathematical justification? The tasks designed would vary based on which objective we choose to pursue. If we pursue (a), then we will be avoiding the thinking underlying such errors made by students. We would be “telling” them what not to do and “by authority” students will accept the corrections. Although this approach might help some students in correcting their mistakes, it does not address their thinking or conceptions underlying such responses. On the other hand, if we decide to address these misconceptions (by following route (b)), then we can begin by thinking about different instances where students make such over-generalisations based on their prior knowledge⁵. It will be useful for us, as teachers, to identify what could be the “counter instances” of common explanations offered to students or the (incorrect) generalisations made by students⁶. In other words, a rule of thumb could be to look for instances where an explanation or generalisation works and where it does not work. For instance, in Error (d) in Fig. 2, the rule for working with whole numbers does not always work with decimal numbers. What are those counter instances? Well, we could make a

table of how the whole number thinking extends (supporting instances) and does not extend (counter instances) to the learning of decimal numbers. Such an exercise would help us realise that generalisations made from whole numbers might lead to errors in other sub-topics within decimals and also in topics such as integers, fractions or algebra. A brief explanation on how we could get started on this exercise is as follows.

- (a) Consider the explanation of comparing the number of digits to identify which number is greater. The explanation works for comparison of whole numbers. It can be generalised to a case where the decimal numbers to be compared are of the type 14.3 and 2.9. However, it does not work for the comparison of 1.436 and 1.9. So, the latter can serve as a counter instance for this kind of an explanation.
- (b) Explanation that a greater number cannot be subtracted from a smaller number. This explanation works for the set of whole numbers but not for integers. Negative integers are generated by subtracting a bigger number from a smaller number. Similarly, we can think of the counter instances for the common explanation that multiplication always increases the number, which is being multiplied.
- (c) Students’ generalisation of adding the numerators and denominators of two fractions to be added. For instance, $\frac{7}{8} + \frac{5}{4} = \frac{12}{12}$ (a similar example has been listed in Prof. Dewan’s paper) can be countered by showing how $\frac{1}{2} + \frac{1}{2} \neq \frac{3}{4}$ (or $\frac{1}{2}$ or 50% or 0.5) but gives 1 whole.
- (d) In algebra, $7s + 5s = 12s$ but $7s + 5r \neq 12sr$ (for details of this error refer to Subramaniam, 2018).

5 It is difficult to get an exhaustive list of all the errors made by learners in the learning of different topics in mathematics. Pradhan & Mavalankar (1994) had made an attempt in this direction through a compendium of students’ errors in middle school mathematics. However, over time, as teachers and researchers, we can create a repository, which would have the common errors made by learners, their possible sources and potential ways of dealing with them. This evolving corpus of knowledge will be available as a resource for novice teachers and can be continuously refined by the experienced teachers.

6 In one of the instances in her study of Chinese and US teachers’ knowledge, Liping Ma (2010) discusses how some teachers used explanations that challenge a student’s generalization that ‘if the perimeter of a closed figure increases, its area also increases’.

Concluding thoughts

In this article, I have argued that it is important not to bunch all the students' errors as "careless mistakes" or "over-generalisations". We can classify the errors to understand – (a) what is their mathematical source and (b) what could be the student's thinking underlying such responses. This will help us in designing appropriate interventions for handling these errors in classroom. This route of making an attempt at understanding students' errors is not easy. However, we have the resource

of (a) the knowledge of experienced teachers gained from paying attention to students' oral and written responses, and (b) the research literature on students' misconceptions. These resources can help us in developing deeper knowledge about students' mathematical ways of thinking. The development of a knowledge base, involving identifying and detailing students' systematic mistakes in specific mathematical topics, and planning suitable tasks that allow for conceptual understanding, might be the potential way forward.

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Acknowledgments

I would like to thank Ms. Jayasree Subramanian from Homi Bhabha Centre for Science Education, Mumbai and Dr. Ritesh Khunyakari from Tata Institute of Social Sciences, Hyderabad for their inputs, which helped in refining this article.



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'CuRe' TRIPLETS

HARA GOPAL R

On observing the triple $(25, 125, 225)$ in which 125 is a perfect cube, 25 and 225 are perfect squares, and the three numbers are in arithmetic progression (AP), I felt that 125 is a very special perfect cube which is guarded by two perfect squares on either side at equal distance.

A surprising discovery we make is that 125 is guarded by two perfect squares in another way, namely: $(81, 125, 169)$; here, 81 and 169 are perfect squares, and the three numbers are in AP as earlier.

I wondered about the existence of other such perfect cubes. If they exist, then on what condition? If not, then why?

I named such triplets 'CuRe Triplets' (**C**ube-**S**qua**R**e).

After a careful search involving many calculations, I discovered many such triplets and found a simple condition regarding their existence (see the theorem listed below).

First let me define a CuRe Triplet.

Definition. A triple (a, b, c) of positive integers is called a **CuRe triplet** if a^2, b^3, c^2 are in arithmetic progression, i.e., a, b, c are positive integers such that $b^3 - a^2 = c^2 - b^3$, which may also be written as $2b^3 = a^2 + c^2$. For example:

- $(5, 5, 15)$ is a CuRe Triplet, since 25, 125, 225 are in AP;
- $(9, 5, 13)$ is a CuRe Triplet, since 81, 125, 169 are in AP.

Keywords: Square, cube, triplet

Theorem. *If b is a sum of the squares of two positive integers, then positive integers a and c can be found such that (a, b, c) is a CuRe triplet.*

To illustrate what this means, note that in the two CuRe triplets displayed above, $(5, 5, 15)$ and $(9, 5, 13)$, the central number 5 is a sum of two squares, $5 = 2^2 + 1^2$.

Proof of theorem. Suppose that $b = x^2 + y^2$, where x and y are positive integers, $x > y$. We shall show that $2b^3$ can be written as a sum of two squares. We have:

$$\begin{aligned} 2b^3 &= 2(x^2 + y^2)^3 = (x^2 + y^2)^2 \cdot (2x^2 + 2y^2) \\ &= (x^2 + y^2)^2 \cdot ((x - y)^2 + (x + y)^2) \\ &= (x^2 + y^2)^2 \cdot (x - y)^2 + (x^2 + y^2)^2 \cdot (x + y)^2 = a^2 + c^2, \end{aligned}$$

where a and c are given by:

$$\begin{aligned} a &= (x - y) \cdot (x^2 + y^2), \\ c &= (x + y) \cdot (x^2 + y^2). \end{aligned}$$

This shows that (a, b, c) is a CuRe triplet.

A few examples.

- Let $x = 2$ and $y = 1$; then $b = x^2 + y^2 = 5$, and:

$$\begin{aligned} a &= (x - y) \cdot (x^2 + y^2) = 1 \cdot 5 = 5, \\ c &= (x + y) \cdot (x^2 + y^2) = 3 \cdot 5 = 15. \end{aligned}$$

We obtain the CuRe triplet $(5, 5, 15)$.

- Let $x = 3$ and $y = 2$; then $b = x^2 + y^2 = 13$, and:

$$\begin{aligned} a &= (x - y) \cdot (x^2 + y^2) = 1 \cdot 13 = 13, \\ c &= (x + y) \cdot (x^2 + y^2) = 5 \cdot 13 = 65. \end{aligned}$$

We obtain the CuRe triplet $(13, 13, 65)$.

- The choice $x = 4$ and $y = 1$ yields the CuRe triplet $(51, 17, 85)$.
- The choice $x = 5$ and $y = 2$ yields the CuRe triplet $(87, 29, 203)$.
- We may generate infinitely many such triplets in this manner, for example:

$$(185, 37, 259), \quad (41, 41, 369), \quad (265, 53, 477), \quad \dots$$

Comment. A property shared by all CuRe triplets (a, b, c) generated by this approach is the following: b is a divisor of both a and c ; i.e., b is a divisor of $\gcd(a, c)$.

More CuRe triplets. Searching for CuRe triplets in an ad hoc ‘brute force’ manner (using a computer) yields all of the above triplets but also yields numerous others which are not generated by the above approach. Here are some such triplets:

$$\begin{array}{cccc} (5, 17, 99), & (8, 10, 44), & (9, 5, 13), & (36, 26, 184), \\ (37, 13, 55), & (72, 20, 104), & (73, 25, 161), & (77, 29, 207), \\ (85, 25, 155), & (188, 34, 208), & (91, 37, 305) & \dots \end{array}$$

We can be sure that these triplets cannot be generated by the method described above because they do not have the divisibility property noted above.

Closing remark

This simply stated problem raises numerous questions which we leave for the reader to explore. One such problem is the following: *Is there a systematic way by which infinitely many CuRe triplets (a, b, c) can be generated which do not have the property that b is a divisor of $\gcd(a, c)$?*



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The mountains beside the lake form a curve and get reflected in water

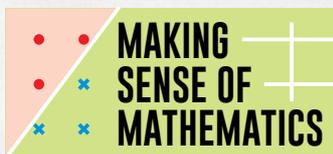


A lake enroute Volos, Greece

Photo & Ideation: *Kumar Gandharv Mishra*

Mathematical Relevance: Reflection on a Plane. Graph similar to a sinusoidal function.

Joining the Dots...



SWATI SIRCAR &
SNEHA TITUS

In our last Low Floor High Ceiling article, we had looked at Squaring the Dots... a series of questions on counting the dots inside squares of different sizes and orientations drawn on dotted paper with the dots as lattice points. The focus of the activity was to tilt squares and try to find a general formula for the number of dots inside the square of a particular tilt, as the side of the square changed.

Naturally, a second question arose. Would it be possible to predict the number of dots inside the square as the tilt changed? Initially it seemed almost impossible, but a change in perspective helped in making sense of this task. And so we moved from counting to generalization.

With the type of squares shown in Figure 1, it is very easy to predict the number of dots in each square. Here we can easily see that for a square with side length n i.e. with $n + 1$ dots on one side, the number of dots enclosed is $(n - 1)^2$. Putting this in words, the number of dots enclosed in a non-tilted square is the number of dots on one side reduced by 2 and then squared.

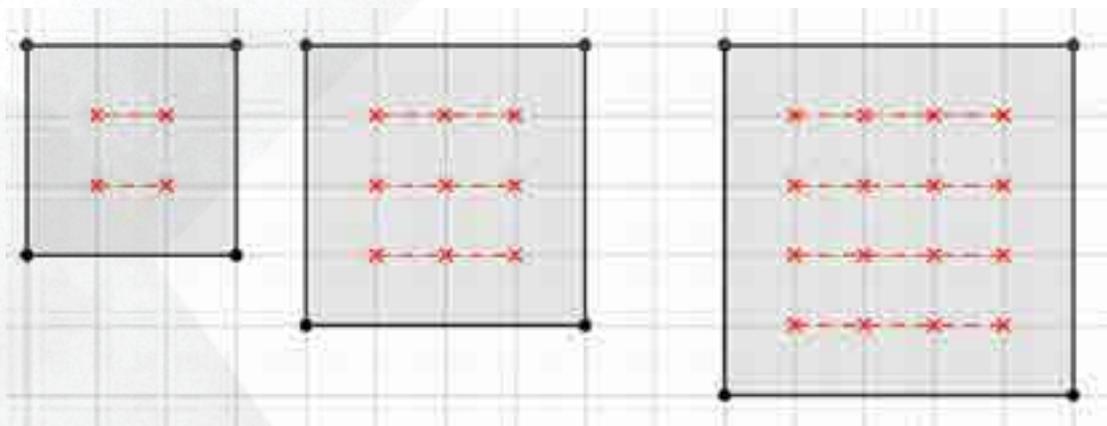
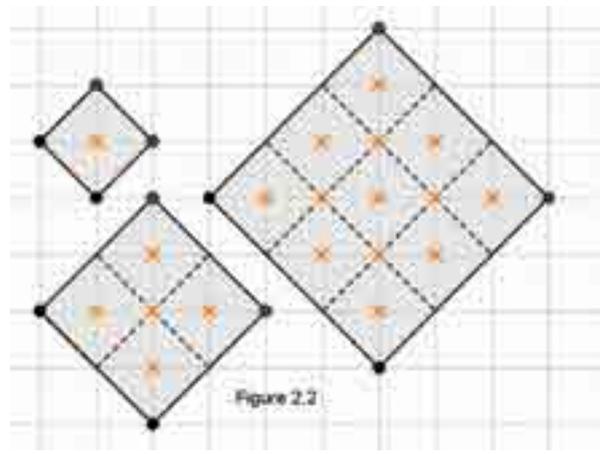
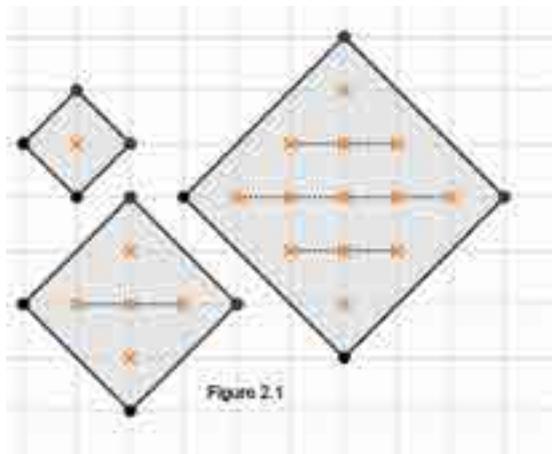


Figure 1. Squares with one side of slope 0

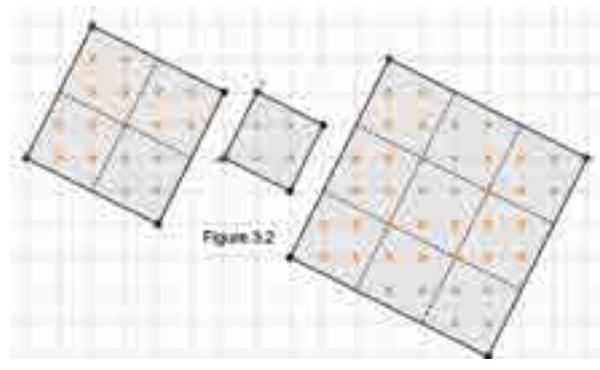
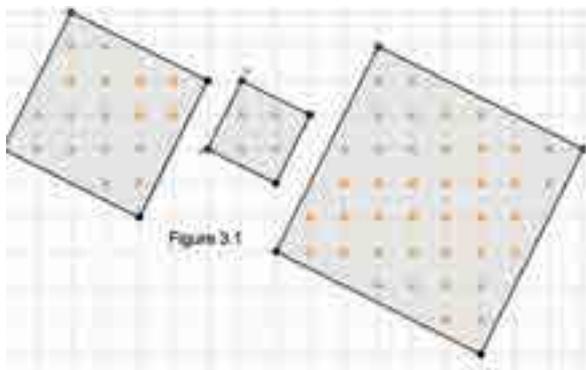
Keywords: Exploration, reasoning, visualisation, conjecture, proof

As we explored squares of increasing tilt, this is the sequence of thought that emerged. As you look at the pictures on the left (Figure 2.1, 3.1, etc.), you will see how difficult it is to get a general formula. But with the picture on the right (Figure 2.2, 3.2, etc.), you might even be able to arrive at a proof without words!



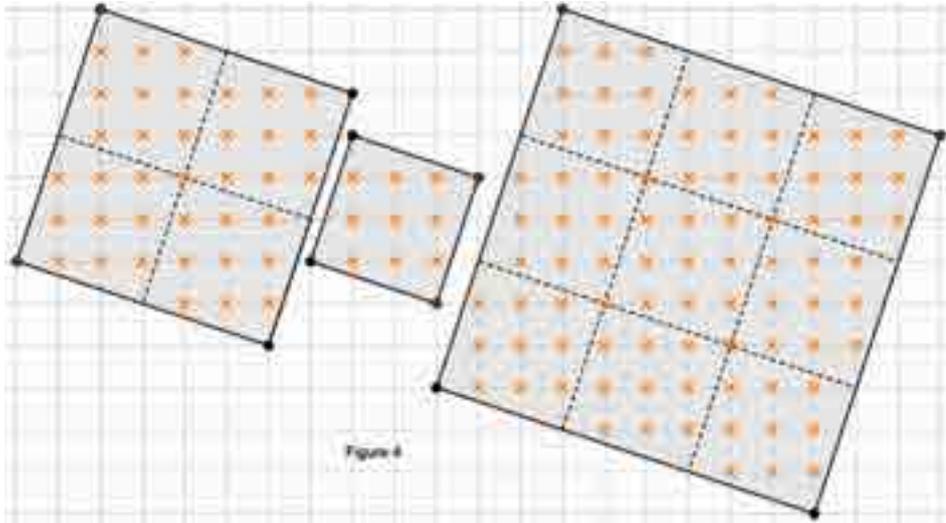
Figures 2.1 and 2.2 show squares of different sizes but all of them have a side of slope 1 (the other of slope -1). And all of them can be tiled with similar squares of side length $\sqrt{2}$. Each of the tiling squares encloses 1 dot and there is 1 dot at each point of intersection of the tiling lines. For a square of side length $n\sqrt{2}$, there are n^2 tiling squares (each with 1 dot inside) and $(n - 1)$ tiling lines along each direction, which intersect at $(n - 1)^2$ dots. So the number of dots inside each square with a side of slope 1 and length $n\sqrt{2}$ is $n^2 + (n - 1)^2$.

Does this continue for squares of greater tilt? Let us explore further.



Figures 3.1 and 3.2 show squares of different sizes but all of them have one side of slope 2 (the other of slope -1/2). And all of them can be tiled with similarly inclined squares of side length $\sqrt{5}$. Each of the tiling squares encloses 4 dots and there is 1 dot at each point of intersection of the tiling lines. For a square of side length $n\sqrt{5}$, there are n^2 tiling squares (each with 4 dots inside) and $(n - 1)$ tiling lines along each direction, which intersect at $(n - 1)^2$ dots. So the number of dots inside each outer square is $4n^2 + (n - 1)^2$.

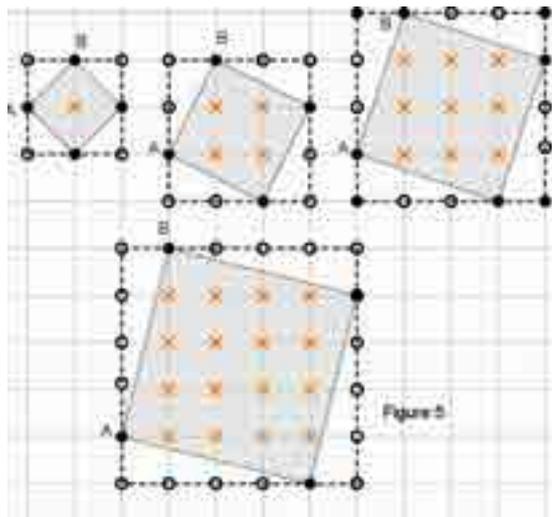
You can see from the figure below for a square with a side of slope 3 that we are beginning to arrive at a general formula.



Using dotted paper or GeoGebra, try drawing squares with a side of slope 4 with different side lengths and see if your conjecture regarding a formula is validated. Did you predict the number of dots and was your prediction correct?

Before we proceed further, let us try to make sense of two things.

- i. How many dots are there in a tiling square in which the slope of one side is m , a natural number?
- ii. How many tiling lines are there in a tilted square which has $n + 1$ dots on each side?



Question 1:

Consider the smallest tilted square of slope m . This is formed by going up by m units and across by 1 unit. Enclose each of these squares in the smallest non-tilted square possible. To do this, we go down from vertex A by one unit and across from vertex B by m units. In each case the side of the non-tilted square becomes $m + 1$ units with $m + 2$ dots on each side. The number of dots enclosed in such a square is therefore m^2 (decrease the number of dots on the side by 2 and then square as explained above). The four right-angled triangles outside the tilted square and within the non-tilted square will not contain any dot (except along the sides) because from each vertex of the tilted square we simply go out to the next dot to get the enclosing non-tilted square. So the number of dots in the smallest tilted square of slope m (where m is a natural number) is m^2 .

Question 2:

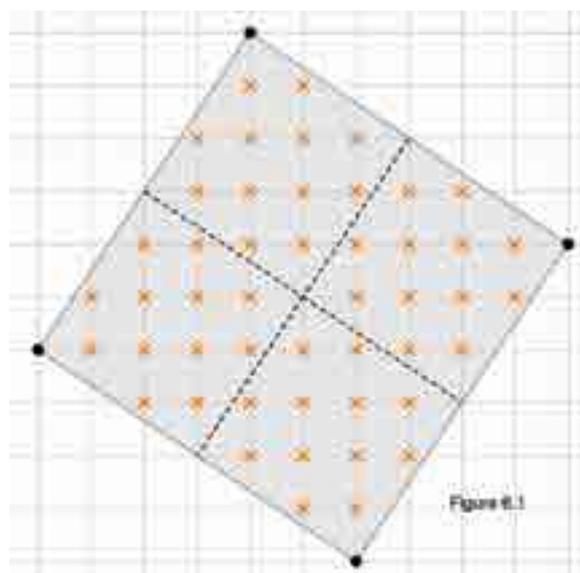
In a tilted square with $n + 1$ dots on each side, there are n^2 tiling squares. There will be $n + 1 - 2$ i.e. $n - 1$ tiling lines in one direction and $n - 1$ tiling lines in the perpendicular direction. Hence there will be $(n - 1)^2$ intersections.

Let us summarize our findings in a table

Slope of 1 side	Side length	Number of dots in each tiling square	Number of tiling squares	No of intersections of tiling lines	Total number of dots in the tilted square
1	$n\sqrt{2}$	1	n^2	$(n - 1)^2$	$n^2 + (n - 1)^2$
2	$n\sqrt{5}$	4	n^2	$(n - 1)^2$	$4n^2 + (n - 1)^2$
3	$n\sqrt{10}$	9	n^2	$(n - 1)^2$	$9n^2 + (n - 1)^2$
4	$n\sqrt{17}$	16	n^2	$(n - 1)^2$	$16n^2 + (n - 1)^2$
m	$n\sqrt{(m^2 + 1)}$	m^2	n^2	$(n - 1)^2$	$m^2n^2 + (n - 1)^2$

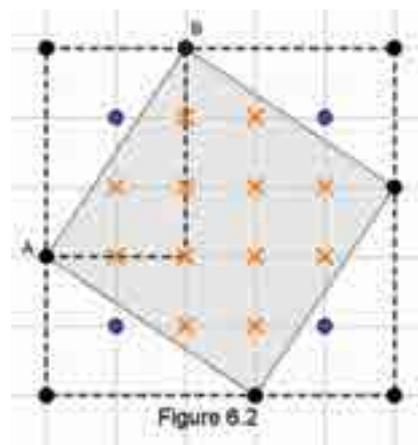
Please note that m and n are natural numbers here.

If m is a positive rational number of the form p/q , where p and q are co-prime with $p > q$, then what would the generalized formula be for the number of dots? Of course, if either p or q is 1, then the above holds. Let us explore what happens if neither p , nor q is equal to 1. In Figure 6.1, we look at a tilted square of slope $3/2$. We see that it too can be tiled, in this case into 4 smaller squares. However, this time, the tiling squares do not have dots in a square array inside.



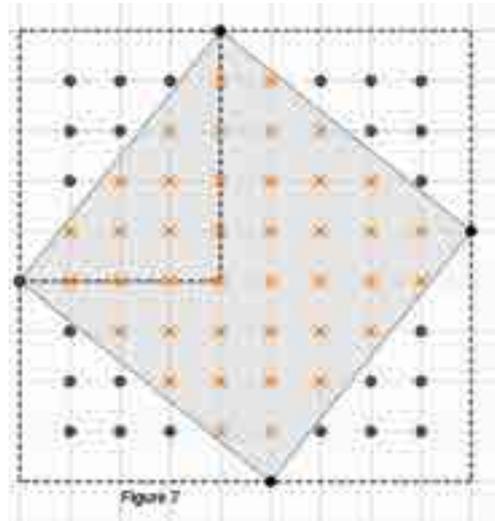
The smallest lattice point square of slope $3/2$ is shown in Figure 6.2. The enveloping non-tilted outer square is formed by going 3 units up from A and 2 units to the left from B. It is of side $3 + 2 = 5$ units and has $(3 + 2) + 1 = 6$ dots along each side and therefore $(6 - 2)^2 = 16$ dots inside.

If we consider the 3×2 rectangle drawn with A and B as opposite vertices, we see that the diagonal AB divides it into 2 congruent triangles each with one dot inside. The rectangle has $(3 + 1)$ dots on one side and $(2 + 1)$ dots on the other. So it has $(3 + 1 - 2) \times (2 + 1 - 2) = 2$ dots inside, with 1 dot on either side of the diagonal AB. Since the four triangles within the outer non-tilted square and the inner tilted square are congruent, the number of dots inside are all equal to 1. So the total number of dots inside the tilted square are $16 - 4 \times 1 = 12$ dots.



In Figure 6.1, there are 4 such squares; so the total number of dots inside is 48 to which we add one dot at the intersection of the tiling lines. This gives us a total of 49 dots.

Figure 7 shows a tilted square with one side of slope $5/4$. See if the same reasoning holds in this case too. Can we begin to generalise?



For a tilted square of slope p/q , where p and q are natural numbers which are prime to each other, the outer enveloping non-tilted square will be of length $(p + q)$ units and have $(p + q + 1)$ dots on each side. So the number of dots inside will be $(p + q - 1)^2$. The p by q rectangle will have $(p - 1)(q - 1)$ dots inside and so the number of dots inside the smallest tilted lattice square with one side of slope p/q is

$$(p + q - 1)^2 - 4 \times \frac{(p-1)(q-1)}{2} = p^2 + q^2 + 1 + 2pq - 2q - 2p - 2pq + 2q + 2p - 2 = p^2 + q^2 - 1.$$

Note that this formula is symmetric in p and q .

We can then calculate the total number of dots enclosed in bigger squares of slope p/q .

Let us summarize our findings in a table

Slope of 1 side	Side length	Number of dots in each tiling square	Number of tiling squares	No of intersections of tiling lines	Total number of dots enclosed by the tilted square
$2/3$	$n\sqrt{13}$	12	n^2	$(n - 1)^2$	$12n^2 + (n - 1)^2$
$3/2$	$n\sqrt{13}$	12	n^2	$(n - 1)^2$	$12n^2 + (n - 1)^2$
$5/4$	$n\sqrt{41}$	40	n^2	$(n - 1)^2$	$40n^2 + (n - 1)^2$
$2/1$	$n\sqrt{5}$	4	n^2	$(n - 1)^2$	$4n^2 + (n - 1)^2$
p/q	$n(\sqrt{p^2 + q^2})$	$p^2 + q^2 - 1$	n^2	$(n - 1)^2$	$(p^2 + q^2 - 1)n^2 + (n - 1)^2$

Note that this formula works even when p or q is 1, i.e., it is a general formula for non-negative integer values of p and q . Of course, with the constraint that $q \neq 0$.

