Issue 13 | July 2022
Triannual | Bangalore

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

RNI No.: KARENG/2018/76619

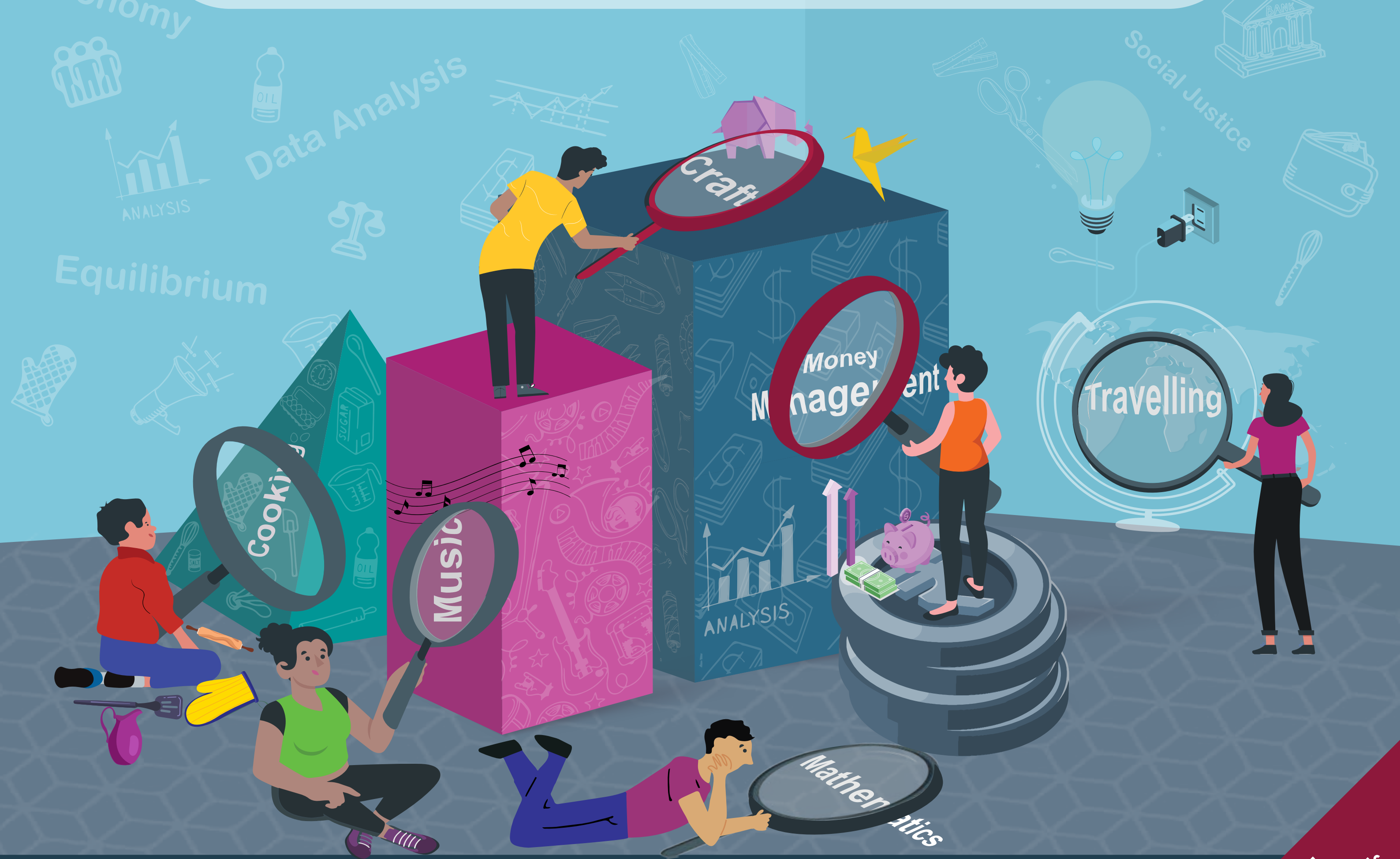
Azim Premji University

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

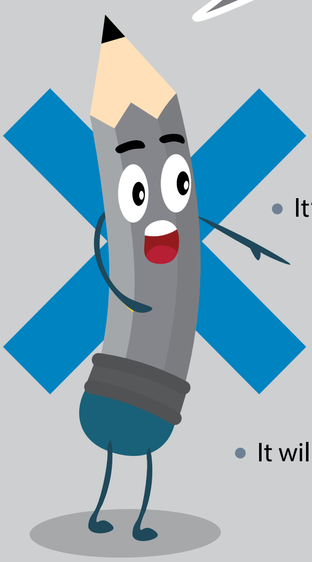
The Lens of Computational Thinking



PULLOUT
Spatial Thinking with 3-D Objects

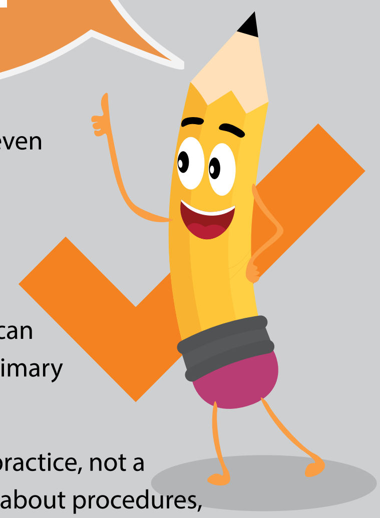
“Computational Thinking is an analytical thinking skill that draws on concepts from computer science but is a fundamental skill useful for everyone”(Wing [2006]).

Don’t dumb it down! These notions are not all there is to computational thinking



- It’s about being able to use technological devices
- It’s all about writing code
- It should be introduced only in middle school
- You have to be a computer science teacher to teach it
- It will just add to the curricular load

Celebrate it! computational thinking has potential energy!



- It can be practised even without a device
- Some students may use it for coding
- It is an approach that can be adopted even in primary school activities
- Computational thinking is a practice, not a subject. It is about reasoning about procedures, about organising data, about how one works, about pattern recognition, finding alternate methods, about thinking.

Tick the correct option(s) below each question

1. Who have you seen or met today who has broken down a complex problem into simpler problems?
   - ☐ Google Maps
   - ☐ The school canteen staff
   - ☐ The music teacher teaching you a new song
2. Who do you think does their work better because they recognise a pattern?
   - ☐ A craftsperson
   - ☐ A musician
   - ☐ A student
3. Who uses algorithms?
   - ☐ A computer programmer
   - ☐ A cook
   - ☐ A travel guide
4. Learning to practise computational thinking will help a student with
   - ☐ Coding and computer skills
   - ☐ Study skills
   - ☐ Troubleshooting

From the Editor's Desk . . .

In the Opening Bracket, Shailesh Shirali rightly points out the importance of teaching students to think- rationally, critically, deeply. The nature of mathematics makes it a powerful vehicle to develop logical reasoning and problem solving in students and the subject has been taught through the ages (ostensibly) toward this end. With NEP 2020 bringing in the aspect of Computational Thinking and aligning it with the mathematics curriculum, it is tempting to think of this as a deep dive into computer science, where coding would be taught just as any other language at the middle school level. It would be a total waste to limit computational thinking in this manner. When we viewed several articles in the July 2022 issue through the lens of computational thinking, we discovered that the content of the articles was no different from that in our other issues. However, a pedagogical stress on the potential to develop computational thinking in students through mathematical investigations and problem solving would enable them to develop a tool-kit and an approach that will stand them in good stead in several situations- from analysing data in the form of inflammatory WhatsApp forwards to reasoning out false claims and, most importantly, verifying for themselves rather than blindly believing others.

Hands on thinking- that's what Amitabh Virmani's article on Making the Great Icosahedron will lead you to try. Mahit Warhadpande and V G Tikekar follow up with Part 2 of The Minimal Instruments of Geometry and Spoof Numbers and Spoof Solutions, respectively. The former explores the measurement and drawing of shapes in Vedic times and the latter describes how changing a few restrictions can open up the solution set considerably. Working with constraints is nothing new, but people show amazing resilience in working around and through them!

ClassRoom begins with a very comprehensive article by R Ramanujam on what Computational Thinking is. You will find answers to many of your questions here, along with useful links to further reading on the subject. A Ramachandran's Investigative Questions has some simple investigations with plenty of scope for teachers to develop more challenges for students along these lines and in the process, teaching them how to decompose a problem, do a systematic search, recognise patterns and verify if solutions are correct. Both the articles on Integers (by Math Space) and the Mean (by the Mathematics Co-development group) come from the space of not accepting algorithms or formulas blindly but understanding and owning them. Encouragingly, Student Corner carries an account by two Class 7 students who did just that - investigated the very stale (but very useful) difference of squares formula and gave their own twist to it. You will find many problems and investigations in both Problem Corner and in Student Corner. And do check out an example of computational thinking - data compression in a knitting pattern (?!!) - in Put Your Thinking Cap On!

TechSpace describes another investigation – The Spaghetti Problem which is linked to very simple mathematics and yet ensures that there is more than enough food for thought. Jonaki Ghosh ends with a beautiful description of mathematical and computational thinking and how they are entwined and yet differ.

Parvin Sinclair's review of Rethinking Mathematics will certainly make you want to get your hands on the book and build your lesson plans around the activities in it - marrying social concerns with mathematical analysis and problem solving. Our manipulative review by Math Space is of the geoboard and this time, the review is accompanied by an activity sheet, to help you understand the value and potential of this popular classroom resource.

We end with the PullOut – which has activities for students to learn about the 3-D world they live in. Talk about math being real! Happy reading! And do revert with your comments on AtRiA.editor@apu.edu.in

Sneha Titus
Associate Editor

The Opening Bracket . . .
Learning how to think

Looking around the world, one sees that things are in an absolute mess. Climate change is becoming more fearsome by the day, and violence and divisions are tearing humanity apart. Wars are being fought at this moment that are going to result in widespread suffering and starvation. We seem to be almost incapable of responding to these problems in a sane manner. What are we to do in such a situation? Rather than seek the solution through negotiation and treaties and agreements, must we not approach the problem through education?

Education is desperately in need of major reform, but where do we start? In NEP 2020 we read: "It is becoming critical that children … learn how to learn. Education must move towards learning about how to think critically" (page 4) and also: "Education must develop not only capacities such as critical thinking and problem solving but also social, ethical, and emotional capacities" (page 5). How would one go about setting up such an education system?

We shall express it this way: *Education must emphasise learning how to think and not what to think.* This has been said by numerous educators. It is repeated so often that it has lost meaning. Who can disagree with the statement that we must emphasise learning how to think? But are we capable of doing so? Are we capable of teaching children how to think?

There is a lot of talk about computational thinking these days. It is obviously important that we learn this skill for ourselves and ensure that children learn it too. This is not merely because of the needs of any particular nation or organization, but it is important to be able to think clearly and objectively and in sequence when we are solving the important problems of life: problems of food distribution, problems of rational budget allocation, problems of transport, problems of supply chains, problems of banking and interest rates, problems of corruption, and so many other problems.