Also note that this general formula is always of the form $4k$ or $4k + 1$ (we had earlier justified this using the symmetries of a square). Let us consider the subcases:

- n even i.e. $n = 2l$ for some natural number l : then $(p^2 + q^2 - 1)n^2 = 4(p^2 + q^2 - 1)l^2$ and $(n - 1)^2 = 4(l^2 - l) + 1$, therefore $(p^2 + q^2 - 1)n^2 + (n - 1)^2$ is of the form $4k + 1$
- n odd i.e. $n = 2l - 1$: then n^2 is of the form $4k + 1$ and $(n - 1)^2$ is of the form $4k$, so we need to consider the factor $p^2 + q^2 - 1$ – note that we need this factor to be of the form $4k$ or $4k + 1$
 - p, q both odd: then p^2 and q^2 both would be of the form $4k + 1$, so $p^2 + q^2 - 1$ would be of the form $4k + 1$
 - p odd, q even or vice-versa: then $p^2 + q^2 - 1$ would be of the form $4k$
 - p, q both even: we leave it to the reader to figure out why this case is not possible!

Further note the possible numbers of dots till 20:

No. of dots	p	q	n	No. of dots	p	q	n
1	1	1	1	4	2	1	1
5	1	1	2	9	3	1	1
12	3	2	1	13	1	1	3
16	4	1	1	17	2	1	2

8 and 20 are missing since there are no (p, q, n) that can make them!

Conclusion

In mathematics, a single question can spark off a series of investigations. Counting is one of the most elementary mathematical operations, taught at the beginning of primary school. Generalisation is a powerful mathematical technique. When counting becomes tedious, the thinking student tries to generalise. But the process needs to be carefully thought out, constraints and exceptions need to be kept in mind. Above all, the process needs to make sense to the student. We hope this train of thought makes sense to you, readers.



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The gap between 'HOW' and 'WHY' in Mathematics...

Contributed by G.R.Veena, M.Sc., PGDHE., M.Phil., CIG., PGDDE, HOD Mathematics, TISB, NAFL Valley,

Number tricks are fun to perform and are an excellent way to enhance mathematical skills. When I was young, we used to discuss a lot of number tricks. One amongst those was as follows:

Take a number with a lot of digits. For example, suppose you think of 2134567.

Add its digits: $2 + 1 + 3 + 4 + 5 + 6 + 7 = 28$.

Subtract 28 from the original number taken, 2134567. You'll get some answer, say $abcdefg$.

Keeping any one digit (say e) as secret, if you say the remaining digits, $abcdfg$ in any order, I can figure out the number ' e ' which you kept hidden from me in less than a fraction of a second...Just one request – that zero not be the hidden digit.

So in this case $2134567 - 28 = 2134539$. If you hide 3 and tell me the remaining digits 1, 2, 3, 4, 5 and 9, I will be able to tell you that the hidden number was 3.

I will illustrate this with one more example. If you thought of 345995, add its digits: $3 + 4 + 5 + 9 + 9 + 5 = 35$. $345995 - 35 = 345960$. If you hide 9 and tell me the remaining digits 0, 3, 4, 5, 6 then I will be able to tell you that the hidden digit was 9.

After learning the trick, I amazed many people with it. Try to figure out the trick...

If you're successful ...good. Otherwise, let me explain. The trick is add the digits and subtract this sum from the next higher multiple of 9. i.e., 'the multiple of 9 just higher than the digit sum'.

The difference is the hidden digit. When adding the digits, all pairs which add up to 9 and 9 itself can be omitted.

$2 + 1 + 4 + 5 + 3 + 9 = 24$ and $27 - 24 = 3$
(Alternatively you could use $2 + 1 + 3 = 6$
(omitting $5 + 4 = 9$ and 9 itself) and then $9 - 6 = 3$.)

In the second example: $0 + 3 + 4 + 5 + 6 = 18$, its digit sum is 9. $9 - 9 = 0$, and as I requested that zero not be the hidden digit, the hidden digit must be 9.

After learning 'How' to get it, frankly, I never bothered to find out 'Why' this happens. Our mathematics curriculum encourages us to learn how to solve a problem and we somehow learn the

tricks of the trade and ace the examination. Rarely do we question 'Why' the solution is obtained that way. 'How' to get the solution is just exercising mathematical thinking. In order to think like a mathematician, we should often ask 'Why'.

Now let's try to think like mathematicians and find out the reason behind this trick. For that let's start with smaller numbers such as this 2-digit number which we shall call XY .

Adding its digits gives $X + Y$. Subtracting this from the original number which is actually $10X + Y$ we get $10X + Y - (X + Y) = 9X$.

Similarly, if we have a 3-digit number: say XYZ . Adding its digits gives $X + Y + Z$. Subtracting this from the original number which is actually $100X + 10Y + Z$ we get $99X + 9Y$, again a multiple of 9.

Similarly, for a 4-digit number say $WXYZ$ we get the difference as $999W + 99X + 9Y$, a multiple of 9.

It is clear from the pattern that we always get a multiple of 9. Any number is a multiple of 9 if the sum of its digits is a multiple of 9. Thus, when the difference is less than 9, the hidden digit should be the difference between 9 and the difference.

There are plenty of situations wherein we use formulae routinely without actually bothering to know 'why' the formula is so. For instance, the volume of a cone or the surface area of a sphere, or even the formula for the area of a circle. It will give us a clear understanding of the concepts/formulae if we make the conscious effort to ask 'why' while learning mathematical concepts.

Admittedly, the gap between 'how' and 'why' is not small. Making an effort and constantly thinking about it may sometimes lead us in the right direction. There are many simple sounding but unsolved problems, for instance, the Goldbach Conjecture, conjectures about Twin Primes, Perfect Numbers, ...; these conjectures are easy to understand but to date we wonder 'why' they should be true. Unless we find out 'why', they remain conjectures that may or may not be true.

<http://www.math.utah.edu/~pa/math.html>

https://artofproblemsolving.com/wiki/index.php?title=Goldbach_Conjecture

TearOut Fun with Dot Sheets

The 'TearOut' series is back, with perimeter and area. Pages 1 and 2 are a worksheet for students, while pages 3 and 4 give guidelines for the facilitator. This time we explore shapes with given perimeter or given area using the dots or the grid.

You will have to draw various shapes in this worksheet. The corners (or vertices) of each shape must be a dot (or where the lines cross each other on the grid). You can use only sleeping (horizontal) lines and standing (vertical) lines. Do not use any slant line. Check Figure 1.

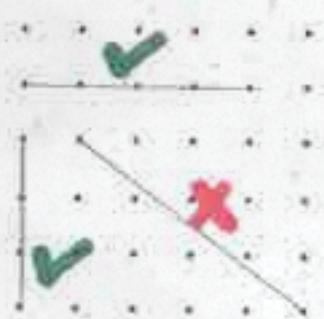


Figure 1

The gap between any two adjacent dots along a horizontal or vertical line should be taken as unit length. Similarly, the area of the smallest possible square with the dots as vertices should be considered one square unit. Likewise, the smallest line segment and the smallest square on a square grid should be taken as unit length and square unit respectively (Figure 2).

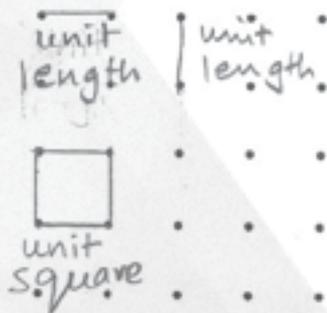


Figure 2

Get some dot sheets (or square grid papers), a pencil, an eraser and, ideally, a scale ... you are all set to go... 😊

1. Draw shapes with area 8 sq. units.
 - a. Find the perimeter of each shape.
 - b. Did any shape have a perimeter of 14 units? If not, draw one such.
 - c. Did you get different shapes with the same perimeter?
 - d. How long is the shortest perimeter?
 - e. How long is the longest one?
2. Draw shapes with perimeter 14 units.
 - a. Find the area of each shape.
 - b. Could you draw a shape with area 8 sq. units that is different from the one you drew for 1.b.?
 - c. Did you get different shapes with the same area?
 - d. What is the smallest area?
 - e. What is the largest area?
3. Draw the following:
 - a. A shape with area less than 8 sq. units and perimeter shorter than 14 units
 - b. A shape with area more than 8 sq. units and perimeter longer than 14 units
 - c. A shape with area less than 8 sq. units but perimeter longer than 14 units
 - d. A shape with area more than 8 sq. units but perimeter shorter than 14 units

	Area	Perimeter
Bigger rectangle		
Smaller rectangle		
L-shaped region		

4. Draw any rectangle, which is at least 6 units long and 4 units wide. Draw a smaller rectangle inside this bigger one such that they share a corner (Figure 3). Shade the L shaped region in between the two rectangles.

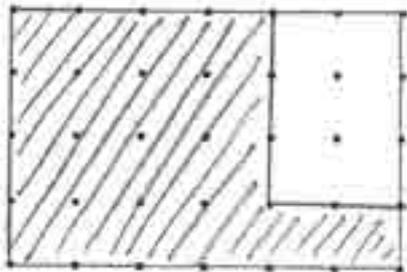


Figure 3

- Find the areas and the perimeters of the rectangles and the L shaped region.
 - How are the three areas related?
 - How are the three perimeters related?
5. Repeat the above with other pairs of rectangles
- Do you see any patterns? What are they?
 - How can you explain these patterns?
6. Draw another big rectangle with the same length and width as the one you drew for 4. Draw another smaller rectangle with the same length and width as the smaller one you drew for 4. This time draw the smaller rectangle inside the bigger one such that they share one side but do not share any corner (Figure 4). Shade the U-shaped region in between the two rectangles.

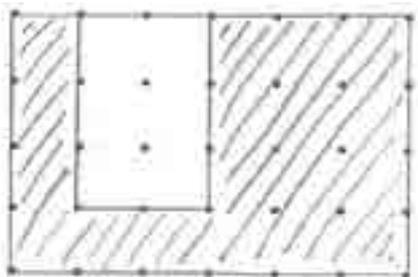


Figure 4

	Area	Perimeter
Bigger rectangle		
Smaller rectangle		
U-shaped region		

- Find the areas and the perimeters of the rectangles and the U shaped regions.
- How are the three areas related?
- How are the three perimeters related?

7. Repeat the above with other pairs of rectangles
- Do you see any patterns? What are they?
 - How can you explain these patterns?
8. Draw any shape and call it Shape 1. Modify it to form a new shape in the following manner. The modified shape and the original one must not differ too much.*
- Shape 2 with same perimeter but smaller area
 - Shape 3 with same area but longer perimeter
 - Shape 4 with smaller area but longer perimeter
- In each case, explain how you modified the shape.

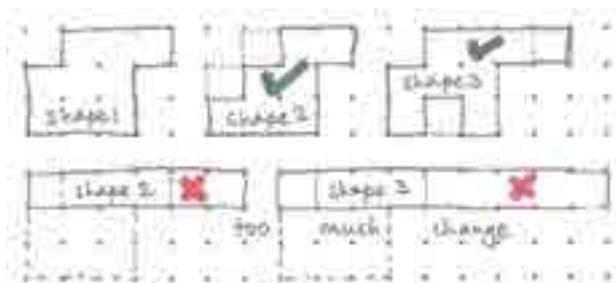


Figure 5

* For example, in Figure 5: top row – shape 1 has 9 squares, shape 2 and shape 1 differ by 2 (out of 9) squares; shape 3 and shape 1 also differ by that much. However, in the bottom row, shape 2 (and shape 3) differ from shape 1 by 6 (out of 9) squares. This ratio should be less than half [i.e., $2/9 < 1/2$ but $6/9 = 2/3 > 1/2$].

This worksheet can be done with Grades 4-5 children once they have an exposure to perimeter and area. The pre-requisite is the understanding of these two measures and not any formula.

The restriction on using only vertical and horizontal lines makes it easier to find the area and the perimeter of the drawn shapes. Perimeter can be found by simply counting the unit lengths along the border of a shape. Similarly, area is given by counting the squares enclosed within the shape. Since dot sheets don't have lines, it is easier to find the perimeter of the shapes drawn compared to a square grid with lines.

The worksheet allows explorations and observations. It encourages one to find patterns and explain them. The last part urges one to use the learnings to create further shapes. This can be used to assess if children can differentiate between perimeter and area. It will help them realize that different shapes can have the same perimeter (Q.2) or same area (Q.1) or both (Q.2b). It breaks the misconception that if perimeter is increased, then area must increase and vice versa (Q.3c, 3d). It also helps them figure out how to reduce area (i) without decreasing perimeter (Q.4, Q.5) and (ii) by increasing perimeter (Q.6, Q.7). They are particularly asked to investigate patterns and articulate the same (Q.5 and Q.7). Finally, they are asked to utilize their learnings to create new shapes with given specifications (Q.8).

1. This question allows children to observe that different shapes can have the same area but possibly different perimeters. All the shapes are essentially octominoes, i.e., polygons made by joining 8 squares. The longest and shortest possible perimeters for octominoes are 18 units and 12 units respectively.
2. The question explores polygons with same perimeter. So it is a relatively harder task on the dot sheet (or grid). In this case, the maximum and minimum possible areas are 12 sq. units and 6 sq. units respectively. In other words, the shapes can be made with 6-12 squares.

3. This question is about testing whether children developed any intuitive idea on how to maximize/minimize perimeter and area. While the first two parts (a and b) are easy to do, the subsequent parts (c and d) pushed them to develop these ideas.
4. This question helps children figure out how an L-ing of a rectangle reduces its area – since that of the smaller rectangle is subtracted – while keeping the perimeter same.
5. This takes it up a notch. The goal is to get children to observe that an L-ing would always reduce the area but would always keep the perimeter intact. They should be encouraged to explain why the perimeter remains unchanged. This will help them see that the perimeters of the bigger rectangle ABCD and the L-shape ABCGFE are the same because ED and DG get exchanged with FG and EF respectively. Since $ED = FG$ and $EF = DG$ being opposite sides of the smaller rectangle EFGD, this exchange causes no change in the perimeter (Figure 6).

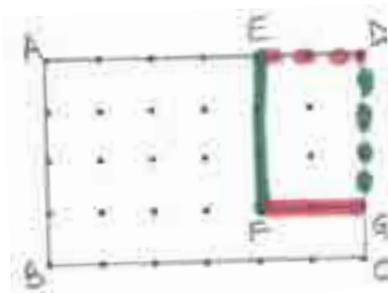


Figure 6

6. As a continuation of the above, this question considers the often un-intuitive situation where area reduces while perimeter increases. This introduces children to the U-ing of a rectangle with the same effective change in the area as in 4.
7. This question is similar to 5. It gets children to conjecture that a U-ing of a rectangle should always reduce the area while increasing the perimeter. Children should be encouraged to observe and argue how the U-ing results in two extra sides FG and EH of the smaller rectangle getting added to the resultant U shape ABCDEHGF, thereby causing a longer

perimeter (Figure 7). Note that FE in the rectangle ABCD gets replaced by GH in the U-shape.

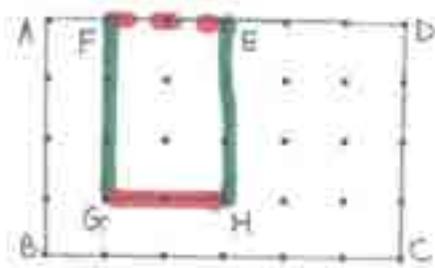


Figure 7

8. Takes this learning further and asks children to apply them to start with a shape and modify it according to given specifications. It may be natural for children to start with a rectangle. This will work fine for 8a and 8c if the minimum dimension is at least 2 units. But it may not work for 8b. So, the starting point has to be a more interesting shape. The first part (a) is a direct application of L-ing, the second one (b) can be achieved by changing an L to a U and the last one can be achieved by applying U-ing.

If this worksheet is used for higher grades, then slant lines can be allowed. The basic aspects of L-ing and U-ing can be generalized by cutting out non-rectangular shapes as well. The starting shape does not need to be rectangular either. Below are some examples of L-ing and U-ing with polygons made with slant lines. Will L-ing always keep the perimeter same? If not, then when will it? Figure 8 includes three cases:

- a. U-ing
- b. L-ing with no change in perimeter – Why?
- c. L-ing with changed perimeter – Increased or decreased? Justify.

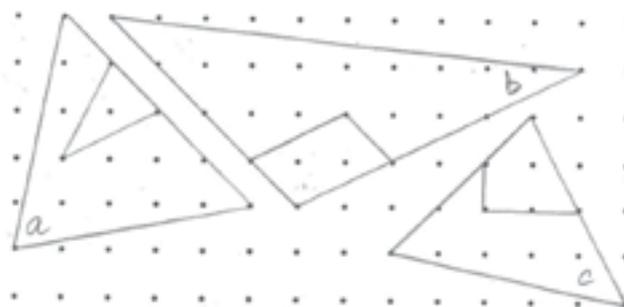


Figure 8



The ground area of this camp has been beautifully designed by removing grass of a fixed square unit on alternate positions



Design of a camp area near Sela Pass, Tawang, Arunachal Pradesh



Photo & Ideation: Kumar Gandharv Mishra

Mathematical Relevance: Tessellation

Exploring Properties of Addition and Multiplication with Integers

SWATI SIRCAR

This is the third in the series of explorations of the properties of addition and multiplication with different number sets. After considering the set of (i) Whole numbers and (ii) Non-negative rational numbers¹, in this article we will deal with the integers. This is a good time to reflect on the series – its need and aspiration.

Why are we doing this series?

The seed idea for this series came from the Pullout on Multiplication in *At Right Angles*, March 2013 issue where Padmapriya Shirali mentioned how the commutative, associative and distributive properties of multiplication can be verified visually for the set of whole numbers.

The more popular approach to justifying them involves taking any two (or three) whole numbers, computing both LHS and RHS of the number fact we want to establish (e.g. $3 \times 7 = 7 \times 3$, $(5 \times 4) \times 9 = 5 \times (4 \times 9)$, etc.) and checking that they are the same. This is an inductive process.

However, Padmapriya's approach can be generalized for any combination of two or three whole numbers regardless of how large they may be. If we agree that any whole number can be represented by that many counters, then her processes would work for any combination of whole numbers. It may be difficult or impossible to arrange counters for big enough numbers, but it can surely be visualised. Her processes free one from computing on a case-by-case basis, i.e., an inductive process and encourages one towards a deductive process. The generalizability is at the heart of the deductive aspect of these visual approaches.

¹ The set of non-negative rational numbers include fractions and whole numbers. Henceforth, we will refer to this set as fractions.

So, we wanted to explore this approach for commutative and associative properties of addition with whole numbers. In addition, we wanted to explore these properties for addition and multiplication for other number sets viz. (i) fractions (including whole numbers) i.e. non-negative rational numbers, (ii) integers, (iii) rational numbers and (iv) real numbers – in short, all the number sets children encounter up to the secondary level. We ran into certain issues with some of these number sets and will discuss them at the end of the series. At the same time, we could come up with ‘almost proof’ for some of the cases – ‘almost proof’ in the sense that we went from some established results through a series of logical deductive steps to the result we wanted to establish. The ‘almost’ refers to the visual approach or pattern approach taken in establishing some of the results. In the remaining cases, we could come up with purely visual approaches similar to Padmapriya’s.

It is important to note that visualisation of specific cases is different from visuals that can be extended to the general case under consideration. The latter is essentially Proof Without Words (PWW). However, many consider PWWs different from deductive proof.

What have we done so far?

We have published the following so far in one article and a poster in the July 2018 and March 2019 issues respectively:

- a. Commutative and associative properties of addition for whole numbers through visualisation and an activity respectively – both generalisable for any combination of whole numbers.
- b. All five properties for addition and multiplication for fractions – all through visualisations with generalisation for any combination of fractions (and whole numbers).

Where do we go from here?

We want to complete the series with three more articles:

- i. All five properties of the two operations (i.e., addition and multiplication) with integers – we have used coloured counters (mentioned and used extensively in the Pullout on Integers in At Right Angles, November 2016 issue) for commutative and associative properties of addition and for distributive property of multiplication. We could provide deductive reasoning for the remaining two properties of multiplication based on the three multiplication facts established through patterns (details later).
- ii. An alternative visual model of multiplication based on scaling with the use of number lines. This is needed to establish similar multiplication facts for non-discrete and negative numbers viz. rational and real.
- iii. All five properties of the two operations with rational and real numbers. This will involve generalization to coloured lengths and coloured areas from the counters. However, coloured volume will not be required since deductive arguments, similar to integers, will work.

Now let us dive into the set of integers. This set doesn’t have the advantage of non-negativity. But it is a discrete set.

Children usually meet integers in their upper primary when they are more capable of understanding patterns and logical arguments. However even though the terms commutativity, associativity and distributivity may appear, they are not usually justified in any way other than case by case computation. Most textbooks conveniently assume the commutativity of multiplication for integers especially while defining positive times negative and negative times positive. We feel that the definition of multiplication should be separated and that it should precede the properties.

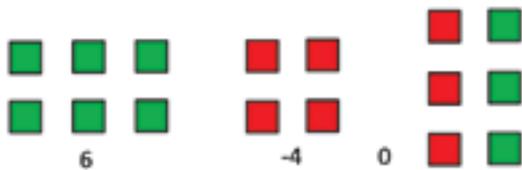


Figure 1

We have used green and red counters as positive and negative units respectively following the Integer Pullout. In this case, square counters have an edge over the round ones. We explain the advantage of squares later. So, six is represented by six green counters, negative four (-4) by four red counters and zero by the absence of any or by equal number of red and green counters (Figure 1).

Addition is modelled in a manner similar to the whole number case. Each integer is represented by that many counters of the right colour. These sets of counters are arranged left to right according to the given addition expression. The combined set of all counters represents the sum. There is an extra step of removing zero pairs. However, this step is not crucial for our purpose.

Commutative Property of Addition

The basic idea behind commutative property is looking at a sum from two vantage points as in the case of whole numbers and fractions.

When it comes to integers, there are four possibilities:

1. Positive + positive
2. Positive + negative (and \therefore negative + positive)
 - a. Sum > 0
 - b. Sum < 0
3. Negative + negative

1 is identical to whole numbers. We have shown an example in Figures 2 and 3 for 2b. The reader can (and should) explore 2a and 3 in similar manner.

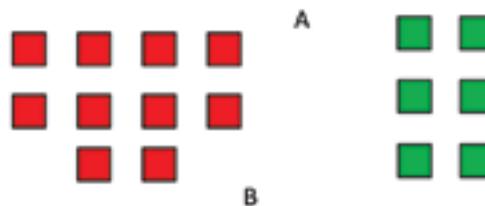


Figure 2: $(-10) + 6$

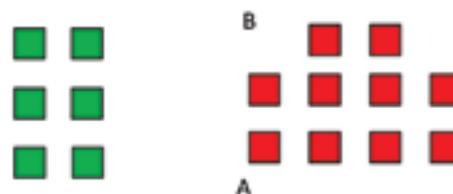


Figure 3: $6 + (-10)$

Figure 2 is from B's perspective and it shows $(-10) + 6$ while Figure 3 shows the same sum from A's view point and it is $6 + (-10)$. Since only the perspective changes, the sums remain unchanged, i.e., $(-10) + 6 = 6 + (-10)$. Note that this holds for any two integers no matter how far from zero, i.e., how ever many counters may be needed to represent them.

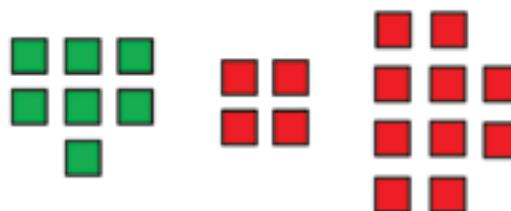


Figure 4

Associative Property of Addition

For this one, the basic idea is that if we have to add $x + y + z$, then it doesn't matter whether we combine x and y first or y and z . This is the same as the whole number case.

It is best done as an activity. Pick any three integers, say 7, -4 and -10 and represent them as piles of appropriate counters, i.e., 1st pile with 7 green counters, 2nd pile with 4 red counters, etc. (Figure 4). Now the sum $7 + (-4) + (-10)$ will be all three piles combined into one. If only two piles can be combined at each step, then step 1 can be combining 1st and 2nd pile, i.e., the

addition $7 + (-4)$ and step 2 can be combining the 3rd pile with this, i.e., adding -10 to the sum $7 + (-4)$ or $[7 + (-4)] + (-10)$. On the other hand, step 1 can also be combining 2nd and 3rd pile or $(-4) + (-10)$ with step 2 as combining 1st pile with this, i.e., adding 7 or $7 + [(-4) + (-10)]$. Both lead to the combination of all three piles into one with no extra counter coming in or going out. So, the two sums must be equal, i.e., $[7 + (-4)] + (-10) = 7 + [(-4) + (-10)]$. The removal of zero pairs, i.e., pairs of green-red counters can happen after the three piles have been combined.

Observe that the three integers were chosen arbitrarily, and this can be extended (at least as a thought experiment) for any three integers regardless of how far they are from zero.

There are 14 possibilities:

1. All 3 positive

2. 2 positive and 1 negative

- Positive + positive + negative
- Positive + negative + positive
- Negative + positive + positive

Each of 2 and 3 has two possibilities
 i. Sum > 0
 ii. Sum < 0

3. 1 positive and 2 negative

- Positive + negative + negative
- Negative + positive + negative
- Negative + negative + positive

4. All 3 negative

The above illustration is an example of 3a with sum < 0 . We leave the rest for the reader to explore.

Defining **multiplication** for integers is tricky since no model does the job completely. There are three cases involving negative numbers and multiplication – (a) negative \times positive, (b) positive \times negative, and (c) negative \times negative. The NCERT textbook does a good job for integers by extending multiplication tables. The Integer chapter of Class 7 textbook deals with the above three cases separately and without

assuming commutativity. However, several other textbooks assume commutativity and treat (a) and (b) as equivalent. We attempt to first define or make sense of each of the above kind of products and then justify the properties.