But surely, learning how to think goes far beyond this. We need to understand for ourselves why we are capable of hatred and violence, why we love power, why we love to compete, why we consume so much, why we need to be entertained, why we deceive ourselves so easily; and so many other such matters. A statement attributed to Albert Einstein is this: "We cannot solve our problems with the same thinking we used when we created them." Do we understand this statement truly? To do so requires great humility; we need to take responsibility for the fact that we are the creators of the problems we see around us, and we need to feel this deeply, in our hearts, and not just our minds. Perhaps then we can begin finding out what it means to think for oneself. And most importantly, we need to learn to think without self-interest. But to do this we need to understand ourselves, we who are the creators of problems. This is what we must, profoundly and with utmost humility, endeavour to learn and to teach.

Can we not create such schools, such centres of learning?

Can we not all join in this endeavour which is so vitally important for the well-being of the Earth?

Shailesh Shirali

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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At Right Angles
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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Problem Corner

Student Corner

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry

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Continue . . .

software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

Features

Making the Great Icosahedron

AMITABH VIRMANI

The great geometer H. S. M. Coxeter wrote in the preface of his book [1], "The chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense." Indeed, although the mathematics involved in discovering and classifying various polyhedra [1] may not appeal to everyone, the beauty and symmetry of these objects [2] appeal to young and old alike.

A good classroom activity across the world is to make models of the five Platonic solids. The history of these objects goes back to the ancient Greeks. Famously, Euclid's *Elements* ends by showing that the only solids with congruent regular convex polygons as faces with all vertices of the same type are the five Platonic solids (Wikipedia [6]; see Figure 1 for the definition of a regular polyhedron).

Note that in the previous paragraph, I used the term regular convex polygons, not regular polygons. In Euclidean geometry, a regular polygon is a polygon that has all angles equal and has all sides of the same length. Regular polygons may be either convex or star[1]. A natural question is then, what are the analogs of the Platonic solids if we use star polygons? Kepler, around 1619, noticed that twelve pentagrams can join in pairs along their sides and meet in fives at their vertices to form a solid. See Figure 2. This is a regular solid: all faces are regular polygons (pentagrams) with the same number of faces (five) meeting at each vertex. But it is not *convex*. (A polyhedron is said to be convex if all its diagonals are inside or on its surface.)

Kepler also noticed that the regular pentagrams can join in another way. They can meet at their vertices in threes instead of fives, and they enclose a different solid. This solid is also regular. It is also not

1 See the glossary for an example of a star polygon.

Keywords: Platonic solids, Kepler-Poinsot solids, great icosahedron, hands-on learning, paper models

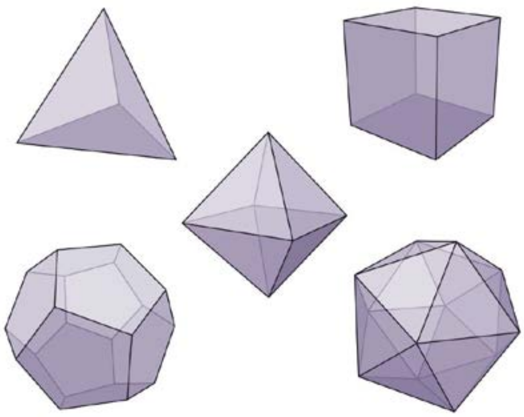

Figure 1. The five Platonic solids. Image from Wikipedia. A Platonic solid is a convex regular polyhedron. Its faces are congruent, convex regular polygons, and the same number of faces meet at each vertex. There are only five such polyhedra.

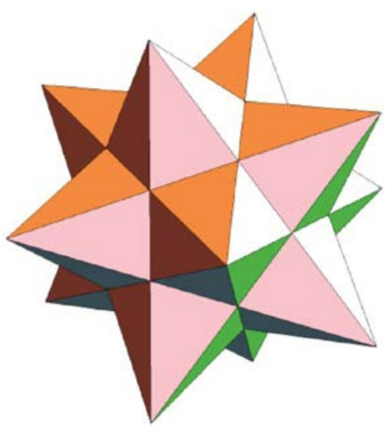

Figure 2. Kepler, around 1619, discovered this solid. It is starlike. Image from Wolfram Demonstrations Project.

convex. The two Kepler solids are called the small stellated dodecahedron and the great stellated dodecahedron [2]. See Figure 3. The reason for the names will become clear shortly.

One can say that Kepler discovered his two new solids by discarding the ancient Greek concern for convexity. Are there more? It turns out there are two more nonconvex regular solids. These are called Poinsot solids, named after Louis Poinsot, who discovered them in 1809. One of them is a version of the dodecahedron, called the great dodecahedron. The faces are simply twelve pentagons, but the pentagons now intersect

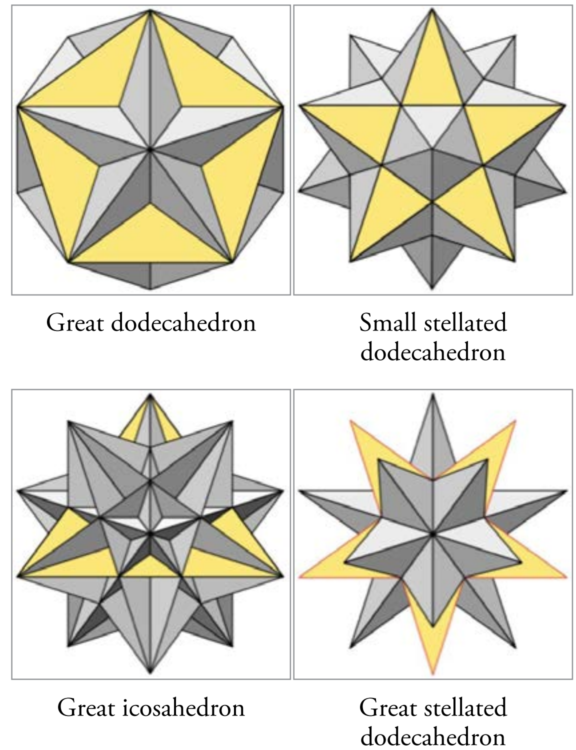

Figure 3. The four Kepler-Poinsot solids. Image from Wikipedia.

each other. The second one is a version of the icosahedron, called the great icosahedron. It is made of twenty intersecting equilateral triangles. The triangles meet along edges at twelve corners as in the icosahedron. See Figure 3.

The Kepler-Poinsot and Platonic solids are members of a bigger class called uniform polyhedra. It is a common mathematical hobby to make models of uniform polyhedra. It is also one of my hobbies. A uniform polyhedron has regular polygons as faces, and all its vertices are equivalent. The faces and vertices need not be convex, as many uniform polyhedra are non-convex, sometimes called star polyhedra because of their star-like appearance. A uniform polyhedron may be regular if all its faces and edges are alike, quasi-regular if all its edges but not faces are alike, or semi-regular if neither edges nor faces are alike. If we do not count prisms and anti-prisms[2], then there are exactly 75 uniform polyhedra [5].

2 See glossary for definitions of a prism and an antiprism.

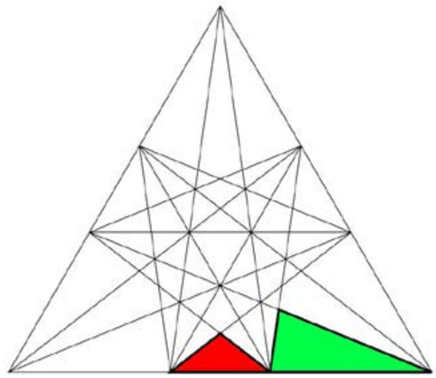

Figure 4. The large equilateral triangle is a facial plane of the great icosahedron. On each side of the triangle we locate two points dividing the sides of the triangle in the golden ratio: $\tau : 1 : \tau$ where $\tau = \frac{\sqrt{5}+1}{2}$. The resulting red and green triangles combined in a fan-shaped pattern as shown in Figure 5 give the basic module for the model.

Father Magnus J. Wenninger (1919-2017) devoted much of his working life to making polyhedra models. The story goes that after he made 65 out of the 75 uniform polyhedra to display in his classroom, he contacted Cambridge University Press to see if there was any interest in a book on polyhedra models. The publishers indicated an interest only if he built all 75.

Wenninger did complete the models. To make the last 10 models, he needed the help of a computer. The difficulty lies in the exact measurements for lengths of the edges and shapes of the faces. This was the first time that all of the uniform polyhedra had been made as paper models. This project took nearly ten years, and the book [4], Polyhedron Models, was published by the Cambridge University Press in 1971.

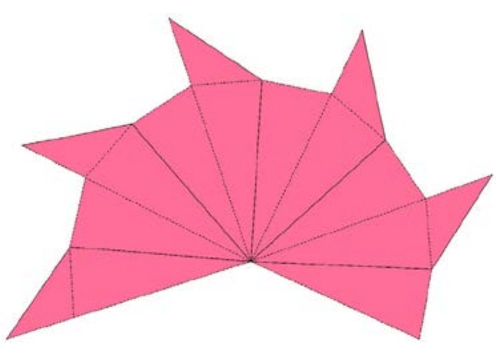

Figure 5. We glue (or put together using a drawing software) 10 of the green triangles and 5 of the red triangles of Figure 4 in a fan-shaped net.

Since then, many enthusiasts have made these models. The Science Museum in London, for example, has a display of all 75 uniform polyhedra models.

During the March 2021 lockdown (working from home), I wanted to make the four Kepler-Poinsot polyhedra. I mentioned one way of thinking about them above. There are several other ways. They can also be obtained by extending the faces of the dodecahedron and icosahedron, a process known as *stellation*. The two Kepler solids and the great dodecahedron can be obtained by stellating the dodecahedron. For this reason, the two Kepler solids are named the small stellated dodecahedron and the great stellated dodecahedron. A concise explanation on stellation can be found in [5] and more details in [1, 2, 4].

All four of the Kepler-Poinsot polyhedra are described inWenninger's book. Making the three stellations of dodecahedron was not difficult. Wenninger's book gives the precise angles for the triangles to be used to make nets: two-dimensional drawings that can be folded into three-dimensional pieces. Various pieces were to be glued together, and we got the models.

The fourth model — the great icosahedron — was not so simple. Wenninger's book tells us to use sketch paper and copy the net from the book. This was unsatisfactory. If I make 120 + 60 triangles in this way, and suppose the angles in the book are slightly off, say, due to the scale of the printing, then the model will not come together. I took a break.

Figure 6. A vertex part for the great icosahedron.

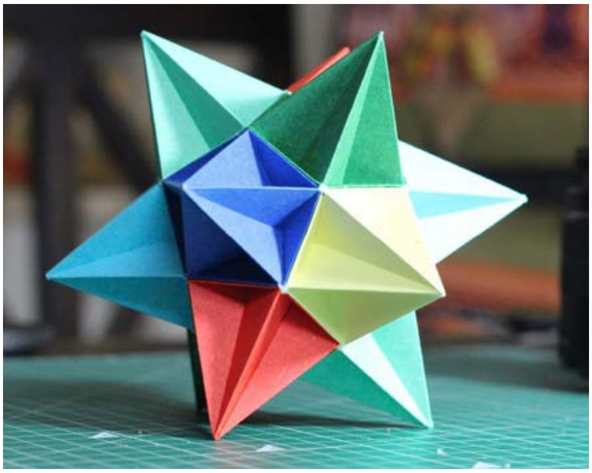

Figure 7. A paper model of the great icosahedron.

Then one day, flipping through the pages, I re-read the preface to the 1978 reprint of the book. It said, "[...] for best results very careful workmanship still demands that you make your own full-scale drawings of all facial planes from which patterns or nets are derived." This is precisely what I did. To make a model of the great icosahedron I began by drawing (in a computer software) a large equilateral triangle: one of the faces of the great icosahedron. On each side of the triangle we locate two points dividing the sides of the triangles (in the computer software) in the golden ratio: $\tau : 1 : \tau$ where $\tau = \frac{\sqrt{5}+1}{2}$. With simple calculations we get the coordinates of the various triangles obtained by connecting those points. For the red triangle of Figure 4 a natural choice (leftmost vertex in the Figure being the origin) turns out to be

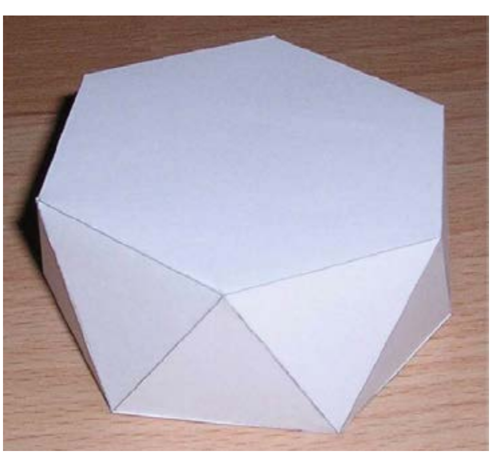

Figure 8. A paper model of a hexagonal antiprism.

$$\left\{\left(\frac{1}{2}\left(1+\sqrt{5}\right),0\right),\left(\frac{1}{2}\left(3+\sqrt{5}\right),0\right),\left(\frac{1}{2}\left(2+\sqrt{5}\right),\frac{\sqrt{3}\left(3+\sqrt{5}\right)}{10+6\sqrt{5}}\right)\right\}, \quad (1)$$

and for the green triangle it turns out to be

$$\left\{\left(\frac{1}{2}\left(3+\sqrt{5}\right),0\right),\left(\left(2+\sqrt{5}\right),0\right),\left(\frac{5}{4}+\frac{13}{4\sqrt{5}},\frac{\sqrt{3}\left(7+3\sqrt{5}\right)}{20+8\sqrt{5}}\right)\right\}. \quad (2)$$

We next glue 10 of the green triangles and 5 of the red triangles of figure 4 in a fan-shaped net shown in figure 5.

Next, we must fold the net of figure 5 in an accordion fashion — up and down — up and down. This gives us a vertex part for the great icosahedron, figure 6. Making 12 such vertices and gluing them in a dodecahedron form gives us the final model, figure 7.

It was a lot of fun to make this model and the other Kepler-Poinsot polyhedra models. I explained the construction to school students in our community's summer camp. It was wonderful to see the kids' eyes light up as they understood what was going on.

This model is a delight to hold.

The short article by Wenninger [5] is highly recommended for a quick tour of the world of polyhedra. For a visual account the book by Alan Holden [3] is highly recommended.

Looking at Figure 7, can you make out twenty intersecting equilateral triangles?

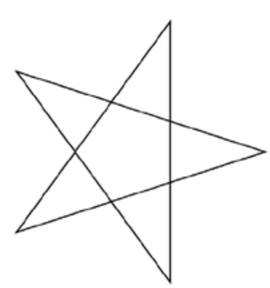

Figure 9. A regular star pentagon has five corner vertices and intersecting edges.

References

[1] H. S. M. Coxeter, *Regular Polytopes*, Dover Publications, 1973.

[2] Peter R. Cromwell, *Polyhedra*, Cambridge University Press, 1999.

[3] Alan Holden, *Shapes, Space and Symmetry,* Dover Publications, April 1992.

[4] Magnus Wenninger, *Polyhedron Models,* Cambridge University Press, 1971.

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[6] Wikipedia, "Platonic solid". https://en.wikipedia.org/wiki/Platonic_solid

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Glossary

antiprism: An antiprism is a polyhedron whose sides are equilateral triangles, capped at top and bottom by a regular *n*-sided polygon. See Figure 8 for an example.

facial plane: A face (or facial plane) is a flat surface that forms part of the boundary of a solid object. A three-dimensional solid bounded by faces is a polyhedron.

golden ratio: Two real numbers a > b > 0 are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities, i.e.,

$$\tau = \frac{a}{b} = \frac{a+b}{a}.$$

The Greek letter τ (tau) represents the golden ratio. It is an irrational number, a solution to the quadratic equation $x^2 - x - 1 = 0$, with value $\tau = \frac{1+\sqrt{5}}{2}$.

prism: A prism is a polyhedron whose sides are squares, capped at top and bottom by a regular *n*-sided polygon.

star polygons: A star polygon is a type of non-convex polygon. See Figure 9 for an example.

stellation: Stellation is the process of extending a polygon in two dimensions or a polyhedron in three dimensions. In three dimensions, starting with a polyhedron, the process extends its faces in a symmetrical way until they meet each other again to form the closed boundary of a new polyhedron.

uniform polyhedra: A uniform polyhedron has regular polygons as faces and is all its vertices are the same.

Features

The Minimal Instruments of Geometry – II

MAHIT WARHADPANDE

In the first part of this article, we introduced three alternative geometrical toolkits: (a) the straight edge and collapsible compass of Euclid, (b) the ruler, compass and protractor of Birkhoff and Beatley, and (c) the rope of the *Shulbasutras*. We also discussed some rope based geometrical constructions. In this second part, let us compare how these toolkits fare against some historical geometrical construction problems. We also ponder the construction of the 'tools' themselves, for example, how might we establish whether a straight edge is indeed straight and so on.

1. Toolkit Capabilities

Let us examine briefly, four constructions of interest during ancient times: doubling the cube, angle trisection, constructing a general regular polygon and squaring the circle [1]. It was eventually proven that none of these could be accomplished by Euclid's toolkit. As Birkhoff and Beatley summarized [2 pp. 165-166]:

' … [the construction of] a desired length x, from its relation to given lengths, $a, b, c,$ etc., using only a straightedge and compasses… [is only possible] whenever the relationship of x to $a, b, c,$ etc., involves ultimately only addition, subtraction, multiplication, division, and the extraction of square roots…'

'In general, it is impossible to divide a given angle into a given number of equal parts by means of straightedge and compasses alone.'

Keywords: Euclid, plane geometry, instrument box, ruler, compass, protractor, rope, geometrical constructions, shulbasutra

In contrast, with the Birkhoff-Beatley toolkit, '… any construction involving the laying off of lengths and angles can be made with scale and protractor, and to any desired degree of accuracy' [2 p. 171]. The Birkhoff-Beatley toolkit can thus achieve all the four constructions under consideration.

Let us now study these four constructions in the context of the rope. It has been shown that the use of a markable straight edge and compass would make it possible to double a given cube and trisect a given angle [3, 4]. These constructions are thus possible with a rope. Angle trisection is also possible using the technique of finding rational multiples of a given angle with a rope as discussed in the first part of this article. This technique also enables the use of a rope to construct any regular polygon. As Figure 1 indicates, the angles that characterize a regular polygon are rational multiples of 360°.

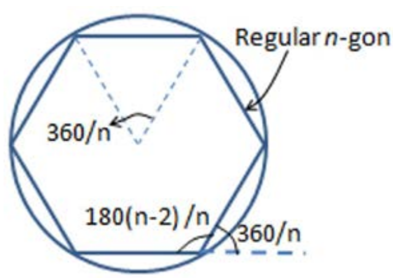

Figure 1. Various angles in a regular n-gon

The construction of a regular n-gon can proceed by constructing a side of the required length and then repeatedly making either the required interior angles $\left(= \frac{(n-2)}{2n} \times 360^\circ\right)$, or the exterior angles $(=(360^\circ)/n)$ and marking off the required length on the new side thus formed. Another possibility is to construct the $(360^\circ)/n$ angles adjacently at the centre of a circle. The arms of these angles will then intersect the circle in points representing the corners of the required regular n-gon.

Finally, the rope can also be used to square the circle. Given a circle of radius r, our problem is to construct a square of area πr^2, i.e., a square with side $r\sqrt{\pi}$. We begin by marking off the lengths of the circumference and diameter on a couple of ropes. Having obtained these two lengths, we can divide the circumference by the diameter using straight edge and compass techniques, to get the value of π (Figure 2).

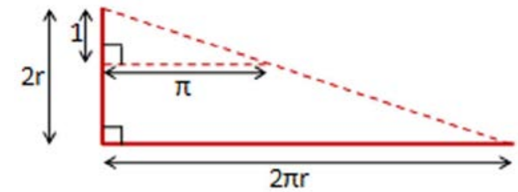

Figure 2. Calculating π from measured circumference and diameter of a circle

As mentioned earlier, given a length, we can find its square root, i.e., we can now get the length $\sqrt{\pi}$. Finally, we can multiply the two lengths r and $\sqrt{\pi}$ to construct the length $r\sqrt{\pi}$ (Figure 3) and subsequently a square with side of that length.

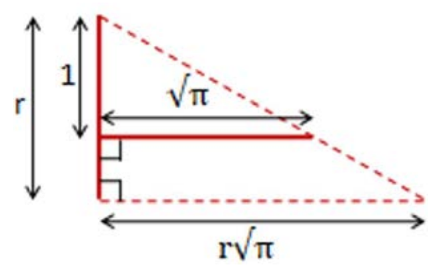

Figure 3. Multiplying the lengths r and √π

We note that though squaring a circle is thus possible using a rope, the *Shulbasutras* only contain approximate methods of squaring the circle [5, 6 pp. 143-149].

In theory, the Birkhoff-Beatley toolkit is still superior to the rope since it can construct a length and angle corresponding to any given number while the rope is limited to addition, subtraction, multiplication, division and square root constructions. However, given any number, we can always find a rational number as close to the given number as we want. Also, in practice, it would be impossible to mark the Birkhoff-Beatley tools with infinite resolution. This means that in practice, both the Euclid and the *Shulbasutra* toolkits should be able to yield geometrical constructions that are as accurate as the Birkhoff-Beatley toolkit constructions.

2. Tool Construction

Finally, we consider some of the practical challenges that inventors may have faced when first making the geometrical instruments discussed in this article.

2.1.1 Rope

The rope is arguably the easiest instrument to construct among those discussed here. In fact, ropes are naturally available in the form of vines, creepers, etc. In contrast, the toolkits of Euclid and Birkhoff-Beatley require a minimal degree of engineering expertise to precisely sculpt rigid objects into a desired shape. The finite thickness, limited flexibility and inconsistent elasticity of naturally occurring ropes pose challenges in accurate rope based constructions and make small scale geometrical constructions (e.g., which fit in a sheet of paper) almost impossible. The invention of cloth or thread making processes would resolve some of these difficulties by making possible ropes of near zero-thickness and high flexibility.

2.1.2 Compass

As we have seen, the rope itself can act as a compass. But even the contemporary compass shown in the first part of this article is fairly easy to make, for example, by using two sticks tied together at one end and the other end of each stick sharpened to a 'point'.

2.1.3 Straight Edge

How do we know whether a 'straight' edge is indeed straight? One way could be to line up a taut rope next to the edge to determine if it is straight. If it is not, it can be pared appropriately.

Another suggestion is to exploit the axiom: 'There is one and only one straight line between two given points' [2 p. 44]. In Figure 4, given two points P and Q, we use our 'straight' edge to draw a line joining them. We then turn our 'straight' edge around to swap its 'top' (T) and 'bottom' (B) ends and draw a 'straight' line again between P and Q. If the 'straight' edge is indeed straight, the two 'straight' lines thus drawn should exactly coincide.