Let us establish the following key facts case by case (m and n are any whole numbers):

a. Negative \times positive = negative

This is done using the notion of repeated addition, i.e., $(-4) \times 3 = (-4) + (-4) + (-4) = -12$. Now repeated addition can be modelled using the counters, which in this case would be negative ones. It can be argued that this is identical to the positive \times positive case i.e., 4×3 except for the colour of the counters. Or in other words, it generates the same array of 4×3 counters except for the colour. So, this array has 4×3 negative counters and thus represents the integer $-(4 \times 3)$. Therefore, $(-4) \times 3 = -(4 \times 3)$. Note that 4 and 3 can be replaced by any whole numbers and the same argument holds. So, we have established the general case $(-m) \times n = -(m \times n) \dots (1)$

b. Positive \times negative = negative

This is explored by extending the multiplication table of m beyond $n = 1$ to $n = 0$ and then towards $n < 0$ maintaining the pattern observed in the table. As we move up the table of m from $n = 4, 3, 2$, etc., we notice that the product is reducing by m in each step. This way or otherwise, we get that $m \times 0 = 0$. So, the next step would be $m \times (-1) = 0 - m = -m$. This approach can be combined with skip counting on the number line to realize that $m \times (-n)$ is n skips of m lengths each starting from zero towards left or the negative side. Now, this is the same as n times repeated addition of $-m$, i.e., $(-m) \times n$. So, $m \times (-n) = (-m) \times n$ and therefore $m \times (-n) = -(m \times n) \dots (2)$. Note that this is not commutativity.

c. Negative \times negative = positive

This combines the previous two cases. It starts with creating a table for $(-m)$ and then extending to $n < 0$. In this case, as we move up from $n = 3, 2, 1$, etc., the product becomes

$-3m, -2m, -m$. So, the product increases by m in each step. So, $(-m) \times 0 = 0$. Combining this with skip counting on the number line, $(-m) \times (-1) = 0 + m = m$. And in general $(-m) \times (-n)$ is n skips of m steps from zero towards right or the positive side which is the same as $m \times n$. Therefore, $(-m) \times (-n) = m \times n \dots$ (3)

However, there is no such model to establish similar meaning for rational numbers – e.g. how should one define $(-3/5) \times (-7/4)$? There is no real-life example suitable for children at an elementary level to make sense of such examples. Therefore, we want to fill this gap through the next article on a visual model of multiplication based on scaling with the use of number lines.

Coming to **properties of multiplication**, we would take a less hands on approach since we have established some key results. This is also a gentle way of showing children how mathematical proofs are done, i.e., how established results can be used step by step to deduce something new. Let us recap the three results:

For any two natural numbers n and m

- (1) $m \times (-n) = -(m \times n)$
- (2) $(-m) \times n = -(m \times n)$
- (3) $(-m) \times (-n) = m \times n$

Commutative Property of Multiplication

This can be done as an extension of the whole number approach with rectangular array using the above three results. However, the proofs are gentle enough for upper primary level and can help children get a flavour of how math builds up logically.

There are three possibilities similar to the ones for commutativity of addition.

We have already established

$$(4) m \times n = n \times m$$

i.e., the positive \times positive case which is identical to natural numbers in the Multiplication PullOut.

Now we want to show the remaining cases: positive-negative i.e. $m \times (-n) = (-n) \times m$ and negative-negative i.e. $(-m) \times (-n) = (-n) \times (-m)$ for any natural numbers m and n .

Commutativity of cases involving zero are trivial.

$$\begin{aligned} m \times (-n) &= -(m \times n) && \text{by (1)} \\ &= -(n \times m) && \text{by (4)} \\ &= (-n) \times m && \text{by (2)} \\ (-m) \times (-n) &= m \times n && \text{by (3)} \\ &= n \times m && \text{by (4)} \\ &= (-n) \times (-m) && \text{by (3)} \end{aligned}$$

Associative Property of Multiplication

As in associativity of addition, there are 8 possibilities:

1. $m \times n \times p$
2. $m \times n \times (-p)$
3. $m \times (-n) \times p$
4. $(-m) \times n \times p$
5. $m \times (-n) \times (-p)$
6. $(-m) \times n \times (-p)$
7. $(-m) \times (-n) \times p$
8. $(-m) \times (-n) \times (-p)$

Note that 1 is the whole number case which was established in the Multiplication Pullout, i.e., we have

$$(5) (m \times n) \times p = m \times (n \times p)$$

We have shown 3, 5 and 8 below and have left the rest for the reader to try.

$$\begin{aligned} 3. [m \times (-n)] \times p &= -(m \times n) \times p && \text{by (1)} \\ &= -[(m \times n) \times p] && \text{by (2)} \\ &= -[m \times (n \times p)] && \text{by (5)} \\ &= m \times [-(n \times p)] && \text{by (1)} \\ &= m \times [(-n) \times p] && \text{by (2)} \end{aligned}$$

2 and 4 can be shown in a similar way.

$$\begin{aligned}
5. [m \times (-n)] \times (-p) &= -(m \times n) \times (-p) \quad \text{by (1)} \\
&= (m \times n) \times p \quad \text{by (3)} \\
&= m \times (n \times p) \quad \text{by (5)} \\
&= m \times [(-n) \times (-p)] \quad \text{by (3)}
\end{aligned}$$

6 and 7 can be shown similarly.

$$\begin{aligned}
8. [(-m) \times (-n)] \times (-p) &= (m \times n) \times (-p) \quad \text{by (3)} \\
&= -[(m \times n) \times p] \quad \text{by (1)} \\
&= -[m \times (n \times p)] \quad \text{by (5)} \\
&= (-m) \times (n \times p) \quad \text{by (2)} \\
&= (-m) \times [(-n) \times (-p)] \quad \text{by (3)}
\end{aligned}$$

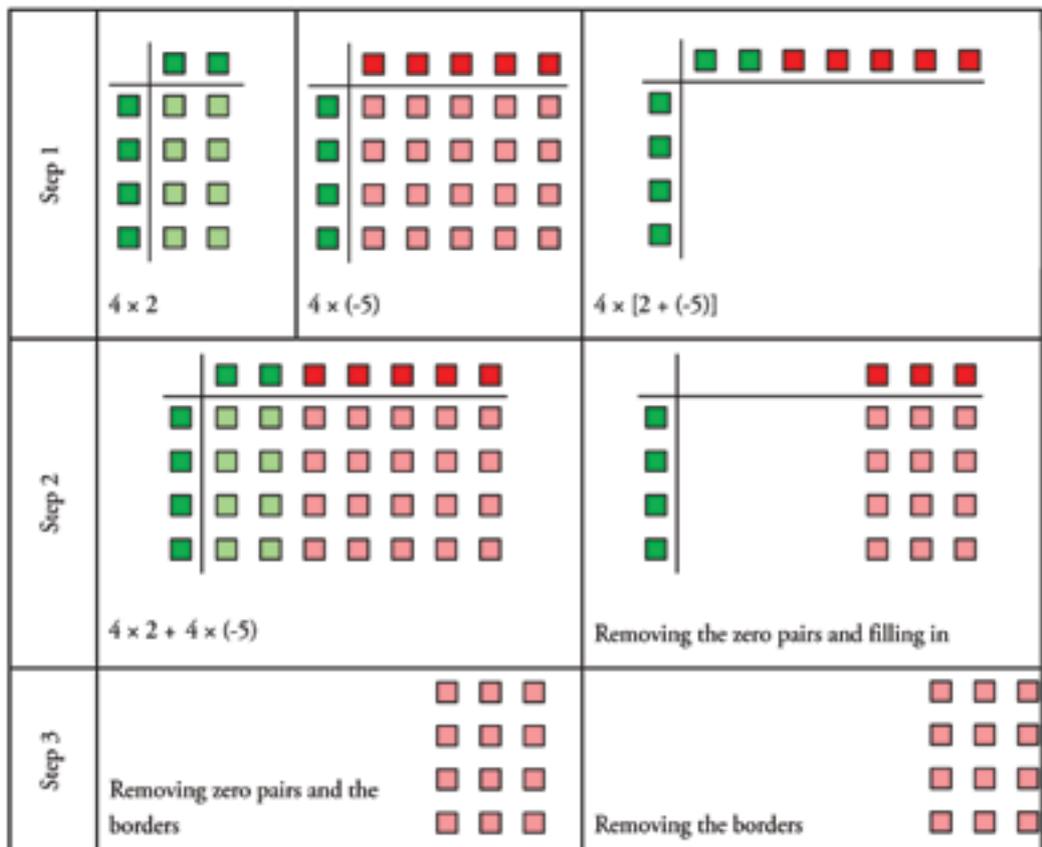
Distributive Property of Multiplication

Unlike the above two, the distributive property involves both addition and multiplication. So, we could not come up with a proof, but the array worked out. Here (1)–(3) will be used to create the array or the product given the integers.

There are 8 possibilities:

1. Positive \times the sum

The counters in the array, representing the product, are in a lighter shade to distinguish them from the counters in the borders, representing the factors.



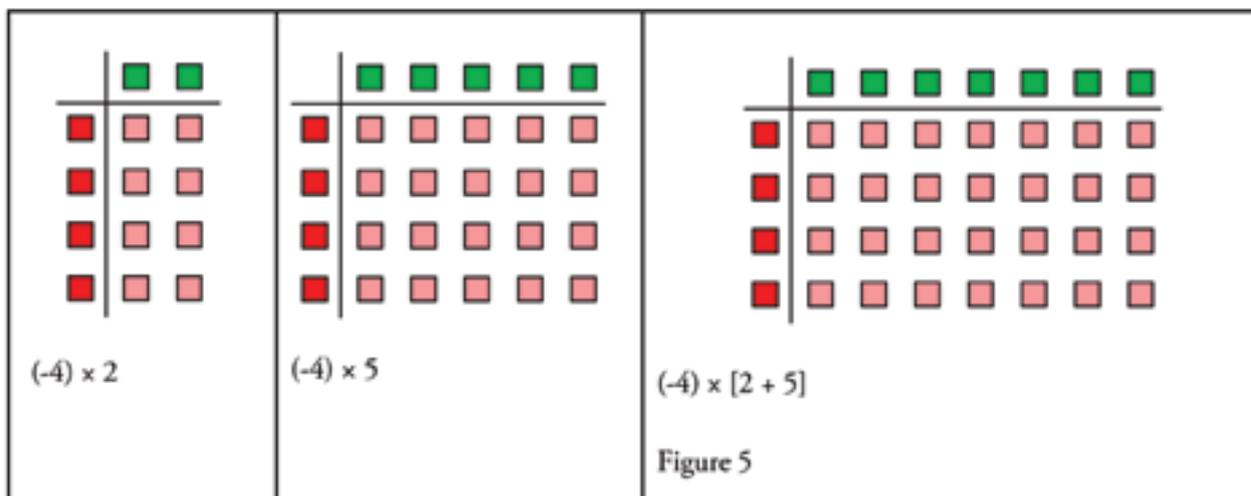
- a. Positive \times (positive + positive)
- b. Positive \times (positive + negative)
 - i. Sum > 0
 - ii. Sum < 0
- c. Positive \times (negative + negative)

2. Negative \times the sum

- a. Negative \times (positive + positive)
- b. Negative \times (positive + negative)
 - i. Sum > 0
 - ii. Sum < 0
- c. Negative \times (negative + negative)

Note that 1a is identical to the whole number case shown in Multiplication Pullout. Also, negative + positive = positive + negative by commutativity of addition which we have established already.

We have shown 1b(ii) and 2a. The reader is encouraged to try the rest.



Since the last arrays are identical for both, $4 \times 2 + 4 \times (-5) = 4 \times [2 + (-5)]$. Note the use of color for the array in $4 \times (-5)$ based on (2).

In the following example of 2a, (1) is used to determine the colour of the arrays.

The 3rd picture of Figure 5 can be easily seen as the sum of the 1st and the 2nd i.e. $(-4) \times [2 + 5] = (-4) \times 2 + (-4) \times 5$.

As in previous examples, observe that the integers chosen could have been arbitrarily distanced from zero. So, this procedure is valid for any three integers. In particular, if the 1st integer is zero then all the products are zero and the identity is trivially true. Similarly, if the 2nd or the 3rd integer is zero then one of the products become zero, i.e., one of the arrays vanishes and again the identity is trivially true.

The square colored counters arranged in a line can be generalized to coloured and therefore

signed length and can be combined with the number line. If these counters are arranged in an array, they can be generalized to coloured, i.e., signed area. We did not face the need for signed volume. Also, this works for any 2-3 integers but not for rational or real numbers.

Rational and Real Numbers

Whole numbers were the easiest since they are discrete and non-negative and therefore can be modelled with counters. Fractions do not have the advantage of discreteness. But thanks to their non-negativity, they can be modelled by area (sometimes proportionate to length) and volume. Integers included negative numbers but could be modelled by coloured counters thanks to their discreteness. However, rational and real numbers are neither discrete, nor non-negative. So, they proved to be the most challenging sets. We will meet them in the next two articles.



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Can there be SSA congruence?

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

The well-known congruence theorems tell us that two triangles are congruent to each other under any of the following conditions:

SAS: *side-angle-side*; the important thing here is that the angle is included between the two sides;

ASA: *angle-side-angle*; the important thing here is that the equal sides are opposite equal angles;

SSS: *side-side-side*;

RHS: *right angle-hypotenuse-side*.

As these congruence rules are well known, we do not amplify on what they mean.

What about SSA congruence?

A question which arises when we examine this list is this: *What about SSA?* That is, if two sides of a triangle have the same lengths as two sides of another triangle, and one angle of the first triangle has the same measure as one angle of the second triangle, what can be said about them? Under what circumstances will they be congruent to one another? Of course, we do not need to consider the case when the equal angles are included between the pairs of equal sides; that would be the SAS situation, which does lead to congruence. (That is why we have labelled it 'SSA'. To spare our feelings, we shall avoid labelling it 'ASS'.)

To make matters more definite, suppose we have two triangles ABC and DEF such that $AB = DE$, $BC = EF$ and $\angle BAC = \angle EDF$ (Figure 1). We wish to ascertain under what circumstances this can lead to congruence of the two triangles.

Keywords: triangles, congruence, corresponding sides, included angle

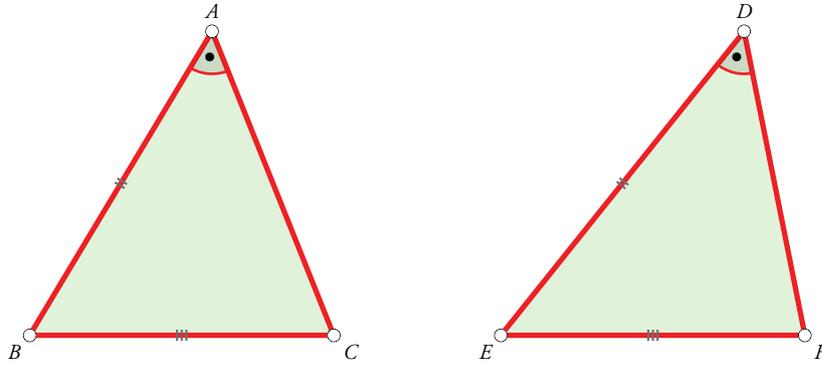


Figure 1.

Invoking the sine rule immediately throws light on the matter. We have:

$$\frac{BC}{\sin \angle BAC} = \frac{AB}{\sin \angle ACB},$$

$$\frac{EF}{\sin \angle EDF} = \frac{DE}{\sin \angle DFE}.$$

Since $BC = EF$ and $\angle BAC = \angle EDF$, we get $BC/\sin \angle BAC = EF/\sin \angle EDF$, and so $AB/\sin \angle ACB = DE/\sin \angle DFE$. Since we also have $AB = DE$, it follows that

$$\sin \angle ACB = \sin \angle DFE.$$

If two angles in the interval from 0° to 180° have equal sines, then two possibilities exist: *either the angles are equal to each other, or they are supplementary to each other.*

We see immediately from this that SSA does not lead to congruence of the two triangles. (Obviously, this is why we do not have a ‘SSA congruence theorem’.) On the other hand, we do obtain some positive information about the situation.

We obtained the above conclusion using trigonometry. But elementary geometry leads to exactly the same conclusion, when we attempt to construct a triangle given the following data: side BC , side AB , $\angle BAC$. Figure 2 illustrates what we mean: instead of one triangle, we get two possible triangles, $\triangle ABC$ as well as $\triangle ABD$. Observe that $\angle ACB$ and $\angle ADB$ are supplementary to each other, in agreement with the trigonometric analysis.

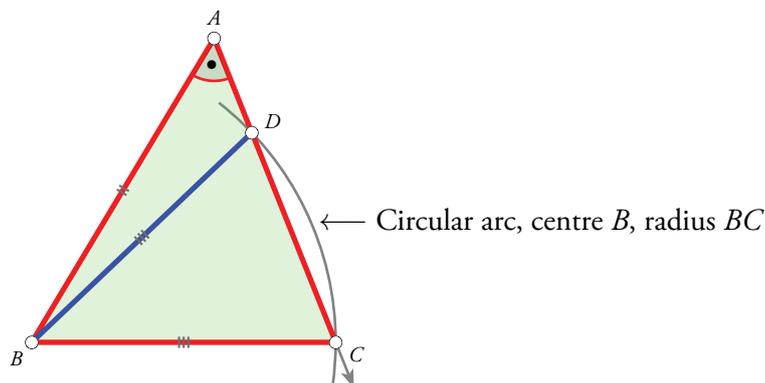
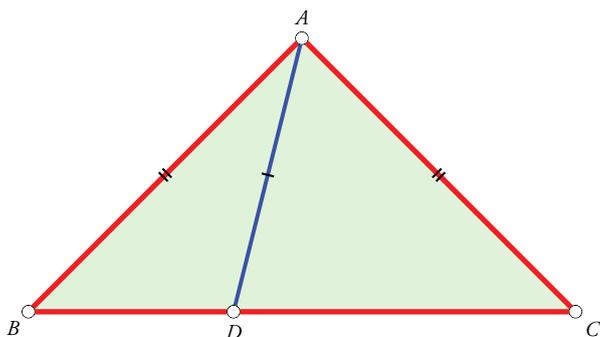


Figure 2. Credit: Discussion on the AtRiUM Facebook page, [1] and [2]

We offer two further constructions that illustrate what we have found.

Construction 1. Let $\triangle ABC$ be isosceles, with $AB = AC$ (Figure 3).



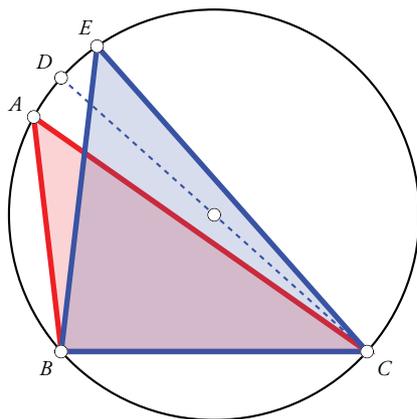
Let D be any point on BC other than the midpoint. Consider $\triangle ABD$ and $\triangle ACD$. We now have: $AB = AC$; AD is a shared side; $\angle ABD = \angle ACD$.

Figure 3. A counterexample to presumed SSA congruence

This fits the ‘SSA model’. But $\triangle ABD$ and $\triangle ACD$ are not congruent; one of them will fit strictly inside the other.

Note how this example illustrates the conclusion obtained above: opposite the equal sides AB and AC in $\triangle ABD$ and $\triangle ACD$ respectively are the angles ADB and ADC , and these are a supplementary pair of angles.

Construction 2. Another such construction makes use of the angle properties of a circle.



Let $\triangle ABC$ be obtuse-angled at B . Draw the circumcircle of $\triangle ABC$ (Figure 4). Let CD be the diameter of the circle through C . Let the image of A under reflection in CD be E ; then E will lie on the circle, and $EC = AC$.

Figure 4. Another counterexample to presumed SSA congruence

Consider $\triangle ABC$ and $\triangle EBC$. They have a shared side (BC), a pair of equal angles ($\angle BAC$, $\angle BEC$) and a pair of equal sides (AC , EC). So this example fits the ‘SSA model’.

Since A and E are images of each other under reflection in CD , it must be that D is the midpoint of arc AE , so DB bisects $\angle ABE$ and therefore $\angle ABC$ and $\angle EBC$ are a supplementary pair of angles. Here too, the two triangles ($\triangle ABC$ and $\triangle EBC$) are clearly not congruent to each other.

Can SSA ever imply congruence?

The answer (surprise) is Yes. *In certain situations, SSA does imply congruence.*

Here is how this might come about. Suppose that $\triangle ABC$ and $\triangle DEF$ are such that (i) $AB = DE$, (ii) $BC = EF$, (iii) $\angle BAC = \angle EDF$. Suppose further that $\angle BAC$ (and therefore $\angle EDF$ as well) is not acute (i.e.,

it is either a right angle or is obtuse). Then necessarily the other angles of both the triangles are acute. We had shown earlier that an application of the sine rule yields the equality

$$\sin \angle ACB = \sin \angle DFE.$$

As already noted, this implies that $\angle ACB$ and $\angle DFE$ are either equal to each other or are supplementary to each other; so there is an ambiguity here, which means that congruence does not follow. But under the additional constraint that the angles must both be acute, the ambiguity disappears and we necessarily have $\angle ACB = \angle DFE$. So congruence does follow in this situation.

Tweaking this line of reasoning, we hit upon another possibility. Suppose, as earlier, that $\triangle ABC$ and $\triangle DEF$ are such that (i) $AB = DE$, (ii) $BC = EF$, (iii) $\angle BAC = \angle EDF$, (iv) $\angle BAC$ (and therefore $\angle EDF$ as well) is acute. Suppose further that $AB < BC$ (which means also that $DE < EF$). Then it must be that $\angle ACB$ and $\angle DFE$ are acute (recall the theorem that in any triangle, the larger side has the larger angle opposite it, and the smaller side has the smaller angle opposite it). In the same way as was described above, it now follows that $\angle ACB = \angle DFE$. So congruence follows in this situation as well.

There may well be other situations where SSA does lead to congruence, but we leave further explorations to the reader.

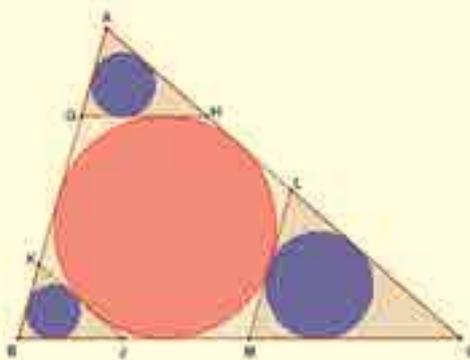
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The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

A Problem with Four Incircles



Shown in the figure is a triangle ABC and its incircle (coloured red). Tangents (GH, JK, LM) are drawn to the incircle, parallel to the three sides of the triangle, thus creating three more triangles (AGH, BJK, CLM) . Incircles are drawn (coloured blue) for these three triangles.

Show that the sum of the radii of the three blue circles equals the radius of the red circle.

HOW to PROVE it

SHAILESH SHIRALI

In this episode of "How To Prove It", we consider two similar sounding terms which have great significance in higher mathematics: contradiction and contrapositive. Both of them arise in connection with proofs. We give several examples of proofs of both these kinds.

A phrase used very often in higher mathematics is 'proof by contradiction.' A vast number of results are proved using this approach. Readers of *At Right Angles* would have seen this proof technique used numerous times.

A less familiar phrase is 'proof by the contrapositive.' But appearances are deceptive; though the phrase itself is not used so often, the approach is very widely used.

Both of these are examples of *indirect proof techniques*. In this edition of 'How to Prove It,' we dwell on proof by contradiction and proof by the contrapositive and explain what is 'indirect' about them.

Matters of notation

We shall use the following standard notation throughout this article.

- If integers a, b, c ($c \neq 0$) are such that a and b leave the same remainder under division by c , then we write $a \equiv b \pmod{c}$. Otherwise put: $a \equiv b \pmod{c}$ means that $a - b$ is a multiple of c . Examples: $27 \equiv 7 \pmod{5}$; $53 \equiv 19 \pmod{17}$.
- If integers a, b ($a \neq 0$) are such that a is a divisor of b , we write: $a \mid b$. Examples: $7 \mid 21$; $17 \mid 85$.
- Negation is indicated as follows: $13 \not\equiv 2 \pmod{5}$; $5 \nmid 12$; $2 \nmid 3$.

Keywords: Proof, direct proof, indirect proof, contradiction, contrapositive

Well-known facts from elementary number theory. We shall make repeated use of some well-known (and easily proved) results from elementary number theory, namely:

- If n is any integer, then $n^2 \equiv 0 \pmod{3}$ or $n^2 \equiv 1 \pmod{3}$.
- In particular, if n is not a multiple of 3, then $n^2 \equiv 1 \pmod{3}$.
- If n is any integer, then $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$.
- In particular, if n is odd, then $n^2 \equiv 1 \pmod{4}$.
- There are no squares of the following forms: $2 \pmod{3}$, $2 \pmod{4}$, $3 \pmod{4}$.

Direct proof and indirect proof

Direct proof. To start with, let us explain what is meant by ‘direct proof.’ Let us say that some statement or proposition q needs to be proved. The direct approach is to start with some statement p of an elementary and basic nature that is clearly true, and then go through a series of deductive steps which yield a sequence of statements, each of which is implied by the previous one, culminating in the statement q to be proved. That is, we start with some statement p whose truth is beyond dispute, then move through a sequence of statements $p_1, p_2, p_3, \dots, p_n, q$ as follows:

$$p \implies p_1, \quad p_1 \implies p_2, \quad p_2 \implies p_3, \quad \dots, \\ p_n \implies q.$$

At the end of this chain of reasoning, proposition q has been proved, as required.

Rather than a standalone proposition that needs to be proved, we may be faced with the task of proving an *implication*, say $p \implies q$. That is, we need to show that *if* proposition p is true, *then* proposition q is true as well. Observe the ‘if then’ nature of what is to be proved.

In some cases, we may be able to show the desired implication in a single step. In case this proves difficult, we may opt to interpose a sequence of

propositions $p_1, p_2, p_3, \dots, p_n$ between p and q , and then establish the following implications:

$$p \implies p_1, \quad p_1 \implies p_2, \quad p_2 \implies p_3, \quad \dots, \\ p_n \implies q.$$

The point of bringing in the additional propositions is that the in-between implications may be easier to establish. So, rather than take one large step, we take a number of relatively small steps, each of which is not too difficult in itself. At the end of this line of reasoning, the desired implication will have been proved. This too is a direct proof.

What is ‘direct’ about these approaches is that we have accomplished the desired task ‘directly.’ In the first situation, proposition q had to be proved; we have done so. In the second situation, the implication $p \implies q$ had to be proved; once again, we have done so.

Indirect proof. The path taken by an indirect proof is very different. It rests on the basic premise that *a proposition is either true or false*. This means that if a proposition is not true, it must be false; if it is not false, then it must be true.

This offers another way of proving a given proposition q to be true: show that it cannot be false! How do we do this? One way would be to assume q to be false, take *that* to be our starting point, and explore its consequences. If at some point we come across a consequence that we definitely *know* to be false, or a consequence that contradicts something we proved earlier, then we can conclude that the assumption made at the start (i.e., that q is false) is *itself* false. So q cannot be false and therefore it must be true!

Observe what has taken place here: the assumption that q is false has caused us to trip ourselves (to put it more colourfully, we have tripped on our own shoelaces), and therefore we are forced to conclude that there must be something wrong with this assumption. This manner of proceeding is known as *proof by contradiction*.