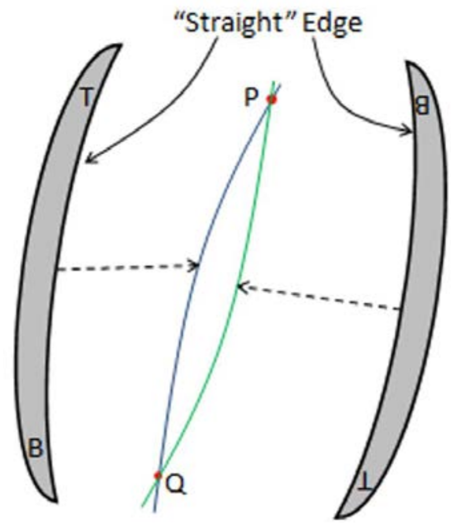

Figure 4. Test for straight edge

In Figure 4, the two lines do not coincide and we conclude that our 'straight' edge is in fact, not straight. We can also use the figure to tell where/how the 'straight' edge should be pared to make it straighter.

Does the ruler in your geometry box pass these tests of straightness?

2.1.4 Ruler

Given a straight edge, how does one mark it with infinite resolution? As mentioned earlier, this is impossible in practice. We can however, find a rational number as close as required to any given number and construct that length using the rope, or the straight edge and compass, which we now know how to make. We can use these constructions to mark the ruler.

One of the early small units of length was the width of a finger. Let us say this is standardised to 2 cm. Then, using the ruler of your geometry box as a straight edge along with the compass, how much further can you divide the 2cm length accurately? Can you construct a 1mm resolution ruler with these tools?

2.1.5 Protractor

The usual geometry box protractor is marked at every 1° angle from 0° to 180°. All these angles, being rational multiples of 360°, can be constructed with a rope.

Can we get a protractor marked at every 1° without using a rope? Euclid's *Elements* demonstrates the straight edge and compass construction of a regular 15–sided polygon. Further, angle bisection could be used to construct a regular $2n$-gon once a regular n-gon has been made. From the 15-sided polygon, we can thus arrive at 120, 240 and 480-sided regular polygons generating 3°, 1.5° and 0.75° angles (i.e., $(360°)/n$ angles) but not quite a 1° angle. In the 19th century it was proved that we cannot construct a 360-sided regular polygon using only a straight edge and compass [8]. However, using a markable straight edge, we can trisect the 3° angle to get a 1° angle.

Another option could be to use the 'table of chords' which had been developed to various degrees of accuracy by Hipparchus, Aryabhata, Ptolemy and possibly others [9, 10]. In modern terminology, this is equivalent to tables of the sine function. Ptolemy's table effectively listed the sine of angles from 0° to 90° in steps of 0.25°. In particular, we have sine(1°) ≈ 0.01745. We can use this to construct the triangle of Figure 5 and therefore a 1° angle. Note that this is still only an approximate construction since the value of sine(1°) can only be constructed to some rational approximation.

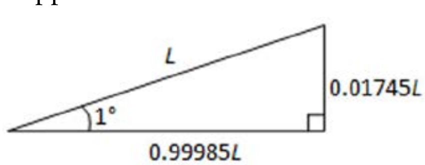

Figure 5. Constructing the required angle knowing its sine value

Ptolemy himself might have done this reverse construction using chords in a circle instead of right angled triangles in keeping with how he derived the sine tables.

Some practical difficulties remain with all the approaches discussed here. Can the length division of a rope be carried out to the extent required to generate a 1° angle with sufficient accuracy? Similarly, how accurate can the construction of a 3° angle with straight edge and compass be and further, how accurate can its trisection be? If we choose to use the table of chords and assume that a ruler with 1mm resolution is available to us, then in Figure 5, we will need to choose $L \sim 10$m to create a 1mm difference in length between the base and the hypotenuse so that they can be accurately constructed to get a 1° angle between them. Is it easy to make a 10m ruler with better than 1mm accuracy and resolution?

3. Conclusion

In theory, the Birkhoff-Beatley toolkit can do many constructions that the Euclid or *Shulbasutra* toolkits can't. However, in practice, both the rope and the Euclidean toolkit should be able to achieve a similar accuracy to that of the Birkhoff-Beatley toolkit for any given construction. In fact, to 'invent' the Birkhoff-Beatley toolkit (i.e., to accurately mark the ruler and the protractor), we will need to use constructions based on the rope or the Euclid toolkit which in turn are constructible using intuitive (axiomatic) ideas. Can you think of a better way?

We also saw that the rope has some added advantages over both the Euclid and the Birkhoff-Beatley toolkits, due to its flexibility.

We reiterate that the ideas and techniques discussed in this article need not be a historically accurate account of how these developments actually took place. Rather, our attempt is to focus on ideas that are mathematically correct. In that spirit, we ask our readers what tools they would like in their toolkit to enable the construction of an increasing variety of curves and shapes? How can these tools themselves be constructed?

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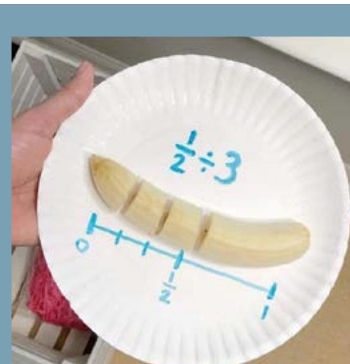
Sharing Food......Fraction-wise

Contributed by Sandeep Diwakar

From this picture, what is ½ divided by 3?

Send in your students' answers to AtRiA.editor@apu.edu.in

We welcome images for ½ divided by ⅓ too!



Spoof Numbers and Spoof Solutions - Part II

V G TIKEKAR

In this two-part article, we consider the curious notion of spoof numbers and spoof solutions which we get when we partially relax the conditions needed to define particular number families.

We recall, from Part I of the article [2], the definitions of *spoof number*, *spoof solution*, *Euler's sigma function* $S(n)$ and *perfect number*.

Definition 1 (Spoof number; spoof solution)**.** While constructing a number belonging to a particular family, if we relax some of the required properties or rules of formation but ensure that all the other properties of that family are satisfied, then such a number is called a *spoof number* of that family. Sometimes, such a number is also called a *quasi number* of that family. We similarly define the notion of *spoof solutions* by considering the spoof numbers obtained in the context of solutions of equations.

Definition 2 (Euler's sigma function)**.** If n is a positive integer, then $S(n)$ = sum of all the divisors of n.

Definition 3 (Perfect number)**.** A positive integer is called a *perfect number* if all its divisors add up to twice that integer, i.e., if $S(n) = 2n$.

The first few perfect numbers are 6, 28, 496 and 8128. We now explore the consequences of bringing these two notions together: spoof number and perfect number.

Keywords: Spoof number, spoof solution, Fermat number, perfect number, Mersenne number, triangular number, Euler

Odd perfect numbers

As of December 2018, 51 perfect numbers are known. Curiously, all of them are even numbers. To the great surprise of mathematicians and mathematics lovers, no one has been able to find any odd perfect numbers. At the same time, and to the still greater surprise of mathematicians and mathematics lovers, no one has been able to prove that there are no odd perfect numbers. All efforts in this direction, even by renowned mathematicians, have failed. Mathematicians have therefore reluctantly put forward the following:

Conjecture 1. *Odd perfect numbers do not exist.*

In passing, we note that the odd perfect number conjecture has been around for nearly 2000 years. The statement that all perfect numbers are even was first made around 100 CE by the Greek mathematician Nicomachus. So this is one of the oldest unsolved problems in mathematics.

Mathematicians have suggested that if odd perfect numbers are so difficult to find, then why not embark on looking for spoof odd perfect numbers? We shall do this later in the article.

Before that, we enumerate certain properties that odd perfect numbers must satisfy, if they exist. They have been proved over the centuries by various mathematicians.

Some properties of odd perfect numbers (OPNs), if they exist

(1) An OPN must have the form $p^{4a+1}m^2$ where p is a prime number of the type 1 (mod 4) and m is a natural number. This is Euler's characterization of an OPN. The prime p is called Euler's prime of the OPN.

(2) An OPN must have the form $12m + 1$ or $36m + 9$.

(3) An OPN must have at least 101 (not necessarily distinct) prime divisors.

(4) An OPN must have at least 7 distinct prime factors.

(5) The number of OPNs with k distinct prime factors is finite.

(6) An OPN with k distinct prime factors must be smaller than 2^{4^k}.

(7) No OPN can be divisible by 105.

(8) If an OPN is not divisible by $3, 5, 7$, it must have at least 27 prime factors.

(9) The largest prime factor of an OPN must exceed 10^7.

(10) An OPN must be greater than 10^{2000}.

But no OPN has ever been found. Mathematicians have therefore looked at the properties that such numbers (if they do exist) must have, in the hope of finding contradictions between some of these properties (which would immediately show that these numbers do not exist). Unfortunately, all such efforts have failed. In fact, in 1888 Sylvester wrote:

> *The existence of an odd perfect number — its escape, so to say, from the complex web of conditions which hem it in on all sides — would be little short of a miracle.*

But nobody has been successful till now in proving the non-existence of an OPN. Since neither existence nor non-existence of OPNs is established, we consider OPNs to be hypothetical numbers and study them by relaxing some of the rules and try instead to obtain spoof OPNs. This thought leads us to the next section.

Spoof OPNs

In view of what has been noted above, let us relax our expectations and look for positive integers that behave like OPNs. In other words, let us look for spoof OPNs.

Descartes's spoof OPN. In 1638, René Descartes discovered the first spoof odd perfect number ("Descartes's number"):

$$D = 198,585,576,189.$$

To explain why it is a spoof OPN, we express the number in factorized form as follows:

$$D = 3^2 \times 7^2 \times 11^2 \times 13^2 \times 22021^1.$$

Among the numbers on the right, the only one which is not prime is $22021 = 19^2 \times 61$. But let us assume (incorrectly, of course) that 22021 is prime (this being our relaxation referred to above), and evaluate the S-function of this number. So we have (since 3, 7, 11, 13 and 22021 are coprime):

$$\begin{aligned}
&S(198,585,576,189)\\
&= S(3^2 \times 7^2 \times 11^2 \times 13^1 \times 22021^1)\\
&= S(3^2) \times S(7^2) \times S(11^2) \times S(13^2) \times S(22021^1) \quad \text{(by Rule 1; see [1])}\\
&= (1+3+3^2)(1+7+7^2)(1+11+11^2)(1+13+13^2)(1+22021) \quad \text{(by Rule 2)}\\
&= 397,171,152,378 = 2D.
\end{aligned}$$

We see that the Descartes number D satisfies the defining property of a perfect number, but as 22021 is not prime, D is not a true perfect number but rather a spoof perfect number. And since D is odd, D is a spoof OPN.

Comment. If we replace 22021 by its prime factors $19^2 \times 61$ and write D as

$$D' = 3^2 \times 7^2 \times 11^2 \times 13^1 \times 19^2 \times 61,$$

then D' will *not* be a spoof OPN.

Comment. Descartes believed that the number D would some day be modified to produce a genuine OPN. Let us hope that such a day will eventually dawn and mankind's efforts in the study of spoof numbers will be rewarded.

Voight's spoof OPN. This number was given by Voight more than three and a half centuries after Descartes, in 1999. The number is

$$V = -22,017,975,903.$$

To verify that this is an OPN:

$$\begin{aligned}
&S(-22,017,975,903)\\
&= S(3^4 \times 7^2 \times 11^2 \times 19^2 \times (-127)^1)\\
&= S(3^4) \times S(7^2) \times S(11^2) \times S(19^2) \times S([-127]^1) \quad \text{(by Rule 1)}\\
&= (1+3+3^2+3^3+3^4)(1+7+7^2)(1+11+11^2)(1+19+19^2)[1+(-127)] \quad \text{(by Rule 2)}\\
&= 2 \times (-22017975903).
\end{aligned}$$

Here the factor -127 is a negative integer and hence not prime (though 127 is prime), but we take it to be prime. So V is a spoof OPN.

Comment. Instead of V, if we consider the number $W = -V = 22,017,975,903$, then W will *not* be a spoof OPN.

Using computers to generate spoof OPNs. Note that the two spoof OPNs studied above (D and V) are big numbers, in contrast with the spoof even perfect numbers (like 60, 90, 84, 840) that we studied in Part I. Note also that it took several centuries to obtain the second spoof OPN (V) after the first OPN (D) came to light. Note further that the properties expected to be followed by OPNs are so involved that it would be extremely difficult to obtain OPNs by mere hand-calculation. All these considerations led mathematicians to use parallel computers (running for a few years non-stop) to look for spoof OPNs. (This mode of research has been tried out for other problems; for example, the four-colour problem, which was proved in 1976 after extensive use of computation.) This resulted in more spoof OPNs being found (including D and V).

A group of researchers ('BYU Computational Number Theory Group' – BYU being 'Brigham Young University') followed a systematic, computation-based research for obtaining more spoof OPNs. We give below three examples of the many spoof OPNs that this team discovered.

Example 1. We relax the usual conditions and assume that 1 is prime. So

$$S(1) = S(1^1) = (1+1) \quad \text{(by Rule 2)},$$

i.e., $S(1) = 2 = 2 \times 1 = 2$. Thus 1 is a spoof OPN.

Example 2. Consider the number $101,411,037$:

$$101,411,037 = 3^2 \times 7^2 \times 7^2 \times 13^1 \times (-19)^2.$$

Here we relax the rules we normally use by (i) writing 19^2 as $(-19)^2$ and treating -19 as prime; (ii) considering the two 7^2s as separate entities rather than considering them together as 7^4; and (iii) assuming that 7 and 7 are coprime so that 3, 7, 7, 13 and -19 are coprime (as is required to use Rule 1). So:

$$\begin{aligned}
&S(101,411,037)\\
&= S(3^2 \times 7^2 \times 7^2 \times 13^1 \times (-19)^2)\\
&= S(3^2) \times S(7^2) \times S(7^2) \times S(13^1) \times S[(-19)^2] \quad \text{(by Rule 1)}\\
&= (1+3+3^2)(1+7+7^2)(1+7+7^2)(1+13)[1+(-19)+(-19)^2] \quad \text{(by Rule 2)}\\
&= 202,822,074 = 2 \times (101,411,037).
\end{aligned}$$

So $101,411,037$ is a spoof OPN.

Comment. Readers can verify that if we take $(-19)^2$ as 19^2 (treating 19 correctly as prime), and/or take 7^4 in place of $(7^2) \cdot (7^2)$, then $101,411,037$ will no longer be a spoof OPN.

Example 3. Consider the number 11, 025:

$$11,025 = 1 \times 9 \times 25 \times 49 = (1^2) \times (-3)^2 \times (-5)^2 \times (49^1).$$

Here we take $9 = (-3)^2$ rather than 3^2; and similarly, $25 = (-5)^2$, and then wrongly assume that $1, -3, -5, 49$ are primes. Further, we take 1^2 as an additional factor, which does not change the value of the given number but we benefit by getting an increased value of the S-function. So:

$$\begin{aligned} &S(11025) = S[(1^2) \times (-3)^2 \times (-5)^2 \times (49^1)] \\ &= (1 + 1 + 1^2)[1 + (-3) + (-3)^2][1 + (-5) + (-5)^2](1 + 49) \quad \text{(by applying Rules 1 and 2)} \\ &= 3 \times 7 \times 21 \times 50 = 22050 = 2 \times 11025. \end{aligned}$$

Hence 11025 is a spoof OPN.

Remark. As mentioned above, spoof OPNs assume importance because we do not yet have any actual odd perfect numbers to exhibit. Mathematicians have adopted this kind of approach in the study of other classes of numbers whose existence is in doubt.

Spoof solutions related to Fermat's Last Theorem (FLT)

The notion of spoof solutions arises naturally when we are dealing with Fermat's Last Theorem (better known as FLT). We have all heard of Fermat's hand-written statement (1637): "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers." In short:

$$\text{The equation } x^n + y^n = z^n \text{ has no solutions } (x, y, z, n) \text{ in positive integers for } n > 2. \tag{1}$$

In particular:

$$\text{The equation } x^{12} + y^{12} = z^{12} \text{ has no solutions } (x, y, z) \text{ in positive integers.} \tag{2}$$

Statement (1) has been proved in its most general form, so (2) is certainly true. That is, the equation (2) cannot be solved exactly over the positive integers. But in the context of our discussion about spoof numbers, the following question becomes meaningful:

Can we find spoof values of x, y, z so that the equation $x^{12} + y^{12} = z^{12}$ is satisfied with those values?

To proceed with the discussion, we must relax some of the normal rules. One way of proceeding is to move out of the set of positive integers, i.e., allow the use of non-integral numbers. This would mean that the above equation is satisfied up to a sufficient number of decimal places. For example, if we take

$$x = 3987, \quad y = 4365, \quad z = 4472,$$

then we find (using regular pocket hand-calculator) that

$$\left.\begin{aligned} x^{12} &= 3987^{12} \approx 1.613447461 \times 10^{43}, \\ y^{12} &= 4365^{12} \approx 4.784218174 \times 10^{43}, \\ z^{12} &= 4472^{12} \approx 6.397665635 \times 10^{43}, \end{aligned}\right\} \tag{3}$$

then we find that the equation $x^{12} + y^{12} = z^{12}$ is satisfied up to 9 decimal places.

Concluding remarks

For many problems exact solutions either do not exist or are known not to exist. For example, (i) as of today, the existence or non-existence of OPNs is not known, and (ii) FLT is known to be true. By introducing the notion of spoof numbers and spoof solutions, we handle these situations and obtain inexact solutions, but they are solutions that imitate the behaviour of exact solutions, and so behave somewhat like them.

In the second part of this two-part essay, we have discussed what are known as spoof odd perfect numbers that behave somewhat like perfect numbers, provided we relax some of the usual rules. We also discussed what can be called spoof solutions to the equation that occurs in Fermat's Last Theorem. It is interesting that we can make some progress in these problems by allowing the relaxation of some conditions.

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Computational Thinking: The New Buzz

R RAMANUJAM

I. The Buzz

As someone working on problems in computer science in the 1980's, I often used to be asked: *What languages do you work in?* I would typically answer, deliberately misunderstanding the question: *Mostly in English, sometimes also in Tamil.* In those days, working with computers meant writing programs in Fortran or Cobol or C, and that was what the questioner was asking about. My answer was about the programming language being irrelevant, the underlying concepts being more important. In fact, a more precise but entirely obscure answer would have been *first order logic*, and to a lesser extent, algebra, these being the languages for abstract reasoning about computation.

All this is to point out that the public perception of computing and computer science may not reflect the *thinking* that underlies these disciplinary domains. (This is quite natural; the public perception of methods used by electrical engineers or archaeologists is unlikely to be accurate either.) The increasing impact of computers on modern living is not necessarily a reason to expect such understanding either: people consult doctors all the time but do not expect to understand medical diagnosis and prescription.

It is when there is advocacy of such "disciplinary thought" in school education that it becomes important to examine such thought, and when it comes to school education, public perception and engagement is critical. Over the last decade there has been an increasingly vocal demand in many countries that Computational Thinking (CT) be a part of the school curriculum. In India, the National Education Policy 2020 (NEP) has not only given importance to CT, but has also coupled it with

Keywords: School mathematics, computational thinking, problem solving, decomposition, pattern recognition, abstraction, algorithm

mathematical thinking. While this has generated quite a buzz in the country, it is quite unclear whether there is a clear perception among the community of educators and teachers what CT is about, why it is being coupled with mathematical thinking at all, and whether promoting CT in schools is necessary or even desirable.

It is in this context that this article seeks to raise and address the following questions:

- Is the omnipresence of computers in modern life sufficient reason to promote (a) learning computing in school, and (b) learning CT in school?
- What does CT have to do with the role of digital technology in the classroom? Given the current massive inequality in the country in the digital space, isn't promoting CT and digital technology going to further deepen the social divide?
- What does CT have to do with learning computer programming, and when should children start learning coding?
- What does CT have to do with school mathematics? With an already crowded mathematics curriculum, are we increasing the burden on students and teachers with additional themes?
- Is the Indian education system equipped to take up CT in schools?

Indeed, this list of questions is not exhaustive, and more questions will arise at all levels of implementation of the NEP in the years ahead. However, a national policy needs to answer these fundamental questions, and provide clarity of direction especially to the teaching community.

II. CT in the NEP

In India, computer science as a discipline is taught principally in the universities, with some preparation for it at the higher secondary level. In the first 10 years of school, so-called *computer classes* have tended to be mainly on usage of computers, platforms and the Internet. Even this is principally in urban private schools; the massive government school system typically introduces computer usage only at the secondary or higher secondary school level.

All this is set to change, with the advent of the NEP, which advocates computational thinking and coding throughout the school years. The relevant item from the NEP is worth quoting:

> 4.25. It is recognized that mathematics and mathematical thinking will be very important for India's future and India's leadership role in the numerous upcoming fields and professions that will involve artificial intelligence, machine learning, and data science, etc. Thus, mathematics and computational thinking will be given increased emphasis throughout the school years, starting with the foundational stage, through a variety of innovative methods, including the regular use of puzzles and games that make mathematical thinking more enjoyable and engaging. Activities involving coding will be introduced in Middle Stage.

The coupling of mathematical and computational thinking is significant, since this suggests completely doing away with the current model of "computer classes" and moving over to teaching the science underlying computing, the emphasis being on thinking. This has important implication for mathematics education as well, shifting the focus from learning "operations", formulas and procedures (to solve equations, etc.) to learning a way of thinking.

On the other hand, the justification for doing so, according to the passage, stems from the importance of mathematics and computational thinking for "upcoming fields and professions such as artificial intelligence, machine learning, and data science, etc." Moreover, coding is advocated from middle school onwards. One can then be pardoned for thinking that the advocacy of CT is merely pandering to fashion, to what is currently considered important in computer science, and has to do with coding. (Indeed, some people have already started advocating the teaching of artificial intelligence and data science at high school!)

The reference to "India's leadership role" further raises doubt: school curricula should be decided by the aims of school education, and children should not be burdened with the responsibility of nationalist priorities. Again, the promotion of CT seems instrumental rather than essential as an educational objective.

This is the only mention of the phrase "computational thinking" in the NEP document. On the other hand, the NEP strongly advocates the use of digital technology in classrooms, devoting entire sections to it.

Meanwhile, the country suffered a huge disruption in school education due to the pandemic, when online education (accessible only to the elite) brought educational technology into prominence. This has led to further confusion, conflating the role of digital technology in the classroom with CT and teaching programming.