A similar line of reasoning can be used if we wish to show the truth of the implication $p \implies q$.

We assume that q is false and check whether we can show, in some way or the other, that p too is false. That is, the falsity of q leads to the falsity of p . But we have already been told that p is true; this is a given. From this, we conclude that q cannot be false; therefore, it must be true. This manner of proceeding is known as *proof by the contrapositive*.

It should be clear now why these two approaches are described as ‘indirect.’

Examples of direct proof

- (i) Prove that if m is an odd integer, then $m^2 \equiv 1 \pmod{8}$.

Solution. An odd integer m can be written in the form $2n + 1$ where n is an integer. Squaring this expression, we get:

$$\begin{aligned} m^2 &= (2n + 1)^2 \\ &= 4n^2 + 4n + 1 \\ &= 4n(n + 1) + 1. \end{aligned}$$

In the last line, we focus our attention on the term $n(n + 1)$. Observe that it is a product of a pair of consecutive integers. One of these integers must be even, so their product is necessarily even. Hence $8 \mid 4n(n + 1)$. It follows that $m^2 \equiv 1 \pmod{8}$. ■

- (ii) Prove the arithmetic mean-geometric mean (AM-GM) inequality: if a and b are any two positive real numbers, then $\frac{1}{2}(a + b) \geq \sqrt{ab}$. (Here, $\frac{1}{2}(a + b)$ is the AM of a and b , while \sqrt{ab} is the GM of a and b .)

Solution. We shall start with a statement that is clearly true: *the square of any real number is non-negative*. Applying this statement to the particular real number $\sqrt{a} - \sqrt{b}$, we deduce that

$$(\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Expanding the bracketed term and simplifying, we get:

$$\begin{aligned} a + b - 2\sqrt{ab} &\geq 0, \\ \therefore \frac{a + b}{2} &\geq \sqrt{ab}. \end{aligned}$$

- (iii) Prove that the cube root of 3 exceeds the square root of 2. (This is generally asked as a question: Which is larger, $2^{1/2}$ or $3^{1/3}$?)

Solution. We shall start with a statement that is clearly true: $9 > 8$. Taking the (real) sixth roots of both sides, we deduce that $9^{1/6} > 8^{1/6}$, hence

$$\begin{aligned} (3^2)^{1/6} &> (2^3)^{1/6}, \\ \therefore 3^{1/3} &> 2^{1/2}. \end{aligned}$$

- (iv) In $\triangle ABC$, sides AB and AC have equal length. Prove that $\angle ABC = \angle ACB$.

Solution. We give the original proof from Euclid’s *Elements* (with the language modified slightly, so as to make it easier to understand).

It is very important to note the placement of this theorem in the original sequence of results proved by Euclid. *The only congruence theorem available to us is what we now call side-angle-side (‘SAS’) congruence*. In particular, the angle-side-angle (‘ASA’) and side-side-side (‘SSS’) congruence theorems are not available as they are themselves proved later on. The ingenuity of Euclid’s proof is striking.

Let ABC be isosceles with $AB = AC$. Extend sides AB and AC to points D and E respectively such that $AD = AE$. (See Figure 1.)

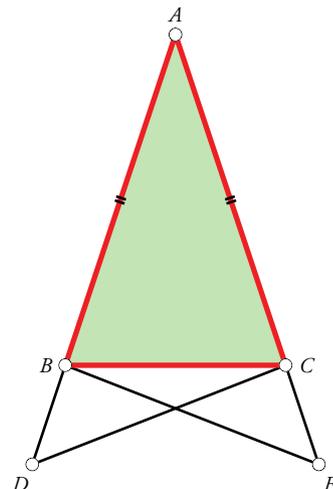


Figure 1.

By the SAS congruence theorem, $\triangle ABE$ is congruent to $\triangle ACD$; for: $AB = AC$, $AE = AD$, and the two triangles have a shared included angle, namely $\angle A$. Therefore $BE = CD$ and $\angle AEB = \angle ADC$.

In the same way, we see that $\triangle BCD$ is congruent to $\triangle CBE$; for: $BD = CE$ (because $AD = AE$ and $AB = AC$), $CD = BE$ (just proved) and $\angle BDC = \angle CEB$ (just proved). Therefore $\angle DBC = \angle ECB$.

Since the straight angle at B equals the straight angle at C , it follows by subtraction that $\angle ABC = \angle ACB$. ■

Remark. There is another direct proof of this result which is not in Euclid's original text; it was found by Pappus a few centuries later; see [1]. It is a very ingenious proof but also counterintuitive. For example, it regards $\triangle ABC$ as a distinct object from $\triangle ACB$. For this reason, it has been described by some authors as "conceptually difficult."

Examples of indirect proof

- (i) Prove that the integer 80000000000007 is not a perfect square.

Solution. A direct proof of this statement would involve computation of the square root of the given integer. The indirect proof is shorter; we make use of the first result proved in the previous section on direct proof: if m is an odd integer, then $m^2 \equiv 1 \pmod{8}$. Note that this implies that any odd square is of the form $1 \pmod{8}$.

Observe that the given integer is odd, and also observe that under division by 8, it leaves remainder 7. However, an odd square leaves remainder 1 under division by 8. It follows that 80000000000007 is not a perfect square. ■

- (ii) Prove that the square root of 2 is an irrational number.

Solution. Just for fun, we give a proof that is slightly different from Euclid's original proof. (However, it is essentially modelled on that proof.)

Suppose that the square root of 2 is a rational number, say $\sqrt{2} = a/b$ where a and b are positive integers. Naturally, we may suppose that a and b share no common factors other than 1, i.e., a, b are coprime.

From the relation $\sqrt{2} = a/b$ we get, by squaring and simplifying, $a^2 = 2b^2$.

Now we ask: is either a or b divisible by 3? Suppose that $3 \mid a$. Then the relation $a^2 = 2b^2$ leads us to conclude that $3 \mid b$ as well. (Please fill in the details of the reasoning used to draw this conclusion.) But then $3 \mid a$ and $3 \mid b$, contrary to what we said above (that a and b are coprime). Hence a is not divisible by 3. If we suppose that $3 \mid b$, then by following the same reasoning, we are led to conclude that $3 \mid a$, and this again goes counter to what we said above; hence b is not divisible by 3. It follows that neither a nor b is divisible by 3.

The last statement implies that $a^2 \equiv 1 \pmod{3}$ and $b^2 \equiv 1 \pmod{3}$. From the second of these statements, we deduce that $2b^2 \equiv 2 \pmod{3}$. But this contradicts the statement that $a^2 = 2b^2$, so we reach a contradictory state of affairs. It follows that the square root of 2 is not a rational number. ■

- (iii) Prove that if a, b, c are odd integers, then the equation $ax^2 + bx + c = 0$ has irrational roots. Otherwise put: if a, b, c are odd integers, then the expression $ax^2 + bx + c$ cannot be factorised over the rational numbers.

Solution. The roots of the equation $ax^2 + bx + c = 0$ are the two quantities

$$\frac{-b \pm \sqrt{D}}{2a},$$

where $D = b^2 - 4ac$ is the discriminant. As a, b, c are integers, the roots are rational if and only if D is a perfect square. Expressed negatively, the roots are irrational if and only if D is not a perfect square.

So we need to establish that $b^2 - 4ac$ is not a perfect square. We need to do so under the

hypothesis that a, b, c are all odd integers. Suppose this is not so; i.e., suppose that $b^2 - 4ac = d^2$, where d is an integer. Note that d^2 is odd, and therefore so is d . We now have:

$$b^2 - d^2 = 4ac.$$

We now use the following (proved above): *if m is an odd integer, then $m^2 \equiv 1 \pmod{8}$* . This fact implies that $8 \mid b^2 - d^2$.

On the other hand, the quantity $4ac$ is of the form $4 \times$ an odd integer, which means that it cannot be a multiple of 8; indeed, we have $4ac \equiv 4 \pmod{8}$.

This means that the equality $b^2 - d^2 = 4ac$ cannot hold. Hence the supposition that the discriminant $b^2 - 4ac$ is a perfect square cannot hold. It follows that the roots of the given equation are irrational. ■

- (iv) Consider any three distinct perfect squares in arithmetic progression. Prove that the common difference of the AP is a multiple of 24.

Solution. Let the three squares be a^2, b^2 and c^2 , where $a^2 < b^2 < c^2$. As they are in AP, we have $b^2 - a^2 = c^2 - b^2$, i.e., $2b^2 = a^2 + c^2$, which also implies that $c^2 = 2b^2 - a^2$. Let d be the common difference of the AP.

We may as well suppose that a, b, c are coprime. For, if they have divisors in common other than 1, we can divide all three of a, b, c by the common divisor and prove the proposition for the smaller squares. If proved, the proposition will then apply as well to the original squares.

We first focus attention on a . Suppose that a is even. If b too is even, then from the relation $c^2 = 2b^2 - a^2$, it follows that c too is even. However, we had already supposed that a, b, c are coprime, so the possibility of all three of a, b, c being even is not allowed. Next, suppose that b is odd. In that case we

have $b^2 \equiv 1 \pmod{4}$, which means that $d \equiv 1 \pmod{4}$; but this leads to $c^2 \equiv 2 \pmod{4}$. However, no square is of this form. As both the possibilities (b even, b odd) have led to contradictions, we are forced to conclude that a is not even; hence a is odd.

Now we focus attention on b . If b is even, then the relation $c^2 = 2b^2 - a^2$ implies that $c^2 \equiv -1 \pmod{4}$; but this is not possible as no square is of this form. It follows that b cannot be even. Hence b is odd. This proves that c is odd as well (again using the relation $c^2 = 2b^2 - a^2$). That is, all three of a, b, c are odd. This implies that all three of a^2, b^2, c^2 are of the form $1 \pmod{8}$, hence d is a multiple of 8.

We again focus attention on a . Suppose that $3 \mid a$. If $3 \mid b$ as well, then by virtue of the relation $c^2 = 2b^2 - a^2$, it follows that $3 \mid c$ too. However, we had already supposed that a, b, c are coprime, so this is disallowed. Hence b is not a multiple of 3. This leads to $b^2 \equiv 1 \pmod{3}$, which means that $d \equiv 1 \pmod{3}$. However, this in turn leads to $c^2 \equiv 2 \pmod{3}$, which is not possible as no square is of the form $2 \pmod{3}$. As both the possibilities have led to contradictions, we are forced to conclude that a is not a multiple of 3. Hence we have $a^2 \equiv 1 \pmod{3}$.

Now we focus attention on b . If $3 \mid b$, then the relation $c^2 = 2b^2 - a^2$ would imply that $c^2 \equiv -1 \pmod{3}$; but no square is of this form. It follows that b is not a multiple of 3. Hence we have $b^2 \equiv 1 \pmod{3}$. From this it follows that d is a multiple of 3.

(From this, one may deduce that c too is not a multiple of 3; we again use the relation $c^2 = 2b^2 - a^2$. So all three of a, b, c are non-multiples of 3. However, we do not need to use this fact.)

Since $8 \mid d$ and $3 \mid d$, it follows that $24 \mid d$, as required. ■

Remark. Here are a few triples (a, b, c) of coprime positive integers for which a^2 , b^2 and c^2 are in arithmetic progression (d is the common difference):

(a, b, c)	(a^2, b^2, c^2)	d
(1, 5, 7)	(1, 25, 49)	24
(1, 29, 41)	(1, 841, 1681)	840
(7, 13, 17)	(49, 169, 289)	120
(7, 17, 23)	(49, 289, 529)	240
(17, 25, 31)	(289, 625, 961)	336
(23, 37, 47)	(529, 1369, 2209)	840
(31, 41, 49)	(961, 1681, 2401)	720

- (v) In $\triangle ABC$, the angles opposite AB and AC have equal measure, i.e., $\angle ABC = \angle ACB$. Prove that sides AB and AC have equal length.

Solution. In Euclid's text, this proposition comes immediately after the proposition that the base angles of an isosceles triangle are equal. So the only congruence result available to us is the SAS congruence theorem. The way Euclid handles this restriction is remarkable. It is a masterly demonstration of proof by contradiction. (As earlier, we have modified Euclid's original proof, to the extent of using words and sentences that would be more familiar to us in the current time.)

We are told that in $\triangle ABC$, $\angle ABC = \angle ACB$. We must prove that $AB = AC$. We shall suppose the contrary

and show that this supposition leads to a contradiction.

Suppose that equality does not hold, i.e., $AB \neq AC$. Then one of them is greater than the other. Without loss of generality, we may suppose that $AB > AC$.

Locate point D on side AB such that $DB = AC$ (see Figure 2). This is possible as we have assumed that $AB > AC$.

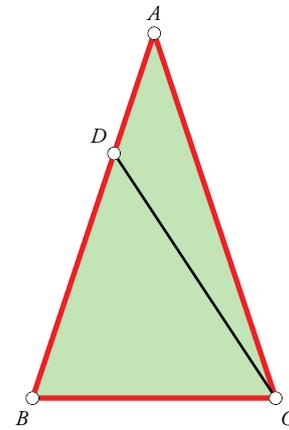


Figure 2.

Now consider the two triangles, $\triangle ACB$ and $\triangle DBC$. We have: $AC = DB$ (by construction); $CB = BC$ (this is a shared side); and $\angle ACB = \angle DBC$ (this is given; remember that $\angle DBC$ is the same as $\angle ABC$). It follows (SAS congruence) that $\triangle ACB$ is congruent to $\triangle DBC$.

But this is absurd, as $\triangle DBC$ is contained strictly within $\triangle ACB$. We have reached a self-contradiction.

The contradiction tells us that our initial supposition is itself incorrect; that is, the supposition that $AB \neq AC$ is false.

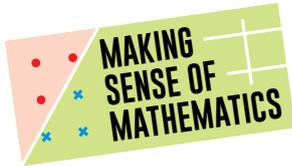
Hence $AB = AC$. ■

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A Visual Proof that $(a + b)^2 \neq a^2 + b^2$

Many students have been drilled to remember that $(a + b)^2 \neq a^2 + b^2$. But a picture that makes sense can create a much more lasting impression and long term learning.

Figure 1 may persuade students that $(a + b)^2 \neq a^2 + b^2$.

The smaller inner square has sides of length $\sqrt{a^2 + b^2}$ since each side is the hypotenuse of a right triangle with legs a and b , so the area of the inner square is $[\sqrt{a^2 + b^2}]^2 = a^2 + b^2$.

The larger outer square has an area of $(a + b)^2$, so $(a + b)^2 \neq a^2 + b^2$.

Furthermore, the combined area of the four triangles is $2ab$, which is how much the area of the large square exceeds that of the smaller square.

That is, $(a + b)^2 = a^2 + b^2 + 4 \frac{ab}{2} = a^2 + b^2 + 2ab$.

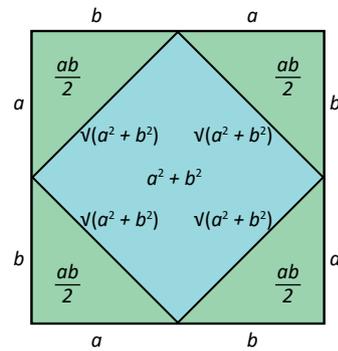


Figure 1: Seeing the difference

Exploring Fractals – The GeoGebra Way

JONAKI GHOSH

In an article published in the November 2016 issue of *At Right Angles* we had seen how geometrical fractal constructions lead to algebraic thinking. The article had highlighted the iterative construction processes, which lead to the Sierpinski triangle and the Sierpinski Square carpet. Further the idea of self-similarity within these fractals was reinforced through the recursive and explicit relationships between various stages of the fractal constructions. After a quick recap of the Sierpinski triangle construction we shall explore how fractals can be constructed using the dynamic geometry software, GeoGebra. The steps for downloading the software are given at the end of the article. Readers will require a basic familiarity with the construction tools of GeoGebra for attempting the construction process. We shall also explore the number sequences which emerge from these fractal constructions.

The Sierpinski Triangle

In order to create the Sierpinski Triangle we begin with an equilateral triangle cut out from a coloured sheet of paper. This may be referred to as stage 0 of the construction process. We now join the midpoints of the three sides to obtain four smaller equilateral triangles and remove the triangle at the centre. This piece with a triangular ‘hole’ will be referred to as stage 1. To obtain stage 2 we repeat the process of joining the midpoints of the three smaller equilateral triangles and removing the triangles at the centre. Figure 1 shows stages 0, 1, 2 and 3 of the Sierpinski Triangle. The construction process may be continued indefinitely, though it is hard to visualize higher stages of the fractal. It will be worthwhile to note that the white portions represent ‘holes’ as these portions have been cut out and removed. It is for this reason the Sierpinski triangle is also referred to as the Sierpinski sieve.



Figure 1. The Sierpinski Triangle - stages 0, 1, 2 and 3

A natural question which arises, is related to the number of triangles at any given stage of the construction process. How many black triangles will there be at the 4th and 5th stages? Can we find a rule for the number of black triangles at the n th stage? If the area of the triangle at stage 0 is 1 square unit, what is the shaded area at stages 1, 2 and 3? At the n th stage? And so on. We will soon answer these questions but before that let us revisit the concept of self – similarity.

The idea of self-similarity

In the Sierpinski triangle construction we note that stage 1 comprises three identical smaller copies of stage 0 where each copy is a smaller equilateral triangle. Similarly, stage 2 has three copies of stage 1 and nine copies of stage 0. Can you identify these? Similarly in stage 3 we will find 3 copies of stage 2, 9 copies of stage 1 and 27 copies of stage 0. Thus fractals have this unique property, where at each successive stage we find parts which are replicas of the previous stages. You may refer to the article in the earlier issue of At Right Angles to see diagrammatic representations of self – similarity within the Sierpinski triangle.

Recursive and Explicit formulae

Coming back to our question related to the number of shaded triangles at any stage, we observe that there are 3 in stage 1, 9 in stage 2, 27 in stage 3 and so on. This leads to the geometric sequence 1, 3, 3^2 , 3^3, that is, powers of 3. To find the next term of the sequence we need to multiply the previous term by 3.

Hence, if S_n represents the number of shaded triangles at stage n then $S_n = 3S_{n-1}$ where S_{n-1} represents the number of shaded triangles at stage $n - 1$. Hence $S_1 = 3S_0$, $S_2 = 3S_1$, $S_3 = 3S_2$ and so on. Thus $S_n = 3S_{n-1}$ is the *recursive rule* or formula for the number of shaded triangles at stage n .

If we tabulate the number of shaded triangles vis-à-vis the stage numbers we obtain the following:

Stage	0	1	2	3	4	n
No. of shaded triangles	1	3	3^2	3^3	3^4	3^n

However to find the explicit rule or formula for the number of shaded triangles at the n th stage we observe that the exponent of 3 coincides with the number of the stage. Thus the number of shaded triangles at the n th stage is given by 3^n .

Let us now consider the shaded areas in the different stages of the Sierpinski triangle construction. Observe that in stage 1, the original triangle is divided into 4 smaller congruent equilateral triangles and the one in the centre is removed. Thus, if the area of the triangle at stage 0 is 1 square unit then the area at stage 1 is $\frac{3}{4}$ square units. We may conclude that the shaded area in the different stages leads to the geometric sequence 1, $\frac{3}{4}$, $(\frac{3}{4})^2$, $(\frac{3}{4})^3$ Here the multiplying factor is $\frac{3}{4}$.

With a little bit of work, the reader can come up with the recursive and explicit rules for finding the shaded area at the n th stage. In fact if A_n denotes the area at the n th stage, then $A_n = (\frac{3}{4}) A_{n-1}$ and $A_n = (\frac{3}{4})^n$ are the recursive and explicit formulas for the area of the shaded triangles at

any given stage of the Sierpinski triangle. The following table shows the shaded area for the various stages.

Stage	0	1	2	3	4	N
Shaded area	1	$3/4$	$(3/4)^2$	$(3/4)^3$	$(3/4)^4$	$(3/4)^n$

A GeoGebra Construction of the Sierpinski Triangle

You may have realized that it is difficult to draw the Sierpinski triangle by hand beyond stage 3. The iterative process thus becomes hard to visualize at higher stages. Surely it would be worthwhile if some software could do the process automatically!

In this section we will construct the Sierpinski triangle using GeoGebra. Here are the steps, which will lead to the Sierpinski triangle in GeoGebra.

Step 1: Open a new GeoGebra file and tap on the **Graphics** view. Remove the grid and the axes and draw a triangle ABC using the **Polygon tool**. We can draw any triangle (it need not be an equilateral triangle).

Step 2: Use the ‘**Midpoint or Center**’ tool available under the **Point tool** to draw the midpoints D, E and F of sides BC, CA and AB respectively. Using the **Polygon tool** draw the triangle DEF.

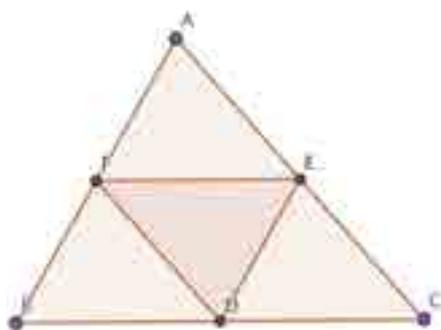


Figure 2: The Polygon tool of GeoGebra may be used to draw a triangle ABC and the midpoint triangle DEF.

Step 3: Right click on the Graphics window, scroll down to ‘**Graphics**’ at the bottom of the menu and go to ‘**Preferences**’. Select triangle t1

(this is triangle ABC) from the left hand side and choose a bright color from the color palette (also increase its opacity). Then select t2 (triangle DEF) and choose white from the color palette and increase its opacity. Further select the segments a, b, c, d, e and f and select the black color from the palette. Close the preferences window. You may hide the vertices A, B, C, D, E and F. You will now see stage 1 of the Sierpinski triangle.

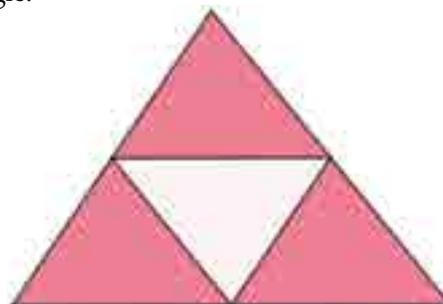


Figure 3: Stage 1 of the Sierpinski Triangle

Step 4: The next step is to repeat the process of marking the mid points of the sides of the three smaller coloured triangles, join them to form the centre triangle and assign the white colour to it. To repeat the process iteratively, we need to create a **Custom Tool**.

Go to **Tools** in the menu bar and select ‘**Create new tool**’. A ‘Create new tool’ window pops up. Note that **Output objects** is already highlighted in the window.

We need to define the **Output objects** by clicking on the arrow icon. Select the following objects one at a time

- Point D
- Point E
- Point F
- Segment e
- Segment f
- Segment g
- Triangle t2

Now highlight the **Input objects**. These are already defined as Point A, Point B and Point C. Next move to **Name & Icon**. In **Tool name** you may enter Sierpinski Triangle. The same

will automatically appear in **Command Name**. In **Tool Help** enter 'Click on the three vertices of the triangle'. Finally click on 'Finish' at the bottom of the window. A message 'New tool created successfully' will pop up and a new tool icon will be available on the GeoGebra tool bar.

Step 5: When you take the cursor on the new tool icon, it will display 'Sierpinski triangle' as the tool name and the tool description will appear as 'Click on the three vertices of the triangle'. Select this tool and click on the vertices of any one of the three coloured triangles. You will notice that it will be divided into 4 smaller triangles and the centre triangle will be 'removed'. Repeat this with the other two coloured triangles. When this is done you will see stage 2 of the Sierpinski triangle on your screen. Using the **Move tool**, you may drag the vertices A, B or C to enlarge or scale down the Sierpinski triangle. This stage has 9 coloured triangles. Using the newly created tool obtain stages 3 and 4 of the Sierpinski triangle.

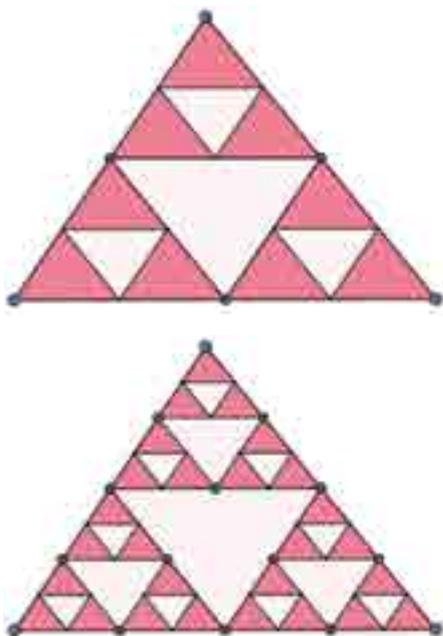


Figure 4: Stages 2 and 3 of the Sierpinski triangle constructed using the custom tool.

The above construction demonstrates how an iterative process leading to a fractal can be carried out in GeoGebra. It would be an interesting exercise to explore different fractal constructions using the same method. In the following section

we are going to construct another fractal called the Pythagorean Tree through a similar process.

The Pythagorean Tree

The Pythagorean tree is constructed iteratively in which squares are erected on all three sides of a right-angled triangle. In subsequent stages, similar right-angled triangles are constructed on the two legs of the original triangle in such a manner that the legs of the original triangle become the hypotenuse of the new triangles. When this process is continued a few times, a beautiful intricate fractal pattern emerges.

The steps leading to the Pythagorean tree are as follows:

Step 1: Open a new GeoGebra file. Remove the axes from the Graphics window but retain the grid. Select two points A and B, which are 4 units apart. Within the circle tool, choose 'Semicircle through two points'. Click on A and then on B to obtain a semicircle with AB as diameter. Use the point tool to select a point C on the semicircle and use the polygon tool to draw the triangle ABC. Note that triangle ABC is right-angled at C. Why? You may drag C so that the two legs of the triangle are of equal length.

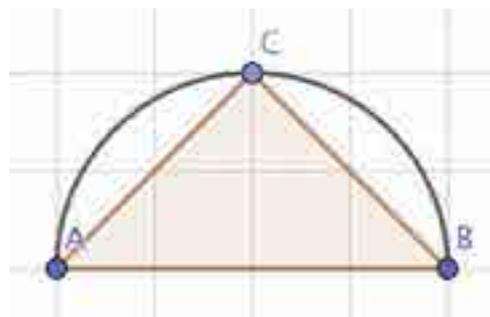


Figure 5: An isosceles right triangle drawn on the Graphics screen.

Step 2: Hide the semicircle and the labels of the vertices A, B and C. Now use the **Regular polygon tool** to draw squares on the sides AB, BC and CA of triangle ABC. Using the Preferences option within the graphics view, select colours of your choice for the triangle and the three squares. This is stage 0 of the Pythagorean tree.

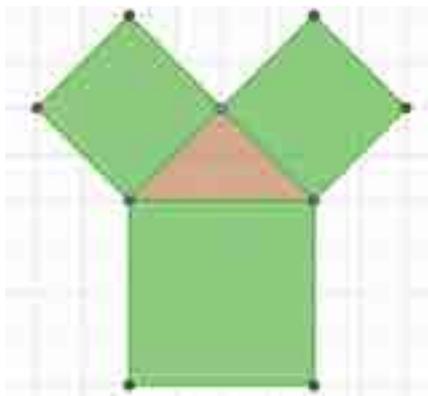


Figure 6: Stage 0 of the Pythagorean tree.

Step 3: Note that we have two smaller squares on the two legs of the right-angled triangle ABC. We would now like to repeat this process (of drawing a right-angled triangle followed by squares on its two legs) on the two smaller squares as shown. For this we need to create a **custom tool**.

Go to **Tools** in the menu bar and select 'Create new tool'. A 'Create new tool' window pops up. Note that **Output objects** is already highlighted.

We need to define the **Output objects** by clicking on the arrow icon. Select the following objects one at a time

- Point C
- Point F
- Point G
- Point H
- Point I
- Poly2
- Poly3
- Triangle t1

(Note that Poly2 and Poly 3 are the two smaller squares.)

Now highlight the **Input objects**. These are already defined as Point A and Point B. Next we move to **Name & Icon**. In **Tool name** you may enter Pythagorean tree. The same will automatically appear in **Command Name**. In

Tool Help enter 'Click on the endpoints of the segment. Finally click on 'Finish' at the bottom of the window. A message 'New tool created successfully' will pop up and a new icon will be available on the GeoGebra icons bar.

Step 5: When you take the cursor on the new tool icon, it will display the tool name as 'Pythagorean tree' and the tool description will appear as 'Click on the endpoints of the segment'. Select this tool and click on the vertices F, G and H, I of the two smaller green squares. You will notice that a brown right-angled triangle along with two squares on its legs will appear automatically. When this is done you will see stage 1 of the Pythagorean tree on your screen.

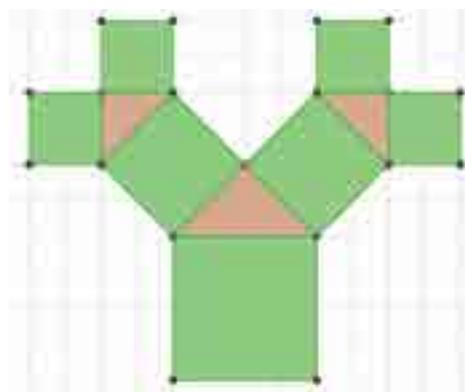


Figure 7: Stage 1 of the Pythagorean tree.

To obtain stage 2, select the new custom tool and click on the outer vertices of the four smaller squares of stage 1.

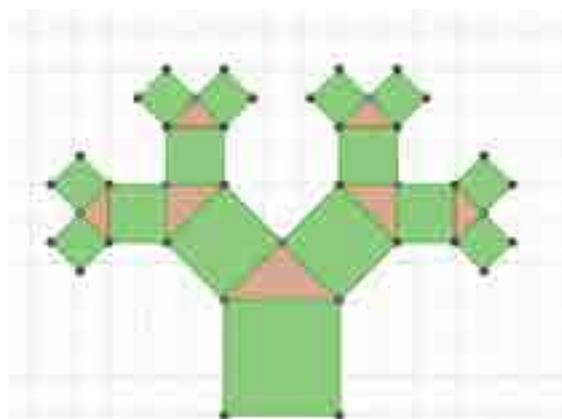


Figure 8: Stage 2 of the Pythagorean tree.

Continuing in the same manner, obtain stage 3. A beautiful symmetrical tree pattern emerges.

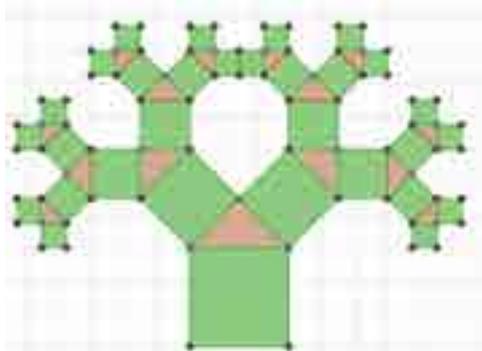


Figure 9: Stage 3 of the Pythagorean tree displaying a symmetrical pattern.

Select vertex C of the original triangle ABC. Note that this vertex moves only on the hidden semicircle. Try to move this vertex and see what happens!

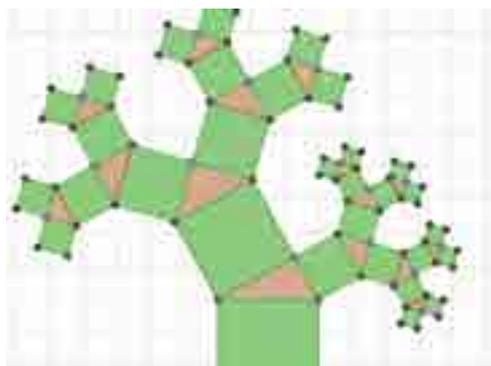
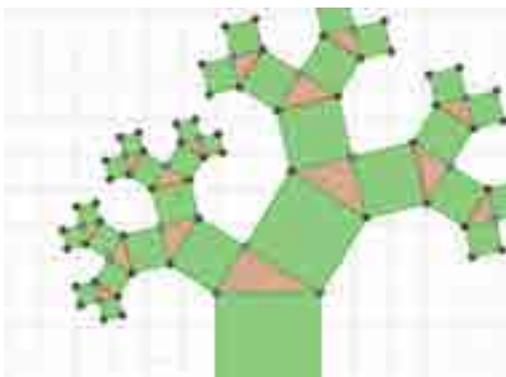


Figure 10: Asymmetrical versions of the Pythagorean tree obtained by dragging vertex C.

The tree actually sways from one side to the other!

A numerical exploration of the Pythagorean tree

The Pythagorean tree is a combination of right triangles and squares. Let us look at the areas of these shapes in Figure 9. For simplifying the

calculations we shall assume that the legs of the original triangle ABC are of unit length each.

We will now compute the areas of the right triangles along one branch of the tree, stage-wise.

Stage (n)	Area of a right triangle at stage n
0	$\frac{1}{2} \times 1^2 = \frac{1}{2}$
1	$\frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}$ (each leg has a length equal to $\frac{1}{\sqrt{2}}$ unit)
2	$\frac{1}{2} \times \left(\frac{1}{2}\right)^2 = \frac{1}{8}$ (each leg has a length equal to $\frac{1}{2}$ unit)
3	$\frac{1}{2} \times \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{16}$ (each leg has a length equal to $\frac{1}{2\sqrt{2}}$ unit)

The sequence emerging from the areas of the triangles is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

This is a geometric sequence with a multiplying factor of $\frac{1}{2}$. As the number of stages is increased, the sum of the areas of the triangles approaches 1!

The explanation lies in the fact that we can compute the infinite sum S of a geometric sequence a, ar, ar^2, ar^3, \dots with $|r| < 1$ using the formula $S = \frac{a}{1-r}$.

In the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $a = \frac{1}{2}$ and $r = \frac{1}{2}$

Thus the sum of infinite number of terms of this sequence gives us $\frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)} = 1$.

In a similar manner we will now compute the areas of the squares along one branch of the tree, stage-wise.

Stage (n)	Area of squares at stage n
0	$(\sqrt{2})^2 + 1^2 = 3$ (the area of the largest square + area of the smaller square)
1	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ (each side has a length equal to $\frac{1}{\sqrt{2}}$ unit)
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ (each side has a length equal to $\frac{1}{2}$ unit)
3	$\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}$ (each side has a length equal to $\frac{1}{2\sqrt{2}}$ unit)

The sequence emerging from the areas of the squares is $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

This is a geometric sequence with a multiplying factor of $\frac{1}{2}$. As the number of stages is increased, the sum of the areas of the squares approaches 4 !

In the sequence $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $a = 2$ and $r = \frac{1}{2}$

Thus the sum of infinite number of terms of this sequence gives us $\frac{2}{1 - \left(\frac{1}{2}\right)} = 4$.

Conclusion

When various attributes of geometrical fractals, such as length of line segments and shaded area are explored numerically, we always arrive at geometric sequences. This makes fractal investigation exciting and meaningful. We hope readers of this article will venture into creating their own fractals using the amazing features of GeoGebra.

References

- [1] Pythagoras tree in GeoGebra, <https://www.youtube.com/watch?v=qVJNXBcobz4>
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DIY Problems for the MIDDLE SCHOOL

A. RAMACHANDRAN

The tasks of this set require you to get down to some actual cut and paste work. Arm yourself with a chart paper, a roll of cello tape and a pair of scissors.

Task 1. Cut out a rectangular piece of chart paper with dimensions 16×9 units. Now this has the same area as a square piece of side 12 units. Your task is to cut the 16×9 piece into just two parts and put them together again to form the square.

Task 2. Take a piece of chart paper the size of a post card (roughly $15 \text{ cm.} \times 10 \text{ cm.}$). Post cards are not much in use now but that is how the author first encountered this problem. Your task is to cut a hole in this, through which you can pass yourself. (You could hang it around your neck and then let it slip down; you then step out of it.)

Task 3. A tetrahedron is a Platonic solid – its 4 faces are congruent equilateral triangles. You could also think of it as a triangular pyramid – a 3-D shape with a triangular base and 3 triangular sides meeting at an apex. Cut out two copies of the net (Figure 1), fold and stick each of these to get two wedge-like shapes, and then assemble them into a tetrahedron which can be sliced to give two identical 3D pieces, revealing a square cross section.

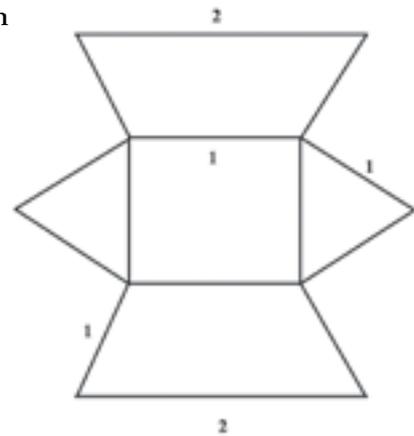


Figure 1

Keywords: Tetrahedron, cube, skew pyramid, foldable map.

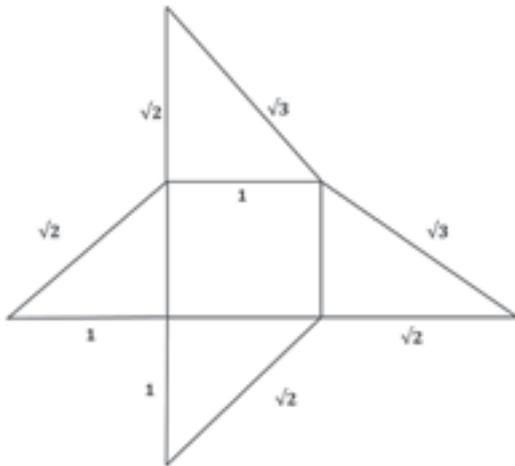


Figure 2

Problem 4. A cube is another Platonic solid – its 6 faces are congruent squares. You could also think of it as a square prism – a 3-D shape with congruent squares at top and bottom and rectangular (here, square) sides. Cut out three copies of the net (Figure 2), fold and stick each of these to get three skew pyramids, and then assemble them into a cube which can be sliced into three identical pyramids.

Problem 5. You may have used foldable printed maps in geography lessons or while traveling. They are usually rectangular in shape, divided into convenient sized smaller rectangles or squares by vertical and horizontal lines. Such maps are generally kept folded along these lines. After opening and using one, you may be at a loss as to how to fold it back. There are many alternative ways to fold a map and this task is an attempt to track them down.

Cut out a square piece of chart paper. Divide this into four squares. Mark them A, B, C, D as in Figure 3. This is your 2 x 2 ‘map.’ Fold this along the dividing lines in as many ways as possible. You could do the vertical fold first and then the horizontal, or vice versa. Every fold can be forward or backward. Once you have folded the map into a square, note down the order in which the letters A, B, C, D appear, reading from top to bottom. How many different ways of folding can you find (by different ways, we mean different orders of the four letters after the folding)?

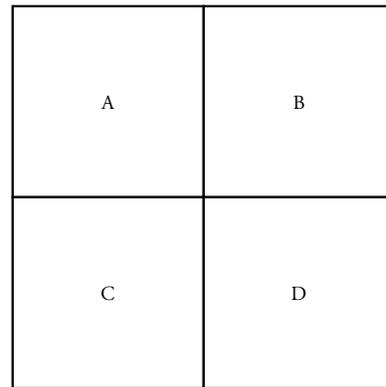


Figure 3

Discussion

1. Cut the 16 x 9 piece as shown in Figure 4. Move the part on the right towards the bottom left and join it up again to get the square.



Figure 4

Exploration: Can you think of other rectangles for which such a dissection can be done? The task is the following. We are given a certain integer sided rectangle, and we are required to cut it into just two pieces and to then assemble the two pieces to form a square. Try it out for different integer sided rectangles and see when it works out and when it does not. What pattern is needed in the sides of the original rectangle for such a procedure to work?

2. Figure 5 should give you an idea as to how to proceed. You need to redraw the figure with more twists and turns. If you cut along the lines shown you get a loop which should be sufficiently long. Theoretically if one settles for a strip width of 1 cm one should get a

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VIII-2-S.1

Let $ABCD$ be a parallelogram. Suppose K is a point such that $AK = BD$ and let M be the midpoint of CK . Prove that $\angle BMD = 90^\circ$. [*Tournament of Towns*]

Problem VIII-2-S.2

Let A be a finite non-empty set of consecutive positive integers with at least two elements. Is it possible to partition A into two disjoint non-empty sets X and Y such that the sum of the least common multiples of the numbers in X and Y is a power of 2?

Problem VIII-2-S.3

The vertices of a prism are coloured using two colours, so that each lateral edge has its vertices differently coloured. Consider all the segments that join vertices of the prism and are not lateral edges. Prove that the number of such segments with endpoints differently coloured is equal to the number of such segments with endpoints of the same colour. [*Romanian Math Competition*]

Problem VIII-2-S.4

Let $a, b, c, d \in [0, 1]$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \leq 3.$$

[*Romanian Math Competition*]

Problem VIII-2-S.5

Let a and n be positive integers such that

$$\text{Frac} \left(\sqrt{n + \sqrt{n}} \right) = \text{Frac} (\sqrt{a}).$$

Prove that $4a + 1$ is a perfect square. (Here $\text{Frac}(x)$ = the fractional part of x .) [*Romanian Math Competition*]

Keywords: Tournament of towns, Romanian math competition, fractional part, prism, median, perfect square, prime number

Solutions of Problems in Issue-VIII-1 (March 2019)

Solution to problem VIII-1-S.1

An altitude AH of triangle ABC bisects a median BM . Prove that the medians of the triangle ABM are side-lengths of a right-angled triangle.

Let AH and BM meet at the point K , let L be the midpoint of AM , and let N and P be the projections of L and M respectively to BC . Since K is the midpoint of BM , it follows that KH is a midline of triangle BMP , i.e., $PH = HB$. On the other hand, by the Thales theorem, $CP = PH$ and $PN = NH$, hence N is the midpoint of BC . Therefore NK is a medial line of triangle BMC , i.e. NK is parallel to AC and $ALNK$ is a parallelogram. Hence $LN = AK$. Also the median from M in triangle AMB is a midline of ABC , hence it is congruent to BN . Therefore the sides of right-angled triangle BNL are congruent to the medians of ABM .

Solution to problem VIII-1-S.2

There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001! + 2, 1001! + 3, \dots, 1001! + 1001$. Does there exist a block of 1000 consecutive positive integers containing exactly 5 prime numbers?

Starting with the given block, create a new block by replacing the largest number of the given block with a number that is one less than the smallest number of the given block. In doing so, the number of primes in the block changes by at most one. By the time we reach the first 1000 positive integers the number of primes is a lot more than 5. Thus somewhere along the way we must have had a block of 1000 positive integers with exactly 5 prime numbers.

Solution to problem VIII-1-S.3

Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose any two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose any non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the numbers x^2 and xy ?

- (a) We choose the numbers x and 1 and then write down $x + 1$ and $x - 1$. Then we can write down $\frac{1}{x-1}$, $\frac{1}{x+1}$ and their difference $\frac{2}{x^2-1}$. The reciprocal of this number is $\frac{x^2-1}{2}$. Adding this number to itself yields $x^2 - 1$ and adding 1 to it yields x^2 .
- (b) First write $x + y$. By (a), we can write down $(x + y)^2$, x^2 and y^2 . Thereafter, the numbers, $(x + y)^2 - x^2 = 2xy + y^2$ and $2xy + y^2 - y^2 = 2xy$. Finally, write down the reciprocal of $2xy$ and add it to itself to obtain the reciprocal of xy . Now xy can be written by taking the reciprocal of $\frac{1}{xy}$.

Solution to problem VIII-1-S.4

For which positive integers n can one find distinct positive integers a_1, a_2, \dots, a_n such that the number

$$N = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is also an integer?

We shall show that the stated requirement is possible for every positive integer n except $n = 2$.

For $n = 1$, $N = \frac{a_1}{a_1} = 1$.

For $n = 2$, suppose that $1 \leq a_1 < a_2$ and assume that a_1, a_2 are coprime. (There is clearly no loss of generality in assuming that the two integers are coprime.) Then if $N = \frac{a_1}{a_2} + \frac{a_2}{a_1}$ is a positive integer, say k , then

$$a_1^2 + a_2^2 = ka_1a_2,$$

which shows that a_1 divides a_2^2 . But, since a_1 and a_2 are coprime, this implies $a_1 = 1$, which in turn implies $a_2 = 1$. A contradiction. Thus, if $n = 2$, we cannot find distinct positive integers a_1 and a_2 such that N is an integer.

For $n \geq 3$, let $a_k = (n-1)^{k-1}$ for $1 \leq k \leq n$. These are distinct integers since $n-1 > 1$ and we have

$$N = \frac{1}{n-1} + \frac{1}{n-1} + \cdots + \frac{1}{n-1} + (n-1)^{n-1} = 1 + (n-1)^{n-1},$$

which is a positive integer. Note that this construction fails if $n = 1$, for it yields the same value for all the a_j .

Solution to problem VIII-1-S.5

In triangle ABC , $\angle A = 2\angle B = 4\angle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

Let I be the incentre of $\triangle ABC$. Let $\angle BCI = \theta = \angle ACI$. Then $\angle ABI = \angle CBI = 2\theta$ and $\angle CAI = \angle BAI = 4\theta$. Hence

$$\angle AIE = \angle BID = \angle BDI = 6\theta, \quad \angle AIF = \angle AFI = 5\theta, \quad \angle AEI = 4\theta.$$

Let $AI = x$ and $DI = y$. Then $AF = IE = x$ and $BD = BI = x + y$.

In $\triangle BAD$ we have $AB/AI = DB/DI$, hence

$$BF = \left(\frac{DB}{DI} - \frac{AF}{AI} \right) \cdot AI = \frac{x^2}{y}.$$

In $\triangle ABE$ we have $EA/EI = BA/BI$, hence

$$AE = (AF + FB) \cdot \frac{EI}{BI} = \frac{x^2}{y} = BF.$$

It follows that $\triangle EAD$ and $\triangle FBD$ are congruent to each other, so that $DE = DF$.

A Geometric EXPLORATION

PRITHWIJIT DE

A question about angle bisectors

Consider a $\triangle ABC$ in which D , E and F are the midpoints of the sides BC , CA and AB respectively. Let G be the centroid of triangle ABC , i.e., the point of intersection of the medians AD , BE and CF . It is well-known that G is also the centroid of triangle DEF .

If, instead of being the midpoints, the points D , E and F are the points of intersection of the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle ACB$ respectively with the opposite sides (BC , CA and AB respectively), then do the incentres of triangles ABC and DEF coincide? They do, if $\triangle ABC$ is equilateral. Are there triangles other than the equilateral triangle with such a property? Let us analyze.

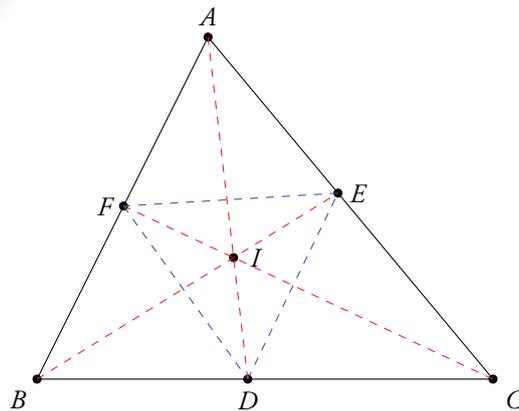


Figure 1.

Let I be the common incentre of $\triangle ABC$ and $\triangle DEF$. Observe that AD bisects both $\angle BAC$ and $\angle EDF$. In $\triangle AFD$ and $\triangle AED$, side AD is common, $\angle DAF = \angle DAE$ and $\angle ADF = \angle ADE$. Therefore, $\triangle AFD \cong \triangle AED$. Hence $DE = DF$ and $AE = AF$.

Keywords: Triangle, median, angular bisector, altitude, concurrence

By a similar argument, $\triangle BDE \cong \triangle BFE$, hence $DE = EF$ and $BD = BF$. Summarising, we obtain $DE = EF = DF$. Thus DEF is an equilateral triangle.

Also observe that $\triangle AFE$ and $\triangle BDF$ are isosceles. Thus, $\angle AFE = 90^\circ - \frac{1}{2}\angle BAC$ and $\angle BFD = 90^\circ - \frac{1}{2}\angle ABC$. But

$$\angle AFE + \angle EFD + \angle BFD = 180^\circ. \quad (1)$$

Therefore

$$90^\circ - \frac{\angle BAC}{2} + 60^\circ + 90^\circ - \frac{\angle ABC}{2} = 180^\circ, \quad (2)$$

whence $\angle ACB = 60^\circ$. Similarly, we can show that $\angle ABC = 60^\circ$ and $\angle BAC = 60^\circ$ and we see that $\triangle ABC$ must be equilateral.

Therefore there does not exist any triangle other than an equilateral triangle with such a property.

The same question about altitudes

What if D , E and F are the feet of the altitudes from A , B and C on to the sides BC , CA and AB respectively? When do the orthocentres of ABC and DEF coincide? In this case one has to do a more careful analysis.

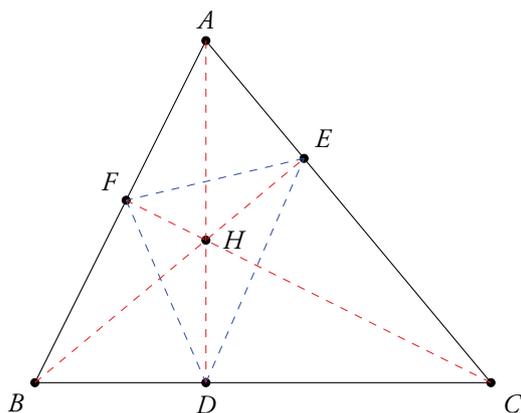


Figure 2.

If $\triangle ABC$ is acute-angled, then points D , E and F lie in the interior of line segments BC , CA and AB respectively. Thus the orthocentre (the point of intersection of the altitudes AD , BE and CF) of ABC , denoted by H , lies in the interior of the triangle.

Moreover, H is the incentre of $\triangle DEF$. To see this, it is enough to prove that AD , BE and CF internally bisect $\angle FDE$, $\angle DEF$ and $\angle EFD$ respectively. Observe that in quadrilateral $BDHF$, $\angle BFH = \angle BDH = 90^\circ$, i.e., a pair of opposite angles are supplementary. Therefore it is cyclic, hence $\angle HDF = \angle HBF = \angle EBA = 90^\circ - A$. Similarly, one may observe that quadrilateral $CDHE$ is cyclic and conclude that $\angle HDE = \angle HCE = \angle FCA = 90^\circ - A$. Hence

$$\angle HDE = \angle 90^\circ - A = \angle HDF, \quad (3)$$

which shows that AD bisects $\angle FDE$. By mimicking this proof, we may prove that BE and CF bisect $\angle DEF$ and $\angle EFD$, respectively.

Now if H is also the orthocentre of $\triangle DEF$, then in this triangle the incentre and the orthocentre coincide. This implies that the internal bisectors of the angles are also the altitudes on the opposite sides, and this leads us to conclude that $\triangle DEF$ is equilateral. This shows that

$$180^\circ - 2A = 180^\circ - 2B = 180^\circ - 2C = 60^\circ, \quad (4)$$

whence $A = B = C = 60^\circ$.

If $\triangle ABC$ is right-angled with, say, $\angle BAC = 90^\circ$, then the points E and F , the feet of the altitudes from B and C to the opposite sides, coincide with A and $\triangle DEF$ degenerates to the line segment AD . If ABC is obtuse-angled with, say, $\angle BAC > 90^\circ$, then the points E and F lie on CA and BA produced beyond A , and their point of intersection, the orthocentre H , lies outside ABC . It also lies outside $\triangle DEF$. Why?

Another class of problems

Now we explore a different class of problems. Given $\triangle ABC$ and a point P in its interior, we draw the lines AP , BP , CP . Suppose that AP intersects BC at D , BP intersects CA at E , and CP intersects AB at F . If $\triangle DEF$ is equilateral, then does it follow that $\triangle ABC$ is equilateral?

We consider this question for some special positions of P inside the triangle, namely, when P is either the centroid (G) or the orthocentre (H) or the incentre (I).

- First, let P be the centroid G . In this case, $\triangle DEF$ is similar to $\triangle ABC$ and its sides are half as long as the sides of $\triangle ABC$. Thus, if $\triangle DEF$ is equilateral, then so is $\triangle ABC$.
- If P is the orthocentre H of $\triangle ABC$, then as we have assumed that P (or H) is an interior point, $\triangle ABC$ is acute-angled and the angles of $\triangle DEF$, as we have seen earlier, are $180^\circ - 2A$, $180^\circ - 2B$ and $180^\circ - 2C$. If each of these is 60° , then it readily follows that each of the angles $\angle BAC$, $\angle ABC$ and $\angle ACB$ is 60° .
- When P is the incentre I of $\triangle ABC$, we claim that if $\triangle DEF$ is equilateral, then so is $\triangle ABC$. Can the reader prove this?



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Architecture always reveals geometrical shapes. Ancient Theatres in Greece are an example, where the play was enjoyed by the audience seated in a semi-circle.



Stone Theatre 'Odeon of Herodes Atticus' Athens, Greece



Photo & Ideation: **Kumar Gandharv Mishra**

Mathematical Relevance: Concentric Arcs, Increase in Length of Arcs with Increase in Height (Elevation).