All this has led a large section of teachers into believing that introduction of CT in schools means the use of digital tools in classrooms and teaching programming from an early age, with perhaps "data science" and "artificial intelligence" as new subjects.

There is a real *danger* that this might indeed be what the NEP's promotion of CT might end up as, in its implementation: promotion of digital tools and an emphasis on teaching coding from an early age. That would be a sorry consequence indeed.

III. What is CT?

Seymour Papert ([4]), an American computer scientist who pioneered computer programming activities for children, coined the term 'Computational Thinking' in 1980. The phrase "computational thinking" appears in Mindstorms (p.182) but without elaboration. We find a detailed exposition of the idea in his 1996 paper [5] - "An exploration in the space of mathematics education". In it he offers the principle of *Object before Operation:* giving mathematical ideas a "thing-like representation" (using the micro-world of programs) helps in thinking about them.

Jeannette Wing, another American computer scientist, popularised the term in 2006 ([6], [7]). She labels it an "attitude and skill set" that everyone can learn and use. Simply phrased, CT is a process that enables us to take a complex problem, understand it and develop possible solutions in a way that a computer, a human, or both, can understand how to implement the solution. If "mathematical thinking" is thinking like a mathematician, for Wing, "computational thinking" is thinking like a computer scientist.

What is critical here is the last part: why should solutions be developed in such a way that a computer can implement them? What do we mean by computer here? Which computer? What capabilities do we assume the computer to have? Asked in another way, how is it different from a human being implementing the solution?

It should be noted here that we do not mean any specific computer with specific capabilities here, but an idealised one. The important property of mechanical computation is that it does not get bored by repetition or start making mistakes. For a calculator adding five 3 digit numbers is no different an effort as adding 500 numbers, some in crores and some in thousands.

These are the two core elements to consider here: data size, and repetitiveness of algorithms. The essential property of computations is *scaling*. Once we devise such procedures, broken up into sufficiently simple steps, they scale up as necessary.

The characteristics of CT, according to Wing, are: *decomposition, pattern recognition, abstraction,* and *algorithms*.

- Decomposition lets us break up the complex tasks into subtasks, then each subtask into sub-subtasks, etc., until each is simple enough to carry out directly.
- While doing so we often find that some tasks come up again and again, perhaps with slight differences; then we consider them as instances of the same task, perhaps with a parameter

capturing the change. These are the processes of pattern recognition and abstraction.

- Algorithms are stepwise procedures that sequence the subtasks in the 'best possible' manner. CT includes not only such a methodology for problem solving but also ways of comparing solutions and evaluating them.

As an everyday example, consider preparing dinner for 4 persons. There is a significant amount of planning involved. Unless we decide a menu, we cannot shop for the ingredients. However, some of the ingredients may not be available, and hence the menu may need to be changed. Once we have the ingredients and the necessary kitchen appliances, if we have two persons cooking, we decide which tasks are to be done in what order, hopefully without one having to wait unduly for the other. Some tasks have to necessarily come earlier than others.

The recipes need to be clear and have to be followed carefully. Safety has to be maintained all along, and some "runtime checks" need to be done for both safety and taste. If, at the end of all this, you think you have a reliable procedure for making dinner for 4, consider how it would scale for a dinner for 20, and one for 100 guests (in a wedding). Further, compare the procedure with another person's recipes for the same menu, or for different menus. The processes involved in all these illustrate CT.

This example serves to illustrate that computational thinking is relevant in everyday contexts. The notion of "computer" here is abstract and does not refer to any particular machine. Likewise, a recipe is not a program written in a programming language but serves a communicative purpose. Indeed this view of CT does not need electronic devices at all and we can speak of *computer science "unplugged"!* ([1])

Thus the central element of computational thinking is **reasoning about procedures**. It is less important to know many algorithms and procedures, or to write specific code, than to be able to design procedures and algorithms, to be able to reason about how they work, assess their performance, to explore alternative methods and compare them. It is less important to know definitions of data structures than to understand how data can be organised in multiple ways, how this affects procedures that use them and which organisation suits which requirement.

From this viewpoint, computation assumes focus rather than computing devices, and developing thinking that *underlies* computation becomes the educational goal. Such thinking helps the student understand data organisation, scaling, assessment of procedures and their comparison, iteration and modular abstraction, independent of whether solutions are sought by computer or implemented on computing platforms. An example of this would be knowledge of multiple procedures for integer multiplication or division, and an understanding of which is best used when.

When we ask a child to perform the addition $53 + 28 + 47$, it is quite alright to add from left to right (or from top to bottom, placing the numbers vertically, as children are used to). But it is computational intuition that suggests grouping as $(53 + 47) + 28$. This gives us the solution not only faster, but more conveniently, since we are used to the decimal system and multiples of 10 carry meaning for us. As "computers" this reformulation is easier on us. That commutativity of addition allows this is the mathematical understanding that underlies such computational thinking.

At the risk of belabouring the point, consider solving the equation: $2(x + 1) + 3(x + 1) = 10$. Again, there is no harm in expanding brackets, carrying like terms involving x to the left, other terms to the right, and then divide, as a standard technique. However, algebraic computation suggests to us that the equation can be rewritten as $5(x + 1) = 10$, giving the solution immediately. Once again, it is not about speed, but about facility: we are the ones computing here; so we consider different methods of computation and choose which one suits us best.

Consider a question such as: which grows faster, $2x^2 - 50$, or $x^2 + 100$? The awareness that as x increases, adding or subtracting a constant amount will not matter is essentially computational intuition, as also the fact that the function $2x^2$ always dominates x^2. The mathematical justification underlying such intuition can be provided by computing the derivatives and comparing them.

What is being stressed here is that computational thinking involves not only devising procedures and (re-)organizing data appropriately but also consideration of alternative procedures and choosing the "best" among them, articulated according to some criterion. Then **reasoning about procedures** assumes central importance.

While the foregoing may be articulated as the essential meaning of CT, the superficial meaning, relating to the use of computers, has relevance as well. These objectives of CT education relate to use of computing platforms, tools and devices, and giving the students not only mastery over their use, but also create in them a predisposition to identify and utilise educational contexts in which the use of computation can help, and employ computations accordingly. An example of this would be generation of data according to some distribution to examine a probabilistic proposition (such as in the Birthday Problem), or plotting the trajectory of a ball in accordance with Newton's laws. We are led to these objectives principally by "compulsions of the day": since such tools are prevalent and easy to use. Educators must concern themselves both with their "right" and safe use, and with use of such opportunities to enrich educational practices. If we are seen to lessen the emphasis on use of such digital tools in pedagogy, it is not because we consider them less important or irrelevant to CT. Set against the danger of CT being entirely reduced to digital tool use (which does not seem merely hypothetical right now), we have tended to stress on reasoning about procedures being the essential meaning of CT.

IV. Curricular components

What does this understanding of CT imply for school education? Teaching CT in school then would include the following components. These relate not only to CT education in itself but also enhanced educational opportunities provided by CT. (Admittedly, they are not fully independent components and admit some overlap.)

1. **Scaling:** Systematic listing and counting of relevant parameters and verifying that all have been counted is essential for the transition from additive reasoning to multiplicative reasoning. This also paves the way for functional variation, and use of symmetries for counting. Comprehending large magnitudes by scaling small ones is a good way of managing complexity.
2. **Iteration:** Looking for patterns, finding a mechanism for pattern generation and modification, and visualisation of new patterns are all not only pleasant processes but also provide a link between aesthetics and formal reasoning. Understanding the power of iterating simple rules creates a foundation, not only for computation, but also for understanding the dynamics of systems.
3. **Data organisation:** The term "data handling" is familiar in school education but ends up only as graphical depiction of data and computing numerical summaries. But data can be represented in multiple ways, and which one is to be chosen when depends on the use such data is to be put to. Moreover, storage and retrieval of data requires memory structures. Designing such data organisation is neatly coupled with understanding of scaling and iterative data access.
4. **Modelling:** Discrete modelling of problems from real-life situations is largely unfamiliar territory in schools. Discrete structures like lists, trees, maps, graphs, lattices and networks arise naturally and provide abstract problem spaces for computation. Working with concrete representations of such structures early on can help in creating mental models for later facility with such models for computational abstractions.

5. **Algorithms:** Starting from two-digit addition in arithmetic, school education provides a variety of procedures for students to learn – so much so that mathematics or science education often degenerates into a mere memorisation of pre-set procedures to be enacted on specific numerical data. *Following* algorithms with a view to understanding them is no doubt desirable, but *devising* procedures is at the heart of computational thinking. This requires a facility with procedures, reasoning about them, consideration of procedural alternatives and selection among them based on a clear rationale.
6. **Programming:** Concrete implementation of data organisation and algorithms on specified platforms to solve given problems is an essential skill. Coding, when accompanied by a feel for program structure, can be exhilarating while coding as translation of informal computing into a given formal language can be painful. Hence creating a good disposition for programming early on is essential for achieving eventual fluency.
7. **Devices:** Computers, smartphones and other devices provide platforms and tools for computation. Children need to learn purposive use of these tools, and develop mastery over them. At the same time, the challenges that such use may pose, due to physical, emotional and intellectual development of children need to be carefully considered. In a country of stark economic inequalities, access to such devices and platforms cannot be taken for granted either. Thus, the guiding principle in this regard has to be safe and nuanced usage based on need.
8. **Social connectivity:** CT provides a unique opportunity for viewing multiple social structures, their communicational infrastructure, identity formation, etc., and thus forges new links between mathematical formalism and society. As of now, even as students are engrossed in social media and lead parallel lives, their educational potential lies largely unaddressed. Importantly, the possibilities of student communities breaking language, regional and other barriers, working together on data creation and algorithm design need to be examined carefully.
9. **Simulation and visualisation:** This has perhaps been the oldest use of CT in school, learning to plot graphs of functions. However, once students understand the computational basis of visualisation and simulation tools, the tools can greatly expand their horizon of exploration across disciplines. (Consider students playing around with bonding structure of atoms in molecules.)

All these components would not carry equal weightage across the curriculum or across the stages, and it is the task of syllabus designers to spell out the weightage provided to each component at every stage.

V. Examples

Educational opportunities for CT already abound in the existing school curriculum, across the stages. In the primary and upper primary stages of schooling such opportunities are principally found in mathematics education, with a more expansive range across disciplines of study in the secondary stage.

We have already referred to reordering and regrouping techniques that often go under the rubric of "mental math." There are also many opportunities for reasoning about counting procedures, You are in a hall where a wedding is taking place with lots of people, anywhere between 100 and 150. How would you actually count how many there are? How would you know you have counted them all? How would you be able to verify whether your answer is correct?

For small children, counting out 20 seeds is sufficient to provide challenges. Counting silently is different from calling out, but why? We can also watch for bunching and grouping, providing for natural data organization. When we ask the child to move aside 15 of them, we can see if she needs to re-start from the beginning.

Consider the question: how many pairs of positive integers add up to 17? Surely there are multiple ways of listing such pairs, but reasoning involves employing some system of listing in order to be sure that all pairs are counted, and each pair counted only once. When a primary school child offers a 'quick' way to add 10, 100, etc., to a given number, and knows that this way is specific only to these numbers, he is employing CT in context. Finding multiple "reasonable" routes between two places on a map, or considering different arrangements of letters, or forming aesthetically pleasing patterns with beads of different colours, can be excellent exercises in CT as well.