A Triangle Area Problem

SHAILESH SHIRALI

Problem

We study the following area problem. In Figure 1 we see a $\triangle ABC$ in which segments have been drawn from two of its vertices (B, C) to the opposite sides. These segments divide the triangle into four parts — three triangles and a quadrilateral. The areas of the four parts are denoted by a, b, c, d , as shown in the figure. The problem is: *Find a relationship between a, b, c, d .* (Note that no other information is given; in particular, the area of the triangle is not given.) Use this relationship to find c in terms of a, b, d .

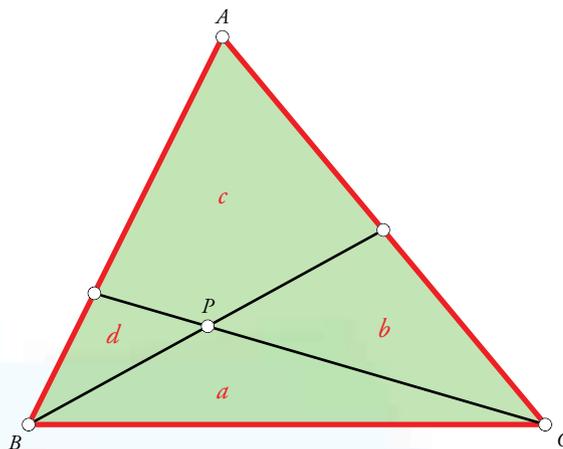


Figure 1.

We urge the reader to tackle this challenging problem before reading further.

Solution

Areal ratio principle. We make repeated use of a simple idea (we refer to it as the ‘areal ratio principle’). Consider any triangle ABC (Figure 2). Let D be any point on BC and let P be any point on AD . Join PB, PC .

Keywords: Triangle, quadrilateral, area, ratio, componendo-dividendo

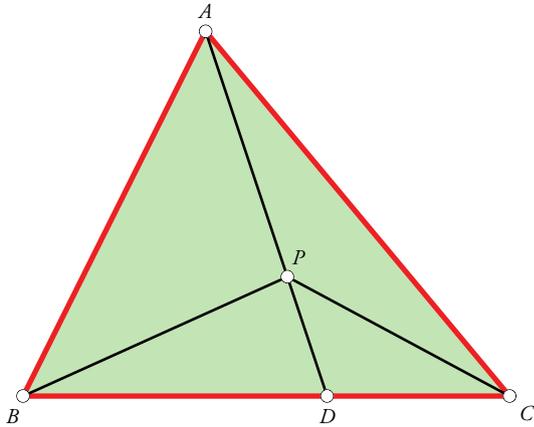


Figure 2.

Then we have the following equalities:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{BD}{CD} = \frac{\text{Area of } \triangle PBD}{\text{Area of } \triangle PCD},$$

and therefore,

$$\begin{aligned} \frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} &= \frac{\text{Area of } \triangle PBD}{\text{Area of } \triangle PCD} \\ &= \frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ACP}. \end{aligned}$$

Here we are using a simple arithmetical principle: if we have two equal fractions $\frac{a}{b} = \frac{c}{d}$ (where $a \neq c$, $b \neq d$), then we also have the equalities

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d} = \frac{a+c}{b+d}.$$

More generally, we have: if $\frac{a}{b} = \frac{c}{d}$ (where $a \neq c$, $b \neq d$), then

$$\frac{a}{b} = \frac{c}{d} = \frac{a+kc}{b+kd} \quad \text{for any real number } k \neq -\frac{b}{d}.$$

The proposition is easy to prove; we leave the details to the reader.

Applying the areal ratio principle to the given problem

In order to apply this principle to our problem, we draw an additional line segment (Figure 3) and denote by y, z the areas of the two triangles into which the quadrilateral (with area c) is divided.

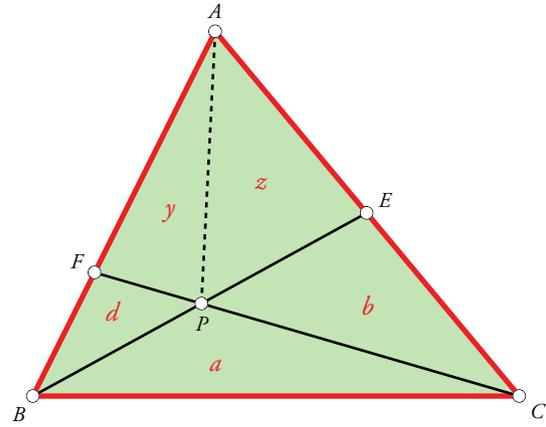


Figure 3. Here $c = y + z$

Invoking the areal-ratio principle stated above, we obtain the following relations:

$$\frac{y}{d} = \frac{z+b}{a}, \quad \frac{y+d}{a} = \frac{z}{b}.$$

Hence:

$$\begin{aligned} ay - dz &= bd, \\ -by + az &= bd. \end{aligned}$$

These equations are readily solved for the unknowns y, z ; we obtain, after a couple of steps,

$$y = \frac{bd(a+d)}{a^2-bd}, \quad z = \frac{bd(a+b)}{a^2-bd}.$$

Since $c = y + z$, we get

$$c = \frac{bd(2a+b+d)}{a^2-bd}.$$

It follows that the desired relation between a, b, c, d is the following:

$$\boxed{c(a^2 - bd) = bd(2a + b + d)}.$$

This is the required solution.

Remarks

Looking at the above relation closely, we may make some interesting remarks.

- Since the quantities a, b, c, d are necessarily positive, it follows from the above relation that

$$a^2 - bd > 0, \quad \text{i.e., } a^2 > bd.$$

It is not immediately obvious why this inequality should be true, other than through this indirect route!

- The above relation expresses c uniquely in terms of a, b, d . However, what if we are given b, c, d and wish to find a ? It would appear now that we have a quadratic equation at hand (in the unknown quantity a):

$$a^2c - 2abd - bd(b + c + d) = 0.$$

This raises the intriguing possibility of obtaining two different values for a . If this truly transpired, it would indeed be very curious, geometrically. However, the possibility cannot happen; for, the sum and product of the two values of a arising from the above quadratic equation are equal to

$$\frac{2bd}{c}, \quad -\frac{bd(b + c + d)}{c},$$

respectively. Of these, the first quantity is positive, whereas the second quantity is clearly negative. Hence one of the values of a must be positive, while the other value must be negative. The negative value is not admissible in a real situation.

Much the same thing happens if we are given a, c, d and wish to find b . The resulting equation is again quadratic:

$$b^2d + bd(2a + c + d) - a^2c = 0.$$

It is easy to check that the sum of the two values of b arising from this equation is positive, whereas the product is negative. Hence one of the values of b must be positive, while the other value must be negative.

By symmetry, we expect that this behaviour will be replicated if we are given a, b, c and wish to find d . The reader is invited to check out this statement.



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Adventures in PROBLEM SOLVING

SHAILESH SHIRALI

In this edition of 'Adventures' we study a few miscellaneous problems. As usual, we pose the problems first and present the solutions later.

Miscellaneous problems

Problem 1. A certain real number a has this most unusual property: it is irrational, yet the two numbers

$$x = \frac{2a^2 - 1}{3a + 2}, \quad y = \frac{3a^2 + 1}{4a + 1}$$

are both rational. Find the value of xy . Can you find a as well? (Based on a problem posed in the Kangaroo Competition.)

Problem 2. Find all pairs of integers, n and k , with $1 < k < n$, such that the binomial coefficients

$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$

form an increasing arithmetic progression.

Problem 3. Find the number of positive rational numbers x such that

$$x^{\lceil x \rceil^{\lfloor x \rfloor}} = \frac{512}{25},$$

where, for any real number z , $\lfloor z \rfloor$ denotes the largest integer less than or equal to z , and $\lceil z \rceil$ denotes the smallest integer greater than or equal to z . (Based on a problem in the 'STEMS 2019' competition organised by students of CMI, Chennai Mathematical Institute.)

Keywords: Kangaroo Competition, Chennai Mathematical Institute, STEMS, rational, irrational, binomial coefficient, arithmetic progression, hexagon

Problem 4. $ABCDEF$ is a regular hexagon; P is an arbitrary point on side AB (Figure 1). Segments DP , EP and FP are drawn, and also segment CF . The points of intersection of DP and EP with CF are G and H respectively. Let $AP/AB = k$. Find the ratio of the area of triangle FPH to the area of quadrilateral $DEHG$ in terms of the parameter k . (Based on a problem posed on the Facebook page of the magazine, AtRiUM. In the original problem, P is the midpoint of AB , so $k = 0.5$.)

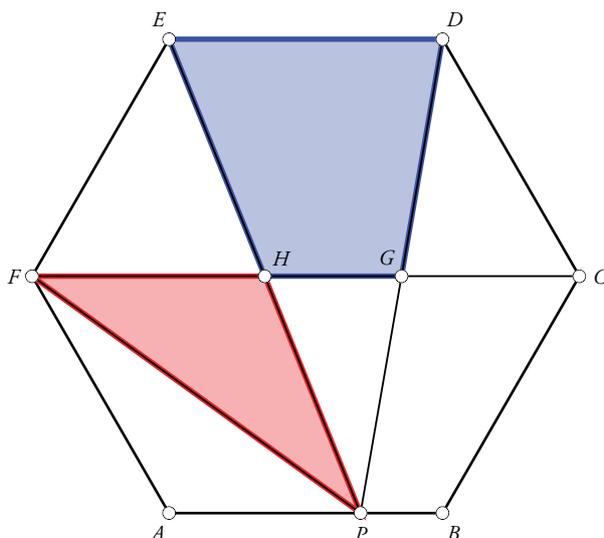


Figure 1.

Solutions to the problems

Solution to problem 1

A certain real number a has this most unusual property: it is irrational, yet the two numbers

$$x = \frac{2a^2 - 1}{3a + 2}, \quad y = \frac{3a^2 + 1}{4a + 1}$$

are both rational. Find the value of xy . Can you find a as well?

Solution. From the two equalities, we obtain, by multiplying through and transposing the terms,

$$\begin{aligned} 2a^2 - (3x)a - (2x + 1) &= 0, \\ 3a^2 - (4y)a - (y - 1) &= 0. \end{aligned}$$

Note that both these equalities are quadratic equations in a . Multiplying the first one by 3 and the second one by 2, we obtain:

$$\begin{aligned} 6a^2 - (9x)a - 3(2x + 1) &= 0, \\ 6a^2 - (8y)a - 2(y - 1) &= 0. \end{aligned}$$

Hence by subtraction we obtain:

$$(9x - 8y)a + (6x - 2y + 5) = 0.$$

If $9x - 8y \neq 0$, then we would obtain, by division, $a = (6x - 2y + 5)/(9x - 8y)$, which would imply that a is a rational number (since x and y are rational numbers); but this is contrary to the given information.

Hence we must have $9x - 8y = 0$, which implies that $6x - 2y + 5 = 0$ as well. We thus obtain a pair of simultaneous equations in x and y :

$$\begin{aligned} 9x - 8y &= 0, \\ 6x - 2y &= -5. \end{aligned}$$

Solving these equations, we obtain:

$$x = -\frac{4}{3}, \quad y = -\frac{3}{2}.$$

Hence $xy = 2$. This is the required answer.

To obtain a , all we need to do is to solve the quadratic equation $2a^2 + 4a + \frac{5}{3} = 0$. This yields two roots:

$$a = -1 \pm \frac{1}{\sqrt{6}}.$$

It may be checked, by substitution, that both of these solutions fit the stated requirement.

Solution to problem 2

Find all pairs of integers, n and k , with $1 < k < n$, such that the binomial coefficients

$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$

form an increasing arithmetic progression.

Solution. The stated condition is equivalent to requiring that

$$\binom{n}{k-1} - 2 \cdot \binom{n}{k} + \binom{n}{k+1} = 0,$$

i.e.,

$$\frac{n!}{(k-1)!(n-k+1)!} - 2 \cdot \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = 0.$$

Multiply throughout by

$$\frac{(n-k+1)!(k+1)!}{n!};$$

we get:

$$\begin{aligned} k(k+1) - 2 \cdot (k+1)(n+1-k) + (n-k)(n+1-k) &= 0, \\ \text{i.e., } 4k^2 - 4kn + (n+1)(n-2) &= 0. \end{aligned}$$

The condition in the last line may also be expressed as

$$n^2 - n(4k+1) + 4k^2 - 2 = 0.$$

We need to find all possible pairs (n, k) of positive integers, $2 < k < n$, satisfying these two conditions. (Note that the two conditions are equivalent to each other.)

We opt to work with the first condition, $4k^2 - 4kn + (n+1)(n-2) = 0$. Observe that it is a quadratic equation in k . The discriminant of this equation is

$$(-4n)^2 - 4 \cdot 4 \cdot (n+1)(n-2) = 16(n+2).$$

Since the quadratic equation $4k^2 - 4kn + (n+1)(n-2) = 0$ has integer solutions (as k is an integer), its discriminant $16(n+2)$ must be a perfect square. This implies that $n+2$ itself is a perfect square. Let us

write $n + 2 = m^2$ where m is a positive integer. This yields $n = m^2 - 2$. Substituting into the equation, we obtain the following:

$$4k^2 - 4k(m^2 - 2) + (m^2 - 1)(m^2 - 4) = 0.$$

Going through the algebra, we obtain the following solutions of this quadratic equation:

$$k = \frac{m^2 - m - 2}{2}, \quad k = \frac{m^2 + m - 2}{2}.$$

Observe that these solutions can be written in a more convenient form,

$$k = T_{m-1} - 1, \quad k = T_m - 1,$$

where T_r is the r -th triangular number, $T_r = 1 + 2 + 3 + \dots + r$.

So we have obtained the required answer: the pair (n, k) must be of the form

$$(n, k) = (m^2 - 2, T_{m-1} - 1), \quad (n, k) = (m^2 - 2, T_m - 1),$$

where m is a positive integer ($m \geq 2$).

Observe that the above two solutions are ‘mirror images’ of each other. (By *mirror image* we mean this: since each row of the Pascal triangle is palindromic, i.e., $\binom{n}{k} = \binom{n}{n-k}$, if three consecutive terms in any row are in AP, then so must be the corresponding three terms counting from the opposite end of the row. The common difference of the latter AP will naturally be negative.) This symmetry holds, since

$$(T_{m-1} - 1) + (T_m - 1) = (T_{m-1} + T_m) - 2 = m^2 - 2.$$

(Here we make use of a well-known identity for the triangular numbers: the sum of any two consecutive triangular numbers is a perfect square.) So we opt to retain only the first of the two solutions, in which k has the smaller value. The following table lists the first few solutions of the problem, corresponding to small values of m .

m	3	4	5	6	7	8	...
(n, k)	(7, 2)	(14, 5)	(23, 9)	(34, 14)	(47, 20)	(62, 27)	...

The three-term APs corresponding to the first three (n, k) pairs in the second row are:

- $(n, k) = (7, 2)$:

$$\left\{ \binom{7}{1}, \binom{7}{2}, \binom{7}{3} \right\} = \{7, 21, 35\};$$

- $(n, k) = (14, 5)$:

$$\left\{ \binom{14}{4}, \binom{14}{5}, \binom{14}{6} \right\} = \{1001, 2002, 3003\};$$

- $(n, k) = (23, 9)$:

$$\left\{ \binom{23}{8}, \binom{23}{9}, \binom{23}{10} \right\} = \{490314, 817190, 1144066\}.$$

It is noteworthy that this problem has infinitely many solutions.

Solution to problem 3

Find the number of positive rational numbers x such that

$$x^{\lceil x \rceil^{\lfloor x \rfloor}} = \frac{512}{25}.$$

Solution. Since x is a positive rational number, $\lceil x \rceil$ must be a positive integer and $\lfloor x \rfloor$ must be a non-negative integer. This implies that the exponent of the x -term in the above equation, namely, the quantity $\lceil x \rceil^{\lfloor x \rfloor}$ is a positive integer. Denote this integer by n ; then the expression on the left side of the above equation is x^n . The equation now takes the form

$$x^n = \frac{2^9}{5^2}.$$

Noting the presence here of the prime numbers 2 and 5, we deduce that x must have the form $2^a/5^b$ for some positive integers a and b . This leads to

$$\left(\frac{2^a}{5^b}\right)^n = \frac{2^9}{5^2},$$

implying that $na = 9$ and $nb = 2$. Since the gcd of na and nb is at least equal to n , whereas the gcd of 9 and 2 is 1, it follows that $n = 1$, which in turn means that $a = 9$ and $b = 2$. This implies that $x = 512/25$. However, this answer is self-contradictory; for, with this value of x ,

$$\lceil x \rceil^{\lfloor x \rfloor} = 21^{20} \neq 1, \quad \text{whereas } n = 1.$$

The implication of this contradiction is that *the given equation has no solutions in positive rational numbers*. Hence the answer is: *the number of solutions is 0*.

Solution to problem 4

$ABCDEF$ is a regular hexagon; P is an arbitrary point on side AB (Figure 1). Segments DP , EP , FP , CF are drawn. The points of intersection of DP and EP with CF are G and H respectively. Let $AP/AB = k$. Find the ratio of the area of triangle FPH to the area of quadrilateral $DEHG$ in terms of k .

Solution. We shall use the vector-based approach described in the article “Arsalan’s Amazing Area Problems” (AtRiA, November 2018). Readers may recall that it had proved extremely effective in solving such problems.

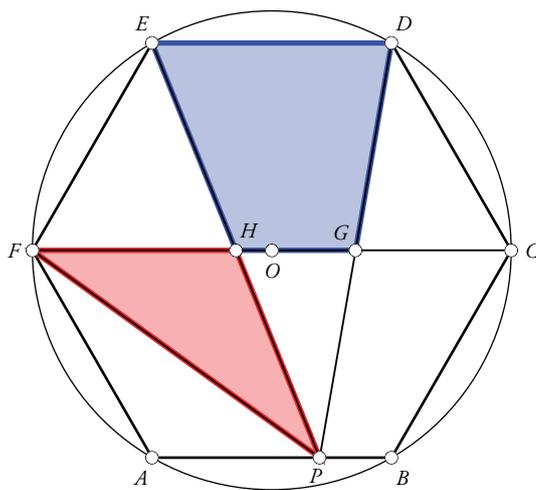


Figure 2.

Draw the circumcircle of hexagon $ABCDEF$, as shown, and mark its centre O . Let O serve as the origin of the vector system, and let the position vectors of A and B be \mathbf{a} and \mathbf{b} respectively. Then the position vectors of C, D, E and F are $\mathbf{c} = -\mathbf{a} + \mathbf{b}$, $\mathbf{d} = -\mathbf{a}$, $\mathbf{e} = -\mathbf{b}$ and $\mathbf{f} = \mathbf{a} - \mathbf{b}$ respectively.

Since $AP/AB = k$, the position vector of P is $(1 - k)\mathbf{a} + k\mathbf{b}$. It follows that the position vector of H (which is the midpoint of segment EP) is $\frac{1}{2}((1 - k)\mathbf{a} - (1 - k)\mathbf{b})$.

Next, we compute the area of $\triangle PDE$. We have:

$$\begin{aligned}\mathbf{PD} &= \mathbf{d} - \mathbf{p} = (k - 2)\mathbf{a} - k\mathbf{b}, \\ \mathbf{PE} &= \mathbf{e} - \mathbf{p} = (k - 1)\mathbf{a} - (k + 1)\mathbf{b}, \\ \therefore \mathbf{PD} \times \mathbf{PE} &= 2\mathbf{a} \times \mathbf{b},\end{aligned}$$

and so

$$\text{vector area of } \triangle PDE = \mathbf{a} \times \mathbf{b}.$$

Observe that the answer is independent of k . This should not come as a surprise (why?). Hence:

$$\text{vector area of quadrilateral } EHGD = \frac{3}{4}(\mathbf{a} \times \mathbf{b}).$$

This is true because the area of $\triangle PGH$ is $\frac{1}{4}$ of the area of $\triangle PDE$.

Next, we find the area of $\triangle FPH$. We have:

$$\begin{aligned}\mathbf{FP} &= \mathbf{p} - \mathbf{f} = -k\mathbf{a} + (k + 1)\mathbf{b}, \\ \mathbf{FH} &= \mathbf{h} - \mathbf{f} = \frac{k + 1}{2}(-\mathbf{a} + \mathbf{b}), \\ \therefore \mathbf{FP} \times \mathbf{FH} &= \frac{k + 1}{2}(\mathbf{a} \times \mathbf{b}).\end{aligned}$$

Hence:

$$\text{vector area of } \triangle FPH = \frac{k + 1}{4}(\mathbf{a} \times \mathbf{b}).$$

It follows that

$$\frac{\text{Area of } \triangle FPH}{\text{Area of quadrilateral } EHGD} = \frac{k + 1}{3}.$$

A surprisingly simple and compact expression!



SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Computing an Angle in an Equilateral Triangle

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

The following geometry problem is simple to state but challenging to solve!

Problem

Within equilateral triangle ABC lies a point D such that $\angle ABD = 54^\circ$ and $\angle ACD = 48^\circ$. (See Figure 1.) Find the measure of $\angle CAD$.

Solution

The author is convinced that there must be a 'pure geometry solution' but has not been able to find such a solution. Instead, the solution offered here uses simple ideas of trigonometry. Readers who find a pure geometry solution are invited to share it with us.

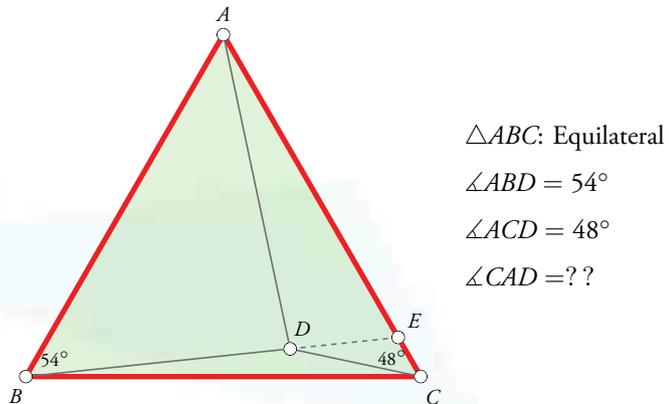
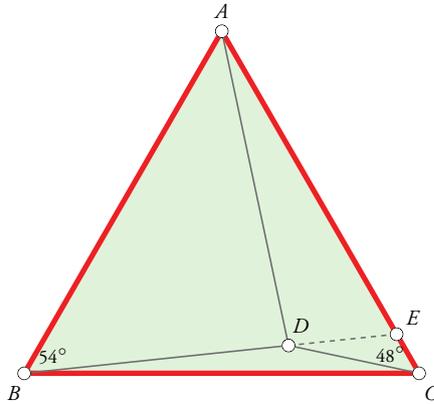


Figure 1.

Let E be the point where BD extended meets side AC . A close study of the figure (assuming it is accurately drawn!) may suggest that triangle CAD is similar to triangle CDE . If this is so, then the required answer is immediately at hand. We shall prove that our surmise is correct and thus deduce the answer.

Keywords: Equilateral, similar triangles, trigonometry, pure geometry.



$\triangle ABC$: Equilateral
 $\angle ABD = 54^\circ$
 $\angle ACD = 48^\circ$
 $\angle CAD = ??$

Figure 2. (The same figure as earlier, redrawn for convenience.)

Claim. $\triangle CAD \sim \triangle CDE$.

Proof of claim. Since $\angle ACD = 48^\circ = \angle DCE$, to establish similarity we only need to show that $CD/CA = CE/CD$. Equivalently, we need to establish that $CE \cdot CA = CD^2$. To do so, we shall make use of the fact that $CA = BC$.

Now we have, from $\triangle BCE$ and $\triangle BCD$, via the sine rule:

$$\frac{CE}{BC} = \frac{\sin 6^\circ}{\sin 114^\circ} = \frac{\sin 6^\circ}{\sin 66^\circ},$$

$$\frac{CD}{BC} = \frac{\sin 6^\circ}{\sin 162^\circ} = \frac{\sin 6^\circ}{\sin 18^\circ}.$$

Hence we have:

$$CE \cdot CA = BC^2 \cdot \frac{\sin 6^\circ}{\sin 66^\circ},$$

$$CD^2 = BC^2 \cdot \frac{\sin^2 6^\circ}{\sin^2 18^\circ}.$$

Hence, to prove that $CE \cdot CA = CD^2$, we must show that $\sin 6^\circ / \sin 66^\circ = \sin^2 6^\circ / \sin^2 18^\circ$.

Equivalently, we must show that $\sin 6^\circ / \sin 18^\circ =$

$\sin 18^\circ / \sin 66^\circ$. That is, we must show that $\sin 6^\circ \cdot \sin 66^\circ = \sin^2 18^\circ$. Transforming the relation yet again, we see that we must establish the following: $\sin 6^\circ \cdot \cos 24^\circ = \cos^2 72^\circ$.

To do so, we draw upon standard trigonometric identities and relations:

$$2 \cdot \sin 6^\circ \cdot \cos 24^\circ = \sin 30^\circ - \sin 18^\circ$$

$$= \sin 30^\circ - \cos 72^\circ$$

$$= \frac{1}{2} - \frac{\sqrt{5} - 1}{4} = \frac{3 - \sqrt{5}}{4},$$

$$2 \cdot \cos^2 72^\circ = 2 \cdot \left(\frac{\sqrt{5} - 1}{4} \right)^2$$

$$= 2 \cdot \left(\frac{6 - 2\sqrt{5}}{16} \right) = \frac{3 - \sqrt{5}}{4}.$$

Hence the equality we hoped to prove ($\sin 6^\circ \cdot \cos 24^\circ = \cos^2 72^\circ$) has been proved. It follows that $\triangle CAD \sim \triangle CDE$, and therefore that

$$\angle CAD = \angle CDE = 6^\circ + 12^\circ = 18^\circ.$$

So the measure of $\angle CAD$ is 18° .



The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Data, Perception and Ignorance

USHA
KRISHNAMOORTHY

“Let the dataset change your mindset” – Hans Rosling

We live in an era of data and information. Right from deciding what to read, what to wear, which restaurant to go to, which city to visit, whom to vote for, we consider ourselves rational human beings who rely on data to make all our decisions. How much of this data is based on facts rather than opinions and/or perceptions?

This review looks at two websites, **Gapminder** and **Our World in Data**, which attempt to provide reliable global statistics and promote a fact-based worldview. While Our World in Data is targeted towards users who have a basic understanding of economics and statistics, Gapminder requires little or no background of either. Examples of my usage of these websites in the classroom are described below.

Gapminder

While looking for information about lifestyles of children around the world as part of a school project, I came across the website Gapminder (<https://www.gapminder.org/>). The website was co-founded by Hans Rosling with his son and daughter-in-law. Rosling, a physician, statistician and public speaker, was a professor of International Health at Karolinska Institute, Sweden. He held several presentations around the world to promote the use of data in exploring issues and trends in global development.

The mission of Gapminder Foundation, as stated on the website, is:

“Fighting devastating ignorance with fact-based worldviews everyone can understand.”

Keywords: Data, statistics, economics, perception, context, facts



Image source: Gapminder Tools

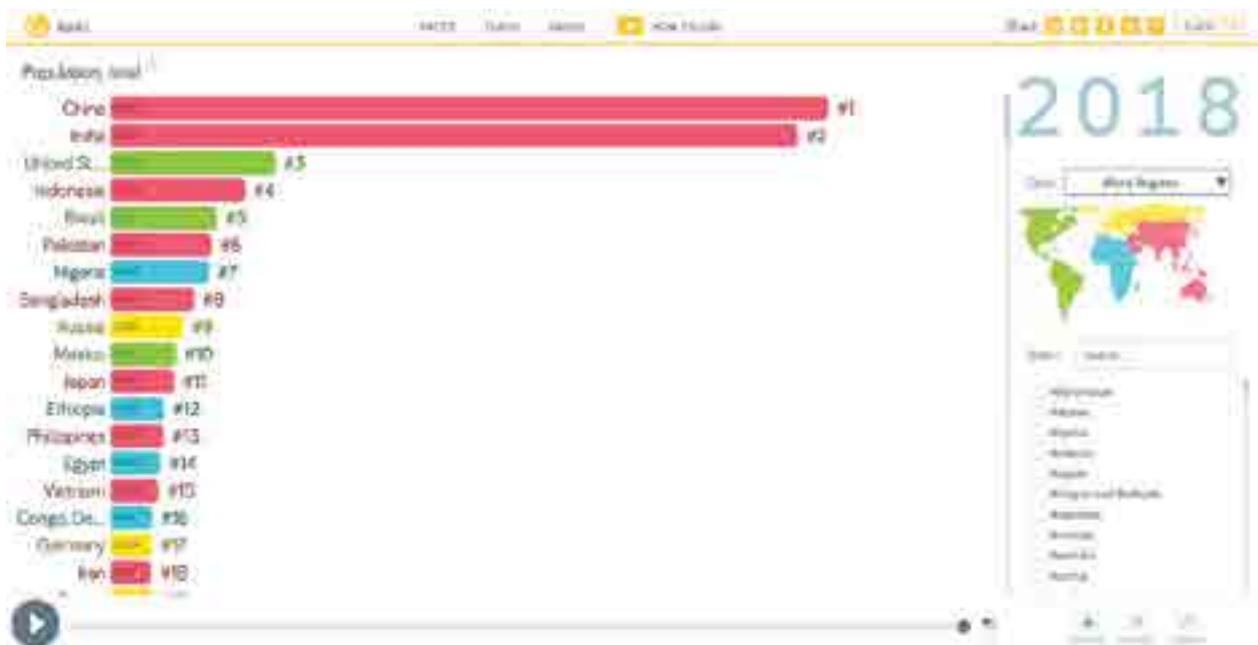


Image source: Gapminder Tools

There is a belief amongst the creators, who are statisticians and/or computer programmers, that people tend to have an overdramatic image and prejudiced notions about the world that can actually lead to bad choices and decisions. In his 2006 TED talk “The Best Stats You’ve Ever Seen”, Hans Rosling stated that university students from Sweden know statistically significantly less about the world

than chimpanzees (who are equally likely to pick the right or wrong answer in response to certain questions). He conducted numerous surveys involving the world’s top decision makers in both the public and private sector and here again he found the “global ignorance quotient” to be quite high. The website is a result of this mission to eradicate widespread ignorance about global statistics.

The website provides a variety of data visualization graphics (some examples below) that allow the user to get a worldview backed by data as well as tools to analyze data for oneself. This greater use and understanding of global statistics, the creators believe, is a way to promote sustainable global development and promote a way of thinking about society and the world that is based on concrete facts, which is what led them to create the Gapminder Foundation in 2005 and subsequently the website.

Gapminder Data

The website combines publicly available data from multiple sources and makes it available in time series format making it possible to analyze trends through the years. Some of the sources of data include the International Labor Organization, the World Bank, the World Health Organization and the OECD QWIDS (<https://stats.oecd.org/qwids/>). There are some

gaps in terms of coverage of the data, which could also be due to a lack of formal records being available for certain countries. While the data is available for download in structured tabular formats, the more interesting use of it is with the tool provided by the website itself. While at first glance they seem to look like simple line graphs or bar graphs plotting a few parameters, the stories that one can get from the data become more apparent as you start to take a closer look. Here's one example.

The parameters shown in this example:

- GDP per capita adjusted to inflation on the x axis
- Life expectancy on the y axis
- Population depicted as the size of the bubbles
- World regions (Americas, Africa, Europe, Asia + Australia) depicted using the color of the bubble

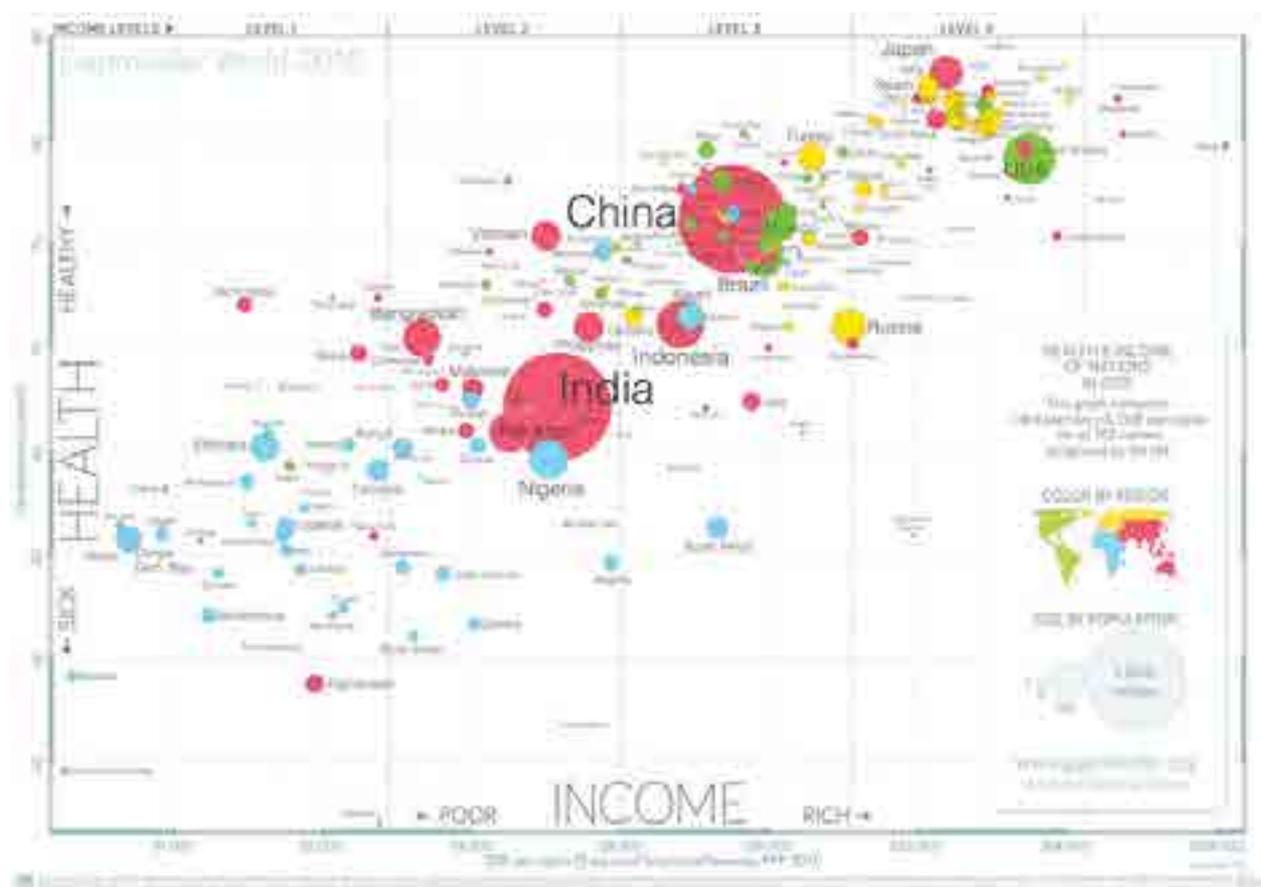


Image source: Gapminder

A slider provided at the bottom of the screen can be used to look at a particular year or you can click on the “play” icon to view change over a period of time.

This section of the website also comes with a set of videos demonstrating how the tool can be used. They do so by taking up questions such as “How does income relate to life expectancy?”, “Will saving poor children lead to overpopulation?” etc. The videos feature Hans Rosling who in a very animated manner demonstrates how data and the tools available can be used to answer the question.

There is an offline version of the tool which includes all data available on the Gapminder website and which also allows you to plug in your own data to use with the visualizations. The offline tool can be periodically updated as new data becomes available on the website. All the material is freely available under the Creative Commons Attribution 4.0 International license, which (amongst other things) permits usage in schools for educational purposes.

Classroom Usage: I have used data (life expectancy and income per person) from Gapminder with senior school students (Class XI) as part of their statistics curriculum. Looking at the data, they were able to not only look at the correlation of these parameters, but also look at the impact of specific events in history (World War II, Indian Independence, economic liberalisation) on these parameters. We also watched the video on the website, “How does income relate to life expectancy”. The Gapminder offline tool opened up a lot of interesting ways in which data can be represented for students, and the whole time-series animation brought in a new dimension to the statistical analysis of data! The biggest caveat however, was the gaps in data which we found while exploring some African countries. While we were able to look at worldwide trends, patterns and impact, zooming in on specific countries made data gaps quite glaring.

Impromptu conversations on how a mental process is quickly triggered to fill in the gaps to agree with the trends brought in an interesting twist to the topic.

Dollar Street

Another key section of the website is “Dollar Street”. While the data visualization tools of Gapminder help visualize trends and correlations between global statistics, they still do not help understand how people in various corners of the world live. To address this, Anna Rosling Rönnlund has come up with Dollar Street.

The idea is to put every family in the world on a street with houses lined up by income: the poorest living on one end, the richest on the other and the rest else in between. A team of photographers have documented over 264 homes in 50 countries so far. In each home, the photographer spends a day taking photos of up to 135 objects, such as the family’s toothbrushes or favourite pair of shoes and also body parts such as teeth, hands. All photos are then tagged by household function, family name and income.

The images allow one to create a mental representation/model of living conditions of a specific income group in a particular country as well as to contrast the conditions across different countries. The website includes details of how the researchers went about the rather complex process of calculating income and gathering data for this. There are definitely gaps, as one would expect in an undertaking of such a scale, but if the project continues, it has the potential to bring out a comprehensive description of the living standards in many countries.

Classroom Usage: This section brings the (however clichéd) saying “a picture is worth a thousand words” to life! Students (Class IX) enjoyed looking at this page and explored everything from toilets to toothbrushes to homes and cars. Some of the pictures confirmed their



Image source: Screen capture

stereotypes while some left them baffled. While looking at countries that they were familiar with, such as India, they got a sense of the gaps in data. Having said that, it is definitely a very unique way of getting a picture of the world and its diversity. The commonly held idea of data as numbers and visuals as graphs and charts was questioned with the use of images as an interesting way of understanding living conditions.

Teaching Resources

In addition to the tools and data, there is a section called “TEACH” for educators who would like to use Gapminder in the classroom. This section includes presentations, videos and quizzes that can be used to explain global development over the years. In addition, there are also lesson plans and handouts available for download. I have used life expectancy and

income data from this website with high school students and we analysed trends for specific countries. We then looked at the same data using Gapminder tools to understand how these parameters correlate with each other.

Factfulness

This section is related to the book with the same name written by the creators of the website. It is described as:

“The stress-reducing habit of only carrying opinions for which you have strong supporting facts.”

The interesting parts about this section are the posters that depict what stories in the media get our attention, trigger our dramatic instincts and how we can look at them. There is also a presentation available that can be used to go over each of these “instincts” in detail.



It probably summarises the raison d'être of the foundation and all the tools and material they have developed.

Website Features

- **Visual appeal:** The layout makes it easy to browse and review various sections. While the homepage doesn't summarise what the webpage is about, it has details about various sections and a video introducing them. In keeping with the current trend of visualisation-driven websites, there is minimal use of text and images and videos are used to explain various tools and concepts. The menus however are not consistent across pages, so the browser "Back" button has to be used a lot to navigate across sections. Dollar Street and the Gapminder tool are easy to access and are probably what one would visit most often. Both these pages render well on mobile devices as well, with no apparent loss of functionality.
- **Loading speed:** Even though the website is graphics intensive, the media seems to be

optimised for a variety of bandwidths and it worked seamlessly in all instances. The embedded videos are hosted on YouTube or TED.

- **Advertisements:** The website is ad free. On some pages there is a section through which people can donate to the Gapminder foundation.
- **Plug-ins/Additional Software:** I have used the website on Google Chrome and no additional software is required except if you want to download the offline version of the tool.
- **Search:** There is a search function that allows you to do a keyword search on the entire website. Additionally you can search for specific datasets on the DATA section of the website.
- **Help/FAQ:** The Help section is mostly FAQ and covers a wide range of questions. In addition to this, the LAB section explains usage of various APIs (Application Program Interface) for advanced users familiar with web programming that can be used to integrate tools within websites.

To summarise, the website provides numerous innovative ways of narrating stories with data. Hans Rosling's enthusiasm in the videos makes data look exciting and beautiful and is a joy to watch! In the classroom, I have found it useful to introduce statistics as a subject and to get students excited about its applications!

What to watch out for?

The data itself. While the sources of data are mostly the experts in the subject matter and official data owners, one still needs to keep an eye out for all the "C"s of data quality: completeness, consistency, credibility, cleanliness! Do not mistake non-availability of a picture of a high-income group person in a particular country to be an indicator of poverty levels in the whole country!

Our world in data

Another website with very similar objectives is Our World in Data (<https://ourworldindata.org/>). Their purpose, as stated in the website:

“Through interactive data visualizations we can see how the world has changed; by summarizing the scientific literature we explain why. Understanding how and why the world has changed up to now allows us to see that a better future is possible.”

The project was started in 2011 by economist Max Roser from the University of Cambridge and it focuses on poverty, health, and the distribution of incomes.

Data Sources

The website used 3 sources for all the data:

- Specialized institutes – such as the Peace Research Institute Oslo (PRIO)
- Research articles – such as Bourguignon & Morrison – ‘Inequality Among World Citizens: 1820-1992’ in the American Economic Review [2002]
- International institutions or statistical agencies – such as the OECD, the World Bank, and UN institutions.

Every chart/visualization has the source clearly indicated and includes details about when it was collated. Some charts, for example “Population by country” (<https://ourworldindata.org/world-population-growth>) indicate Gapminder as the source for certain years. The data is available for download in CSV (comma separated value) format.

Analysis

With every data type, the website provides detailed analysis by a team of researchers. Unlike Gapminder, which has short videos to explain data trends and to answer FAQs, Our World in Data has detailed reports with each data type. The report is accompanied by charts and other visualizations such as maps to illustrate various statistics that have been computed and analysed.

While Gapminder provides tools to review and analyse data by oneself, this website provides that analysis compiled from many sources including news agencies and the United Nations. Some sections like the one on Child Labor have a detailed section on data and research gaps and also on why those gaps exist.

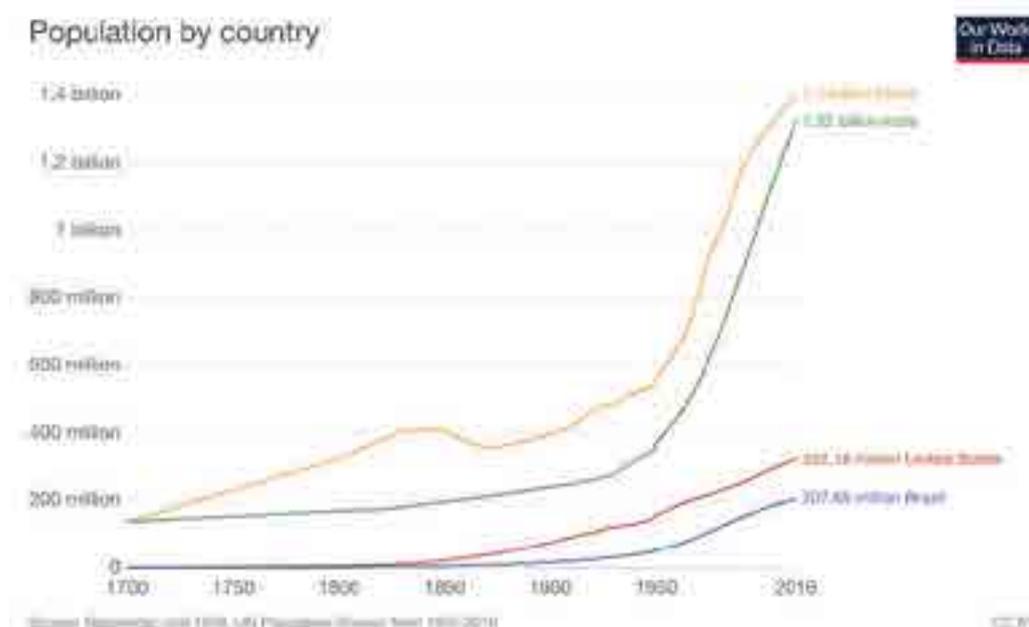


Image source: Screen capture

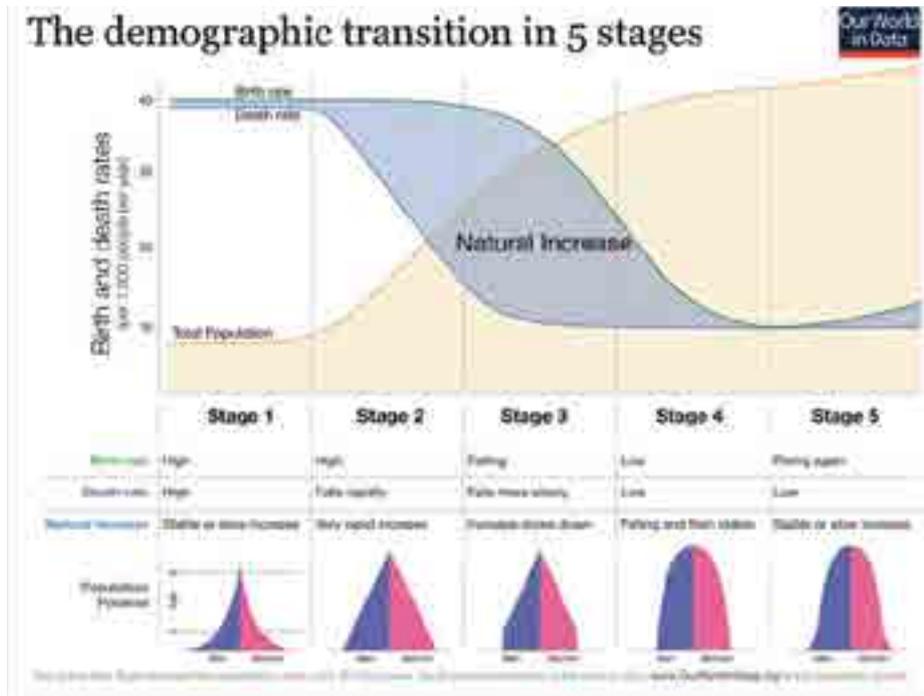


Image source: Screen capture

Some of the interesting statistics and statements mentioned include:

- *Recent estimates suggest that today's population size is roughly equivalent to 6.9% of the total number of people ever born*
- *A country's level of education attainment is a key determinant of the emergence and sustainability of democratic political institutions*
- *It is estimated that North Korea spends roughly one-third of their national income on defense.*

Just like Gapminder, all the material in this website is also freely available under the Creative Commons Attribution 4.0 International license.

Teaching Resources

This section provides notes, presentations, links to related blog posts and a list of books/articles related to various topics. The presentation includes relevant charts to illustrate various ideas.

Classroom Usage: We used this website while researching child labour for a school project with class X students. It not only provides data but also provides pointers on what correlations one should focus on and how data should be

read. However, some parts are for a much older audience and require a background in economics and/or statistics to interpret.

Website Features

- **Visual appeal:** The website is very clean with just a few menus which are clear and easy to navigate through. Unlike Gapminder, this website uses a lot of text interspersed with charts and other graphics. For this reason, it's probably less appealing to a younger audience. This website, just like Gapminder, renders well on a mobile device, but it is a lot easier to interact with the charts on a computer.

- **Loading speed:** The website loads without any obvious lags, and navigation is quite seamless.
- **Advertisements:** The website is ad free. There is a separate section on how one can donate.
- **Plug-ins/Additional Software:** I have used the website on Google Chrome and no additional software is required.
- **Search:** There is a search function that allows you to do a keyword search on the entire website.
- **Help/FAQ:** The “How to use” section answers some FAQs on how the website, charts and data can be used.

Both these websites demonstrate unique stories data can narrate. While accuracy and completeness of data of an event or a situation is crucial, it is perhaps appropriate visualization and representation that provide insight and understanding of the worlds that this data is meant to portray.

Many independent initiatives are also being undertaken to produce eye-catching visuals to recalibrate popular assumptions and beliefs about global statistics. Commemorating this is the Kantar *Information is Beautiful* Awards, which recognises excellence and beauty in data visualizations, infographics, interactives and information art. In February 2019, they announced the winners of the World Data Visualization Prize

(<https://informationisbeautiful.net/2019/winners-of-the-world-data-visualization-prize/>). This year’s topic (much like the two websites reviewed above) focused on how governments around the world are improving citizens’ lives, and the innovations that drive and measure success in this realm. (<https://wdvp.worldgovernmentsummit.org/>). Award categories included interactive, static and hand-drawn “napkin” and a final grand prize, the winner of which is An Alternative, Data-Driven, Country Map by Nikita Rokotyan. It uses an artificial intelligence technology called t-SNE (t-Distributed Stochastic Neighbor Embedding) to discover clusters of nations that may not be physically next to each other but are related by happiness score, health expenditure, investment in education and many other variables. This technology provides a methodology for visualising high-dimensional data in a two or three dimensional map. The aim of dimensionality reduction is to preserve as much of the significant structure of the high-dimensional data as possible in the low-dimensional map. This is done by using all dimensions to allocate a location to each datapoint. You then end up with a different image of the world, in which Japan, UK, USA, France are neighbours and so are Russia, Brazil and Argentina!

Just like Gapminder and Our World in Data, all the entries to this contest have truly innovative ways of representing data, making information, however dismal, truly beautiful!



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The Closing Bracket . . .

I wonder why India is lagging so far behind in the matter of teacher education. Not lagging behind in the sense of being behind other countries, but lagging behind in the sense of not addressing the task at hand – the task of preparing teachers who will take on the most important job that there is, that of educating children.

It is a matter that should have been addressed soon after independence, along with another matter of equal importance – that of primary education. Strangely, both these tasks were given very little attention for many decades after independence. It seems hard to believe that so obvious a mistake could have been committed; but it was. We are paying the price for that right now.

If the situation different right now? Only a little. There is a little more conversation about these matters these days, government proposals get put out in the public domain and responses are invited from the public, and so on; but very little action, it seems. Perhaps there is too much discussion and too little action.

Perhaps we are also paying the price for something for which it is only we – the people of this country – who can be blamed. These are the social attitudes that we in this country have developed, over the past several decades, with regard to success and failure, with regard to status, with regard to material possessions; and, crucially, with regard to the teaching profession. (This list can be added to; there are also our attitudes to caste, to colour, to gender, and so on. Probably, these various prejudices are not unrelated to each other; they are not independent variables. But we can hold off that discussion to another day.) As long as our culture continues to devalue the teaching profession, as it has been doing for over half a century, we will continue to pay the price for our shortsightedness.

But let's return to the question of policy and action. To me, it seems rather a mystery. Is the teaching of a subject – any subject – a task of such complexity that all manner of research has to be done to unravel it, that numerous committees have to be constituted to tackle it? That does not seem true. Of course there are fine points involved – e.g., in the case of mathematics teaching, there are fine points involved in the understanding of errors: the way concepts get formed incorrectly, and the overlay of these concepts with subsequent inputs; all these play havoc with students' learning and therefore need to be understood. (A similar situation may hold in the case of the physical sciences.) But this can hardly be given as an excuse for holding up action.

Surely, good teaching is largely a matter of common sense resting on a base of affection for children and an interest in their learning, and also on a base of love for the subject. Surely, anyone who wishes to teach mathematics well will realise that learning mathematics does not mean merely knowing how to solve problems in mathematics; that learning mathematics means also exploring and investigating patterns and delighting in them and talking about them to fellow explorers; it means also posing questions and inquiring into them and coming up with conjectures based on patterns that we find; it means also figuring out ways to justify claims and (sometimes) to disprove claims, made by us or others; and it means also bringing the spirit of experimentation and the joy of inquiry and learning into the classroom. All this, surely, is the work of the teacher; this is the base on which we all have to build.

The sooner we realise that addressing this task is our common responsibility – that it is the work of individuals like you and me to create such a culture – the better for all of us and for the health of this planet.

I would like to conclude this little piece by reproducing a wonderful little quote that has been attributed to Girish Karnad:

As a student of mathematics, initially, I was not very loyal to the subject. It was to score high marks that I took it up. But when I got immersed in it as months passed, I began to understand its rhythm, its pitch, its progression and crescendo. Its beauty danced in front of my eyes. A character in Aldous Huxley's novel weeps at the beauty of the Binomial Theorem. There is nothing surprising about this reaction when numbers unravel their mystical attributes wave by wave, branch by branch. I realised the impact that mathematics had on me when I started to write Tuglaq in Oxford. I solved the structural issues like I would while working on theorems. I first figured out what internal network and relationships different aspects and characters of the play had, what its balance at various points were, and what happens to that balance if the play progresses in a certain direction, just like it happens in a theorem. The technical training I needed to write plays came from Mathematics.

Would that more of us weep at the beauty of learning...

Shailesh Shirali
Chief Editor

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Editors

Currently drawn from Rishi Valley School, Azim Premji Foundation, Homi Bhabha Centre for Science Education, Lady Shri Ram College, Association of Math Teachers of India, Vidya Bhavan Society, Centre for Learning.