There are many opportunities for data representation. Consider a village in Maharashtra where 61 families are Marathi speaking, 13 speak Kannada, 12 speak only Hindi, 8 are Tamilian, 5 are Gujarati and there is one lone Bengali family. One boring table will surely suffice. But suppose that we use one flag for each family, a colour representing a linguistic group. How would we depict this information? The flags can even be organized in a 10×10 grid, but a chaotic distribution of colours does not help. Once we group colours together in the grid, we suddenly get a great deal of visual information, not only about how numerous a group is, but also the relative size of groups in the village. The histogram, presented then, elucidates this structure further. The point here is not so much about using the histogram to give numerical answers to questions, but to consider alternative data representations and arrive at the one that is most felicitous for answering questions. This is at the heart of data structuring in computational thinking, and prepares students for wonderfully creative notions such as *codes* (and error correction) that can come later on.

Iteration, essential to CT, provides an excellent tool for explorations. Consider starting with a square. Join the mid-points. A new shape comes up, repeat the process. This simple recipe leads to beautiful figures. When the student realises that the procedure is abstract and can be applied to any polygonal figure as "input," she begins to have a taste for CT. We can then begin to see tessellations, *kolams* (or rangoli), and fractals as opportunities for creation of patterns by iterative procedure, reasoning about their variations and communicating such understanding in formal terms. In senior school, we can then describe the change of physical, biological and economic systems over time modelled by simple equations applied repetitively, and use these models to predict the long-term behaviour of such systems.

VI. Mathematics education and CT

One natural question that arises is whether computational thinking is actually different from mathematical thinking. This is really a question for foundational thought to be addressed by philosophers, One can narrow it down to the context of school education and assert that it helps pedagogically to meaningfully distinguish the two.

Before we explain this in detail, it is useful to consider university mathematics for a moment. Real analysis abounds with examples that distinguish mathematical thinking and CT. Bolzano's theorem asserts the existence of a root in an interval when a continuous function has values of opposite sign in that interval. The Newton - Raphson method of successive approximations provides a computational method for finding a root. The Brouwer fixed point theorem asserts that for any continuous function f mapping a compact convex set to itself there is a point x such that $f(x) = x$. Computing such a fixed point is a challenge and a general algorithm had to wait until recently to be formulated. Extracting the algorithmic (or constructive) content of mathematical statements and proofs is a greatly interesting challenge.

At primary school level, we consider that it is not especially useful to distinguish mathematics from CT, but it is relevant to highlight opportunities for CT within the mathematics classroom, as we did above. At secondary level, it does become useful to distinguish the two. For instance, consider solving a system of n linear equations (with integer coefficients) in n unknowns. We can learn an algorithm to do this, namely Gaussian elimination. Nonetheless it is required to develop some intuition into when the system given does not have a solution, or has more than one solution. Indeed, in the latter case, one can ask whether the system must have infinitely many solutions. A further question relates to rational numbers arising at intermediate steps. Should we retain them as rationals and employ rational arithmetic as we proceed further, or convert them to their decimal representations? Does this matter? What do we gain by using a matrix representation for the system of equations? Raising and answering such questions is essential for CT.

CT has relevance not only for mathematics education but also for science and other subjects in school. Giving prominence to data, understanding data qualitatively and quantitatively, and interpreting data are essential skills not only for science but also for geography, and though less appreciated this way, for history as well. The fine arts provide many creative opportunities for CT, and conversely CT can greatly enhance educational contexts in the graphic arts as well as in music and dance.

VII. Revisiting the questions

While we have discussed what CT means and how it can enrich school education, we have not answered the question of *why* we should do so. One thing is clear. Advocacy of online education and the use of digital technology in classrooms arise from premises very different from what we have been discussing here. Our reasons are different.

- Developing *critical thinking* is a central aim of education, and a critical outlook on algorithms is the need of the twenty-first century. Algorithms increasingly run our lives and developing a mature understanding of how data is created and processed by algorithms requires a foundational knowledge of how algorithms work. Mastery over these processes is best developed slowly, over the school years.
- Developing *autonomy* in the learner is again a central aim of education, and computing provides a powerful new addition to the learner's toolkit for understanding the world. Not only is this new tool versatile, it adds capabilities as yet unexplored in school.
- *Resource consciousness* is a crucial need for modern life, and while this is an ecological imperative, instilling such consciousness needs to be attempted in ecologically sensitive practices; formal thinking on such lines is to be nurtured as well. Computing science provides a new such opportunity by bringing in a sensitivity to scaling and complexity of resource utilisation.
- Education embodies the spirit of *modern democracy* in preparing the citizen for participation in social development and directing the path of development. In the contemporary world, this is impossible without the citizen gaining *democratic control over data*, all data that involves her, and all data that is a determinant of her welfare. Educating the citizen on the relationship between data and democracy is thus a curricular imperative.

Viewed thus, the aims of CT education at school are about utilising the tremendous new potential brought by computation for autonomy and empowerment, and at the same time developing a critical outlook on data and algorithms, and a sensitivity to resource use as practices scale up.

These are broad statements of aims. What should be learnt at which school age is best decided on the basis of research on psychology of children's learning, not by availability of technological tools. Indeed, digital technology can be seductive in its glamorous manifestation and we need to be wary of children becoming enslaved by devices. Such considerations again suggest that the relationship between CT and mathematics education as we have discussed provides more safe and meaningful opportunities for CT than a technology-based understanding of CT.

Lastly, whatever the NEP may advocate, and however it gets implemented, we need to ask whether we have the capability for introduction of CT in the education system at all levels. The teaching community, especially in mathematics, is alive to the possibilities and needs support by way of pedagogic resources. The experience of **CSpathshala**, a voluntary initiative of *ACM India*, providing a complete CT curriculum ([2]) for schools and reaching nearly a thousand schools in the country, offers a strong foundation from which many future initiatives can take off. We note here that the insights into CT presented in this article largely stem from the *cspathshala* experience.

The 2019 mathematics textbooks of Tamil Nadu State Board include an *information processing* track incorporating elements of a CT curriculum. The largely positive response from teachers to this initiative again offers hope.

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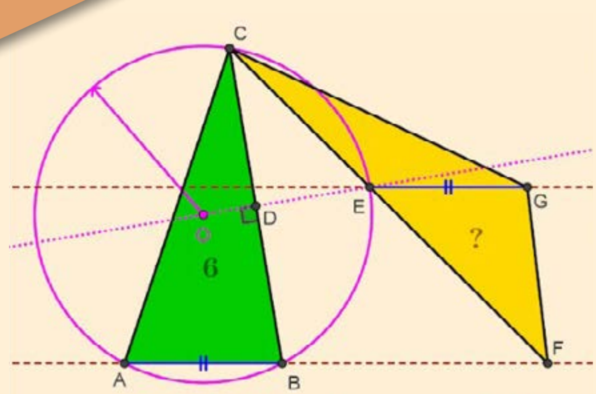
Triangle Area Puzzle

If the area of the green triangle is 6 sq. units, what is the area of the gold triangle?



Dashed brown lines are parallel lines.

(if you would like a hint..........please look at page 48)

@AmareshGS1
ಅಮರೇಶ್. ಜಿ. ಎಸ್., ಬಳ್ಳಾರಿ, ಭಾರತ.

ClassRoom

Investigative Questions for the Middle School

A. RAMACHANDRAN

How is an investigation different from a problem? A problem generally has a unique solution that can be reached by applying standard procedures. In the case of an investigation, one is not sure at the outset if there is a solution at all or if there are multiple solutions. We may face novel situations requiring new approaches.

Here are two such questions. They require only middle school level arithmetic and algebra, but one has to search for possible solutions in a systematic way. The second one is much more demanding than the first.

I. Can a 3-digit number and the two 3-digit numbers obtained from it by cyclic permutation of its digits, form an arithmetic progression? That is, can the three 3-digit numbers *abc, bca* and *cab* form an A.P.? We should leave out trivial solutions such as 111, 222, etc.

   A variation on this theme would be to allow one of the digits to be zero. In that case one of the numbers in the set of three would actually be a 2-digit number, but one could consider that as having three digits, with zero in the hundreds place.

II. Can a 3-digit number and the two 3-digit numbers obtained from it by cyclic permutation of its digits, form a geometric progression? That is, can the three 3-digit numbers *abc, bca* and *cab* form a G.P.?

If you are not familiar with the term 'cyclic permutation,' here is an explanation. The triangle in Figure 1 has its vertices marked A, B, C. You may read the letters A, B, C in clockwise order, starting with each letter in turn. This gives the arrangements ACB, CBA and BAC. These three arrangements are cyclic permutations of each other. If the same is done moving in anti-clockwise direction, we obtain the arrangements ABC, BCA and CAB. These are again cyclic

Keywords: Investigations, middle school, numbers, digits, decomposition, arrangements, patterns

permutations of each other. Together they account for the six permutations of the three letters.

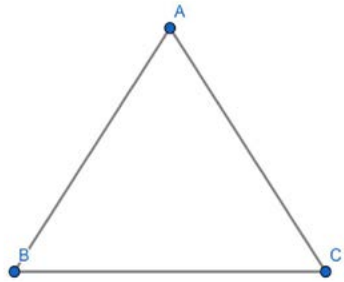


Figure 1

Solutions

b	a	c
1	1	1
2	2	2
3	3	3
4	4	4
	1	8
5	5	5
	2	9
	8	1
6	6	6
	9	2
7	7	7
8	8	8
9	9	9

Table 1

I. If *abc, bca* and *cab* form an A.P., then $2(100b + 10c + a) = 100a + 10b + c + 100c + 10a + b$, which simplifies to $7b = 4a + 3c$. We now need to find values of a, b, c from the set 1-9 that satisfy the above equation. We can assign values 1-9 to b in turn, and search for corresponding values of a and c. The possible values of a, b, c are given in Table 1.

We obtain the following solutions: (259,592,925), (148,481,814), (851,518,185), (962,629,296).

Allowing one of the digits to be zero, we have the following solutions: (037,370,703) and (740,407,074).

The common difference is ±333 in all cases.

Now, why is this so? In all cases the common difference is $(100b + 10c + a) - (100a + 10b + c) = (-99a + 90b + 9c)$. We also know that a, b, c must satisfy the relation $7b = 4a + 3c$. Substituting for b from this relation, the expression for the common difference becomes, after simplification, $\frac{333}{7}(c - a)$. Since this quantity has to be an integer, $(c - a)$ must be a multiple of 7. Since both are single digit numbers, the value of $(c - a)$ can be only ±7; the only solutions for (c, a) are (7, 0), (0, 7), (8, 1), (1, 8), (9, 2) and (2, 9), which is reflected in our six solutions.

The reader is invited to find alternative approaches to solving the above problem.

II. If *abc, bca* and *cab* form a G.P., then

$(100b + 10c + a)^2 = (100a + 10b + c)(100c + 10a + b)$, which, after multiplication, combining like terms, and cancelling common factors, reduces to $a^2 + 10ac = 10b^2 + bc$, or $10(b^2 - ac) = a^2 - bc$. As the RHS in the last equation is a multiple of 10, it suggests a way to check for possible solutions.

There are only two solutions: (432,324,243) and (864,648,486).

The numbers in the second set are all twice those in the first set. Naturally they have the same common ratio of 3/4.