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RATIO

PADMAPRIYA SHIRALI



**Azim Premji
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A publication of Azim Premji University
together with Community Mathematics Centre,
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INTRODUCTION

How does one introduce a topic like ratio, which is so widely present in daily life and so intimately connected with human experiences? Our cherished cultural achievements are permeated with it: music is full of ratios, as is art. Our daily existence involves cooking and shopping, and these are filled through and through with the usage of ratio. Shadows, which are present with us all through the day, offer a visual depiction of ratios in action. In mathematics as a subject, the notion of ratio is embedded into many topics – sometimes in an obvious way, at other times not so. Fractions, scale drawing, enlargement, trigonometry, tables, linear equations... are all illustrations of this.

Considering that ratio is linked in an essential way to so many concepts in mathematics, it is important to lay a strong foundation and develop the necessary conceptual base when teaching this topic. A basic error that often arises while studying ratio is the application of *additive thinking* to a context that requires *multiplicative thinking*. This is an error that students make quite frequently while solving ratio problems. There is a clear need to point this out in various ways and by using different examples, so that students clearly see that ratio and proportion are based on a multiplicative relationship.

Often the problems that students encounter in textbooks are limited and become repetitive.

In addition, lack of exposure to sufficient variety inhibits students from noticing the proportional situation in varied contexts. They may not develop the ability to distinguish between proportional and non-proportional situations. Again, most problems involve whole number scale factors and students get flustered when they come across real life problems that do not fall into that category; i.e. the scale factor is a fraction. Examples taken up should cover diverse areas and bring in fractional scale factors as well.

While students encounter the topic in different forms as they go to higher grades, we need to reiterate for ourselves what we expect the student to achieve during the introductory phase.

Expectations

- Understand the relationship of ratios to fractions: While fractions compare part to the whole, ratios compare both part-to-part and part-to-whole.
- Understand the importance of order in reading and writing ratios.
- Explore and generate equivalent ratios.
- Understand that a proportion is based on a multiplicative relationship and generalise the numerical relationship, i.e., if $a : b :: c : d$, then $ad = bc$.
- Recognise proportional and non-proportional contexts.
- Understand the usage of unitary method in relation to proportion problems.
- See rate as the ratio of two quantities having different units (rate being a special kind of ratio).
- Divide a number into a given ratio.
- Solve a proportion problem using different strategies and diagrams.

So, where do we begin?

Students have an intuitive understanding of Ratio and Proportion. It may be good to begin there, to let them use their intuitive understanding and then help them to notice the principles they have applied, study the principles, experiment with them and articulate them. I shall lay out gradually what I mean by 'studying and experimenting'.

Activities 1 to 5 and Game 1 build on this intuitive understanding.

Keywords: Ratio, proportion, fraction, part to part, part to whole, equivalent ratio, unitary method

ACTIVITY 1

Objective: To draw out intuitive multiplicative thinking through a visual poser.

Materials: Hexagon pieces

Lay out a design pattern for the students as shown in Figure 1.

If I have 12 yellow hexagons, how many blue hexagons do I need to make designs as in Figure 1?

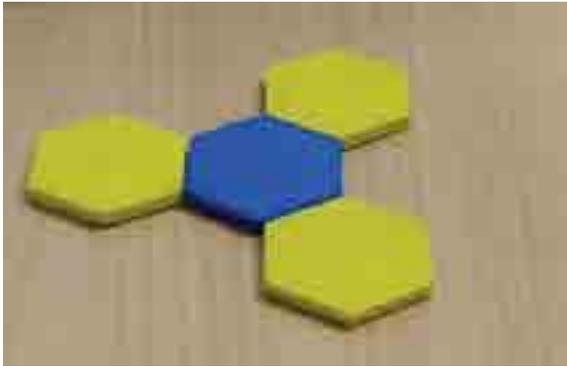


Figure 1

If I have 6 blue hexagons, how many brown hexagons do I need to make designs as in Figure 2?



Figure 2

ACTIVITY 2

Objective: To point out multiplicative thinking used by students.

Pose a problem based on the classroom situation. Let students use their intuition in responding to it.

Tanvi and Mouli are solving problems. For every two problems that Tanvi solves, Mouli solves three.

If Tanvi solved eight problems, how many did Mouli solve?

Let the students think it over for a few minutes and allow them to come up with their responses. Ask them to explain their answers.

Some may give the right answer, 12, but it is possible that some student will respond by saying 9.

'Mouli solved nine (six more than three) as Tanvi solved eight (six more than two)'.

How do the other students respond to such reasoning? Do they see the flaw in the reasoning?

Once the discussion is over, the teacher should reiterate the significance of the words 'for every' and then point out to the students the multiplicative reasoning that they applied.

For every two problems that Tanvi solved, Mouli solved three. If Tanvi solved eight problems which is four times 2, then Mouli will solve four times 3, which is 12.

ACTIVITY 3

Objective: To develop intuitive understanding of proportion through enlargement.

Materials: Square dot paper

Give a drawing of a letter of the alphabet or a numeral, as shown in the figure.

Ask students to double it. Were they able to do it correctly?

Did their drawing seem right to them?

What kind of errors did they make?

What could have caused the errors?

Ask them to triple it. Did they find any difficulty?

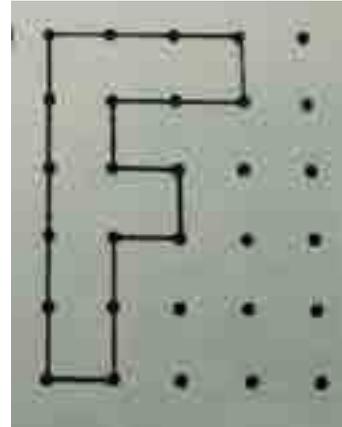


Figure 3

ACTIVITY 4

Objective: To develop intuitive understanding of proportion through model building.

Materials: Unifix Cubes

Give the class a $2 \times 2 \times 2$ cube.

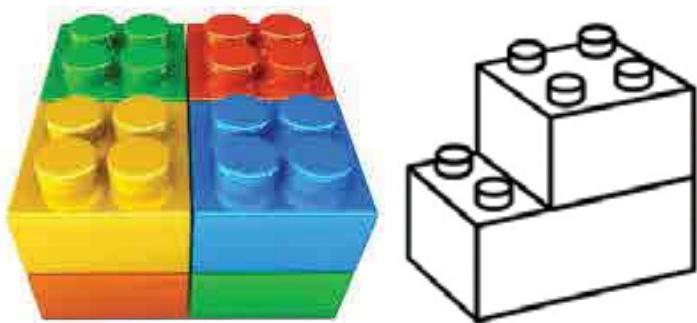
Ask them to double it.

How did they approach the problem?

How many cubes will they need? Did they forget to double any dimension?

Ask the class to triple this shape.

Did they find this challenging?



ACTIVITY 5

Objective: To spot similar shapes

Materials: Sets of rectangles, isosceles or right-angled triangles and trapeziums (each set to contain one pair of similar figures). Note that we do not permit use of tools of measurement.

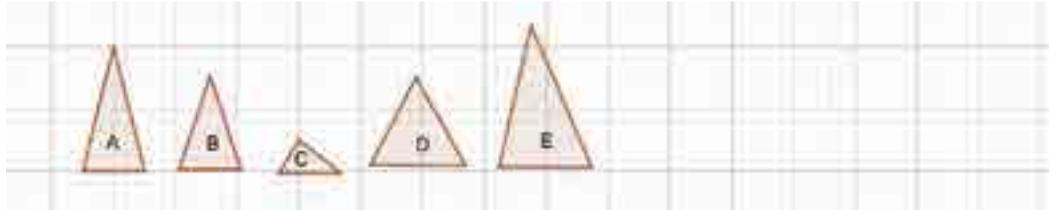


Figure 6



Figure 7

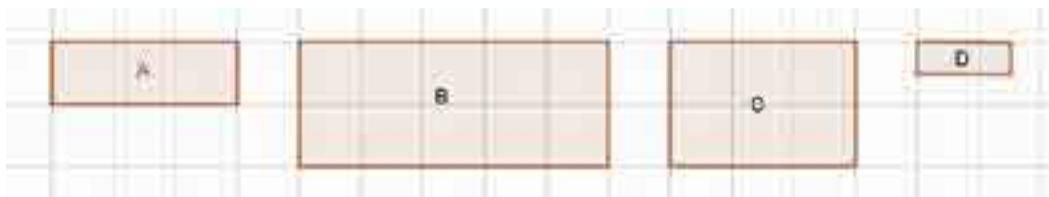


Figure 8

The teacher can prepare these using square grid papers and mount them on plain card sheet. Each shape can be labelled using alphabets. We must take care in selecting the sizes, so that spotting similarity is neither too easy nor too difficult.

Divide the students into three groups. Give the sets one at a time to the groups so that they can discuss and select the pair that looks similar to them. They can use either the plain side of the shapes or the side with square grid in deciding about similarity. By rotation, each group looks at the three sets and notes down the similar pairs in each set.

At the end of the activity, all groups share their findings and present their reasoning. Now verify the answer using a ruler.

You could keep the challenge level high by insisting that they use only the plain side of the shapes.

Follow this up by a worksheet that requires testing for similarity.

Are the figures in each set similar?

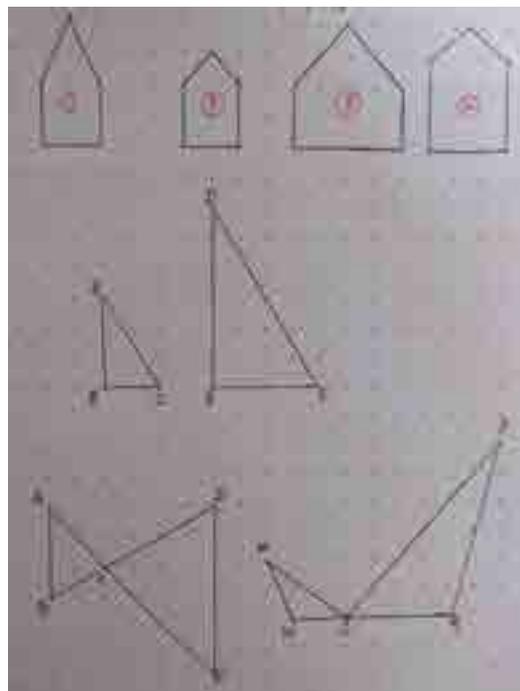


Figure 9

GAME 1

Objective: To spot equivalence of ratios and group them

Materials: 16 Cards with appropriate pictures to demonstrate equivalent ratios
(for example, four equivalent ratios of 3:1, 1:4, 3:4 and 2:3)

Here are two pictures for equivalent ratios of 1:2



Figure 10



Figure 10.1



Figure 10.2

Divide the class into two teams. Ask each team, one after the other, to sort the cards and group them. If they manage to group the four sets correctly, then they get four points.

Analyse how they have grouped them. Ask the students the basis of their groupings.

Does everyone agree with the reasoning behind the grouping?

ACTIVITY 6

Objective: To introduce what a ratio is and to show how we read and write ratios.

Materials: Classroom

Point to the furniture arrangement in the classroom.



Figure 11

In this picture, each table has two stools with it.

The ratio of tables to stools is 1:2.

Teach students how the ordering of words affects the way we write a ratio.

For instance, in this picture the ratio of stools to tables is 2:1.

Get students to describe this cube design using ratios.

What is the ratio of black to grey cubes?

How do the black cubes compare with the white cubes?

Are the students able to use multiplicative thinking?

They can record their findings in sentences.

There are twice as many ____ cubes as ____ cubes.

As a second step, they can write it in the form of ratio statements.

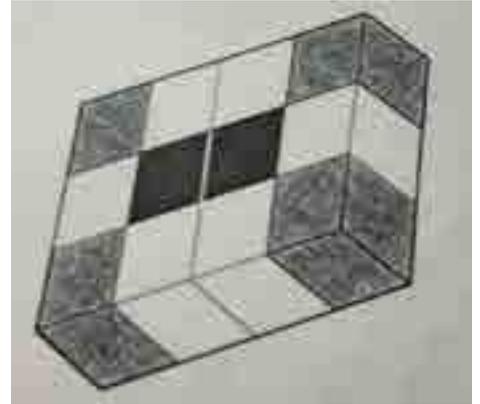


Figure 12

ACTIVITY 7

Objective: To study and experiment with a given ratio.

Materials: Cubes or counters

Here are 2 yellow cubes and 4 black cubes.

The ratio of yellow to black is 2:4.

There are 2 times as many black cubes as yellow cubes.

If you add 2 cubes of each colour, will the black cubes still be 2 times the yellow cubes? What is the new ratio of yellow to black?

What will happen if you take away 1 cube of each colour from the original pattern? What is the new ratio of yellow to black?

What will happen if you double each colour? What is the new ratio of yellow to black?

What will happen if you halve each colour? What is the new ratio of yellow to black?

Do the students form any conclusions based on this experimentation?



Figure 13

ACTIVITY 8

Objective: Explore the multiplicative relationship of ratios by generating equivalent ratios.

Materials: Square grid paper

The teacher needs to show the students the relationship of fractions to ratios. She also needs to point out the differences. They may not have recognised before that every fraction is in fact a ratio.

While studying fractions, we compare the part to the whole, but while studying ratios we compare both part-to-part as well as part-to-whole.

Also in the case of fractions, the order in which we write the numbers is fixed, the numerator is the part and the denominator is the whole. In the case of ratios, there is no such fixed order.

The teacher can show the close connection between equivalent fractions and equivalent ratios through drawings.

This figure shows a part-to-whole ratio.

Students should record the work as shown in this figure to internalise the fraction, ratio connection.

Students also need to understand that, just as with fractions, we can reduce ratios by common factors of the quantities, to make the simplest form.

Example: 4:12 can be divided by 4 on both sides

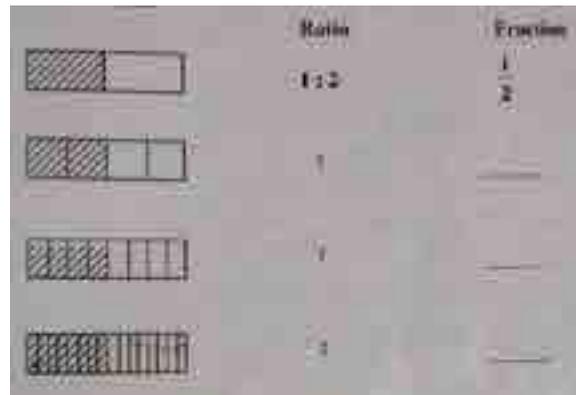


Figure 14

to give 1:3.

8:10 can be divided by 2 on both sides to give 4:5.

Discuss part-to-whole and part-to-part ratios in the same context to make the difference clear.

A class has 18 boys and 12 girls.

The ratio of boys to girls is 18:12 or 3:2. This is a part-to-part ratio.

The ratio of boys to total number of students is 18:30 or 3:5. This is a part to whole ratio.

ACTIVITY 9

Objective: To show that a proportion is based on a multiplicative relationship and generalise the numerical relationship.

Materials: Five equivalent ratios of 3:1, 1:4, 3:4 and 2:3 written out on separate cards

Distribute the cards to students. Ask them to sort the ratio cards and group them. Through inspection, they will spot the equivalent ratios and group them. Verify that they are right.

The teacher can now draw their attention to the multiplicative relationship within each group of ratios, explain what proportion means and point out that the ratios in each group are in proportion.

Raise the question: "If we take any two ratios which are in proportion, how do the numbers relate

to each other? Say, 6:8 to 15:20." One obvious relationship is that we can reduce both pairs to a common ratio. Is there any other relationship? Can we see a relationship if we write them as fractions? Help the students notice that 6 times 20 equals 8 times 15. Ask them whether this happens with every pair of ratios in proportion.

Through discussion, the teacher can generalise the procedure, i.e., when $a:b::c:d$, then $ad=bc$.

The class can then discuss practical examples involving proportion like recipes, heartbeat, physical exercise like runs.

However, at the same time, the teacher must also expose the students to non-proportional contexts so that they are able to identify and differentiate between proportional and non-proportional contexts.

A simple example of a non-proportional context is the comparison of the ages of two people.

Aarav is 12 years old. His sister Ami is 6 years old. How do these numbers compare? Aarav is twice

as old as Ami. What will be the case 6 years later? Would Aarav be twice as old as Ami?

One day I'll catch up! My Dad is four times as old as I am. When will he be three times as old? When will he be twice as old? When will I catch up?

An interesting exploration to do would be to examine how the scale factor behaves, as the daughter and father get older. Can the students see why this happens?

Reflection: What line of inquiry can the student now pursue? What tasks can the teacher set?

ACTIVITY 10

Objective: Understand the usage of unitary method in relation to proportion problems.

Materials: Price lists of products, food items etc.

Unitary method, i.e., calculating per unit is a very common way of solving proportion problems. Many students find it natural and easy-to-understand and take to it without any difficulty.

Example. 3 chocolate bars cost Rs. 45. How much will 7 bars cost?

They may reason it out as, if 3 bars cost Rs. 45, then 1 costs Rs. 15, so 7 bars will cost Rs. 105.

It seems reasonable and efficient to solve it this way.

Is the unitary method always the best way?

If 500 balloons cost Rs. 2745, what will be the cost of 800 balloons?

Finding the cost of one chocolate bar from the price of three bars seems perfectly sensible, but finding the cost of one balloon from the cost of 500 balloons does not. A far more workable method here would be to find the cost of 100 balloons and then to multiply by 8.

Naturally, we need to modify the unitary method according to the situation.

Ice Cream - Single Scoop	
Butter Scotch	₹ 35
Real Strawberry	₹ 35
Pina Lemon	₹ 35
Paan	₹ 40
Choco Chums	₹ 40
Rose Petal	₹ 40
Coffee Mocha	₹ 40
American Nuts	₹ 45
Fruit Cocktail	₹ 50
Kaju Anjeer	₹ 50
Kesar Pista	₹ 50
Belgian Chocolate	₹ 50
Coffee	
Plain Coffee	₹ 30
Black Coffee	₹ 50
Cappuccino Coffee	₹ 50
Cold Coffee	₹ 90
Tea	
Plain Tea	₹ 25
Masala Tea	₹ 30
Ginger Tea	₹ 30
Black Tea	₹ 30
Green Tea	₹ 30
Lemon Ice Tea	₹ 55
Soups	
Tomato Soup	₹ 80
Hot n Sour Soup	₹ 80
Sweet Corn Soup	₹ 80

Figure 15

ACTIVITY 11

Objective: Aids to solving problems (modelling)

Materials: Square paper

In the initial stages, the teacher can give problems accompanied by visuals.

In the second stage, the student can model a given problem.

Modelling is an important tool for solving word problems.

Ex. In a class, the ratio of girls to boys is 2:3. If the number of students in the class is 30 what is the number of boys? What is the number of girls?

Here is one way of modelling it.

For every 2 girls there are 3 boys in the class.

Each such group contains 5 students.

How many groups of 5 in 30?

6 such groups.

Therefore, in 6 such groups, the number of girls is 12 and the number of boys is 18.

Reflection: An important question while solving problems is this. *What imagery does the student use to understand a problem?*

Can we teach imagery? Is not imagery a subjective experience that may be difficult to articulate?

At the same time, helping students to find their own imagery requires that the teacher expose them to visuals,

multiple strategies and approaches to problem solving. Imagery is often closely linked to the way a concept has been developed. Otherwise, simplifying ratios or finding proportions can become highly mechanical.

11. In the class room the ratio of tables to stools is 1:2

- If there are 13 tables in the class, how many stools will be there? _____
- If there are 20 stools in the class, how many tables will be there? _____

12. Ashwathi is washing his mother make tea. She uses 2 tea bags for 3 cups of tea.

She will use 4 tea bags to make _____ cups of tea.

She will use _____ tea bags to make 12 cups of tea.

13. In the dining hall the ratio of jugs to plates is 1:8

If there are 12 jugs in the dining hall, how many plates will be there?

If there are 120 plates how many jugs will be there?

Figure 16



Figure 17

ACTIVITY 12

Objective: Aids to solving problems (rephrasing)

Another important aid in solving ratio problems is the ability to rephrase a problem in ratio language. What is 'ratio language?' It involves the use of phrases and words such as 'for each', 'for every', 'per', 'each time' and so on.

In the beginning, it is good practice for students to rewrite the information of a problem in ratio language.

Help the students to rephrase the problems.

Example: At an ice-cream shop, the ratio of chocolate cones sold to vanilla cones sold was 4:3. If the shopkeeper sold 84 ice-cream cones in a day, how many chocolate cones did he sell?

For every 4 chocolate cones sold, there were 3 vanilla cones sold.

This makes a group of 7 objects.

How many groups of 7 are there in 84? 12 such groups.

Therefore, the number of chocolate cones will be 12 times 4, i.e., 48.

Writing Mathematics

Students should be encouraged to write out their reasoning for a solution.

Example: One-fourth of the class is absent today. 30 students are present today. How many students are in that class?

"If one-fourth of the class is absent, then three-fourth of the class is present. Three-fourth stands for 30. Therefore, one-fourth stands for 10. So the total number of students is 40.

ACTIVITY 13

Objective: Aids to solving problems (table making)

In a library, the number of non-fiction books is one-fourth the number of fiction books.

How many fiction books would be there if there are 50 non-fiction books?

How many non-fiction books would be there if there are 120 fiction books?

Non-fiction	20		50	100
fiction	80	240		120

Figure 18

What approaches would you use for these problems?

Aunty Elizabeth decides to give her nephew and niece Rs.1500, to be shared between them. Because the niece is the older of the two, she decides that she must divide the money between the niece and the nephew in the ratio 5:3. How much does each child get?

For every 4 kilometres that Jaleel jogged, Kian jogged 3 kilometres. If Jaleel jogged 1 km, how far would Kian have jogged?

A student finished 8 homework problems in class. If the ratio of problems finished to the problems still left was 4: 1, how many homework problems did she have in total?

Occasionally, pose an **open-ended** question.

Think!

The ratio of two even numbers is 3:7. What are they?

It takes 3 minutes to boil 1 egg. How many minutes does it take to boil 2 eggs?

GAME 2

Objective: Interpret given ratios and construct a feasible design.

Materials: Square grids of 5×5 size

25 tiles of 4 different colours

The teacher constructs a design and shares it with the students using ratios.

Examples.

The ratio of green to red is 5:4.

The ratio of green to blue is 10:1.

The ratio of red to yellow is 4:3.

The pattern has two lines of symmetry.

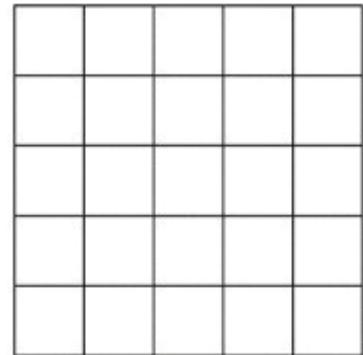


Figure 19

Are the students able to construct a design that satisfies all these conditions?

After playing the game a few times, students can form teams and create challenges for one another.

Creating a pattern that lends itself to good clues is an educative and challenging experience in itself, particularly if the solution is unique.

Extension: We can also use grids of size 4 × 4.

ACTIVITY 14

Objective: Studying ratio applications (maps from an atlas)

Materials: Atlas or map charts

Teacher can help students read a map and observe the way scale is used in the map, the way the scale is denoted on the map and the need for accuracy in the representation.

Students can apply their knowledge to estimate the distances between pairs of places and the lengths of rivers. They can also compare the lengths of the coastlines of different states of India. And so on.

ACTIVITY 15

Objective: Meeting real life ratio challenges (scale drawings)

Materials: Measuring tapes and scales

Each time a teacher draws on the blackboard or writes on the blackboard, she is implicitly using a ratio. A 20 cm long pencil when drawn on the board may become 30 cm long.

That is an example of **enlargement** or multiplication, by a scale factor of $1\frac{1}{2}$.

Maps and scale drawings, in contrast, depict reduction in scale.

Divide the students into groups and ask each group to make a scale drawing of a classroom, library or the playground.

Discuss what would be a sensible scale to use.

In the process of making a scale drawing, they will encounter fractional ratios and at times need to do rounding off.

ACTIVITY 16

Objective: Highlight the need for a common unit or conversion in a ratio.

Example. Madhu says, "I will take 10 days to complete my project." Vishal says, "I need 2 weeks." Ashok described the ratio of time needed by Madhu and Vishal as 10:2.

What do you think?

Let the students discuss the implication of a 10:2 ratio in this context.

If it is wrong, why is it wrong? What is the right way of expressing it?

ACTIVITY 17

Objective: See rate as the ratio of two quantities having different units (rate is a special kind of ratio).

Materials: Price list, purchase bills, electricity bills, travel time

What is a rate? A rate compares different kinds of measures, such as rupees per kilogram (cost per unit bought), kilometres per hour (speed of a vehicle), rupees per day (wages for a worker), heartbeats per minute (state of health). So many aspects of our life involve some kind of rate.

Plotting of rate in a table as shown demonstrates the multiplicative relationship very clearly.

We can also show this data as a graph.

Time (h)	1	2	3	4	5
Distance (mi)	60	120	180	240	300

Figure 20

ACTIVITY 18

Objective: Meeting real life ratio challenges (recipes)

Materials: Actual recipes!

No discussion on ratios can end without discussing the use of ratios in recipes, on which our very survival rests!

Strong coffee? Weak coffee?

Teacher can distribute four recipes for making coffee. The students have to arrange them according to the strength of the decoction.

If coffee is not welcome, we can substitute it with orange juice!

Ordering four recipes requires comparison of four different strengths.

However, we may also make comparisons within a recipe.

Here is one such example.

Comparison within one recipe between the quantity of juice and water.

To make a jug of orange juice, I use two glasses of orange juice and five glasses of water.

What happens to the strength of the mixture if I add an extra glass of water?

What happens if I add an extra glass of orange juice?

What happens if I add one of each, two of each, etc.?



Figure 21

ACTIVITY 19

Objective: Solving proportion problems as algebraic equations.

Materials: Word problems

In a bag of red and green sweets, the ratio of red sweets to green sweets is 3:4. If the bag contains 120 green sweets, how many red sweets are there?

For every 3 red sweets there are 4 green sweets.

Number of red sweets = y

Number of green sweets = 120

Hence, 3:4:: y :120

$$3 \times 120 = 4 \times y$$

$$4y = 360$$

$$y = 90$$

Project Ideas

- Study of shadows
- Sports data
- Human body (comparative study)
- Currency exchange rates
- Microscope enlargements

History: The ancient Greeks originally posed this problem:

‘How can a line be divided into two parts so that the ratio of the larger part to the smaller is the same as the ratio of the whole line to the larger part?’

Historically, what kind of ratio problems did people study? What challenges would they have faced which led to the development of topics like trigonometry? It would be of interest to the students and teachers to study this question.

To close, a few questions...

- How does one support and scaffold the students’ learning?
- Has one exposed the students to various strategies so that we have been able to achieve the intended goals or learning outcomes?
- By studying incorrect solutions, can one understand the misconceptions that the students hold and so address them?



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