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Integers: Extending the Number line with Coloured Counters

This article is intended for students – as hands-on play with integers by extending the number line and combining it with coloured counters. For this activity, rectangular dot sheets are better than square grid sheets since the dots can be joined by horizontal lines to form number lines and will not get mixed up with existing lines. We also recommend sketch pen/crayon/colour pencil of two contrasting colours to draw the counters.

MATH SPACE

Take some rectangular dot sheets, sketch-pens of two contrasting colours (for example, red and green), a ruler and a pencil, and you are good to go. Join the dots along the second line from the top. Then do the same for the 5th, 8th, 11th, … lines so that there are 2 lines of dots followed by a line connecting them. Pick a dot somewhere in the middle and label it 0. Then label the dots on its right successively 1, 2, 3, etc. Label each horizontal line similarly. Make sure all the zeros are along the same vertical line (Figure 1).

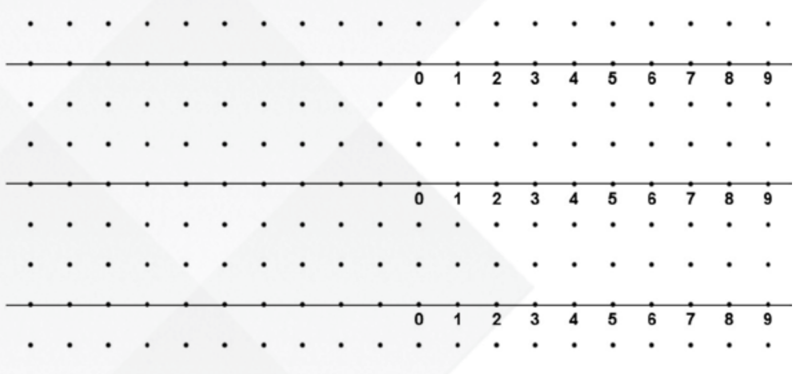


Figure 1

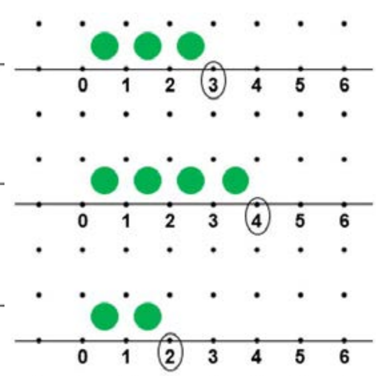


Figure 2

Keywords: integers, addition, subtraction, number line, modeling, word problems

1. Successors and Predecessors

(a) Pick any number between 2 and 5. Circle the corresponding dot. Draw as many counters as the number represents, equally spaced as shown (Figure 2 - chosen number here is 3).

(b) In the next line, draw the successor of your number and circle the corresponding dot.

(c) In the next line, draw the predecessor of your number and circle the corresponding dot.

(d) Draw the predecessor of 1. How many counters did you draw? Why?

(e) What would be the predecessor of 0? Where and how can we draw it?

For predecessor, we move one step ___________ (right/left).

> To find the dot corresponding to the predecessor of 0, we extend the number line to the left. Since this is one unit away from zero, we must mark it as 1, but since it is on the left side, we mark it as "-1" to distinguish it from the 1 on the right. As before, we draw one counter, but since it corresponds to movement towards the left, we use the other colour and we draw it below the number line (Figure 3). We call counters of this second colour **negative** and those of the first colour, **positive**. Accordingly, the numbers on the right of zero are called positive and those on its left are called negative.

2. Newer numbers with predecessors

(a) Draw the predecessor of -1. Using the same pattern, how do we write this number?

(b) Now, mark the part of the number line which is to the left of 0 accordingly, i.e., -2, -3, -4, etc.

(c) Draw -4. How many _________ (red/green) counters did you use ___________ (above/below) the line to the _________ (left/right) of 0?

> To draw the predecessor of any negative number, we have to add a negative counter as we shift one step left.

Try these:

I. Draw the predecessor of -3

II. The predecessor of -19 is _____. It is shown with _____ (18/20) __________ (red/green) counters. It is to the _________ (left/right) of -19.

Think:

A. Which city is colder? Shimla with a temperature of -7°C or Leh at -8°C.

B. If sea level is at 0m, and a pole of height 5m is shown by 5, then a hole of depth 5m is shown by ____. Does -6 show a hole deeper than a hole of depth 5m?

C. Titir with a debt of ₹300 is ________ (richer/poorer) than Tinku who has a debt of ₹200.

3. Successor of -1

For successor, we take one step to the ________ (right/left).

(a) So, which number is the successor of -1?

(b) How many counters (and of which colour), are used to show the successor of -1?

(c) For successor (and therefore to step right), we have always added a positive counter. When we do that here, we get a ______________ (red counter/green counter/red-green pair)?

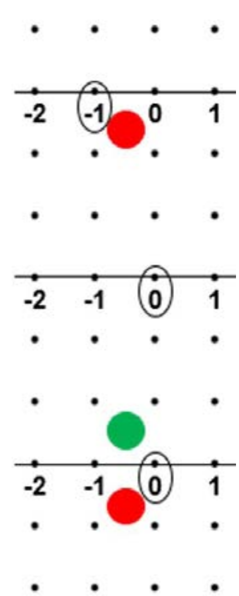


Figure 3

This positive-negative counter pair is equivalent to zero (Figure 3). Therefore, we will call such a pair **zero pair** from now on. Since they are equivalent to zero, they can be removed (or added) as and when needed.

Try this:

III. 0 = ___ + ___ = ___ + ___ = ___ + ___

4. Adding positive numbers

When we add positive numbers, we take not one, but several steps to the _____ (right/left).

(a) Add 3 to 4 (b) Add 2 to -5 (c) Add 5 to -3

(a) 4 + 3: Take ___ steps to the _______ (left/right) of 4, adding 1 _______ (red/green) counter at each step.

(b) (-5) + 2: Take ___ steps to the ______ (left/right) of -5, adding 1 ______ (red/green) counter at each step.

(c) (-3) + 5: Take ___ steps to the ______ (left/right) of -3, adding 1 ______ (red/green) counter at each step.

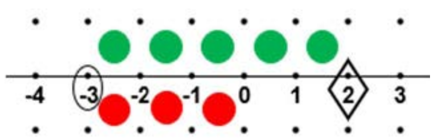


Figure 4

You used the same strategy for all three sums. Did you get zero pair(s) for any sum?

Is adding a green counter the same as removing a red counter? Try the same sums by doing this (when you can). Do you get the same answers?

So, **adding a positive number** is equivalent to:

(i) **Adding** positive counters (and removing zero pairs if any) OR

(ii) **Removing** negative counters

Try these:

IV. (-5) + 5 = ____

V. (-8) + ___ = 0

VI. ____ + 4 = 0

Think:

D. Titir got ₹100 as birthday gift and paid off some of her debt of ₹300.

a. How much debt remains?

b. So, how much debt was paid off, i.e., subtracted?

c. If the original debt of ₹300 is written as -3, then the gift is ___ (1/-1) and the remaining debt is ___.

d. Debt remaining = Original debt + gift = original debt – debt paid off, i.e., ____ = (-3) + ___ = (-3) – ____

e. What would have happened if the gift was ₹500?

E. Tintin had a debt of ₹200 but made a profit of ₹600.

a. How much did Tintin have after making the profit?

b. If a debt of ₹200 is -2, then a profit of ₹600 is ___ (6/-6) and the amount Tintin had after profit is ___.

c. So, (-2) + ___ = ___

5. Subtracting positive numbers

When we subtract positive numbers, we take not one, but several steps to the __________ (right/left) and you remove __________ (red/green) counters.

(a) Subtract 2 from 6 (b) Subtract 3 from -2

(c) Subtract 5 from 3

(a) 6 – 2: Take ___ steps to the __________ (left/right) of 6, removing 1 counter at each step.

(b) (-2) – 3: Take ___ steps to the ______ (left/right) of -2, removing 1 counter at each step.

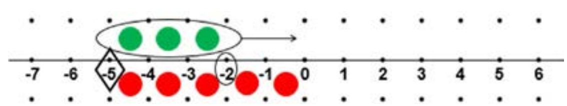
What did you do when you ran out of counters?

(c) 3 – 5: Take ___ steps to the __________ (left/right) of 3, removing 1 counter at each step.

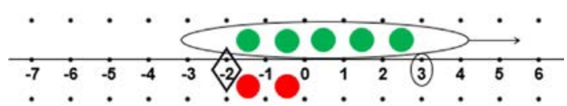
What did you do when you ran out of counters? See Figure 5 for some ideas.

You used the same strategy for all three sums. Did you find that removing a green counter is the same as adding a red counter? Try the same sums by doing this, when you can. Do you get the same answers?

> So, **subtracting a positive number** is equivalent to:
>
> (i) **Removing** positive counters (and removing zero pairs if any) OR
>
> (ii) **Adding** negative counters
>
> Did you try the zero-pair strategy? What happens if we add zero-pairs till there are sufficient positive counters and then remove them?



(b) (-2) – 3 by adding zero pairs



(c) 3 – 5 by adding zero pairs OR

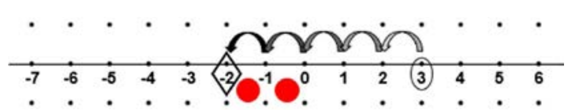


Figure 5

Try these:

VII. (-5) – 7:

a. Do you reach/cross 0 on the number line for this?

b. How many zero pairs do you need to add?

c. Try this one step at a time. How many negative counters do you add?

d. To which number do you add these negative counters?

e. So, adding ___ negative counters to ____ is _____ + (-_____). How is this related to (-5) – 7?

f. How are the number of zero pairs and the number of negative counters related?

VIII. 10 – 14:

a. Do you reach/cross 0 on the number line for this?

b. How many (minimum) zero-pairs do you need?

c. Try this one step at a time. How many negative counters do you add?

d. To which number do you add these negative counters?

e. So, adding ___ negative counters to ____ is ____ + (-___). How is this related to 10 – 14?

f. How are the number of negative counters and the minimum number of zero pairs related?

6. Adding negative numbers

When we add negative numbers, we take not one, but several steps to the _____ (right/left).

(a) Add -2 to -5 (b) Add -3 to 4

(c) Add -5 to 3

(a) (-5) + (-2): Take ___ steps to the _______ (left/right) of -5, adding 1 ______ (red/green) counter at each step.

(b) 4 + (-3): Take ___ steps to the _______ (left/right) of 4, adding 1 _______ (red/green) counter at each step.

(c) 3 + (-5) Take ___ steps to the _______ (left/right) of 3, adding 1 _______ (red/green) counter at each step.

Did you get zero pair(s) for any of the sums?

Is adding a red counter the same as removing a green counter?

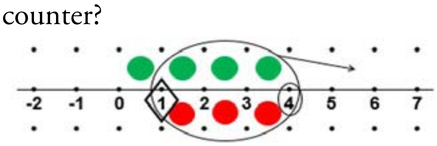


Figure 6

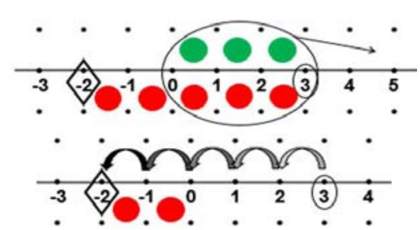


Figure 7

The direction of steps comes from point 2 above if we consider that adding -3 can be as adding -1 three times. Another way of looking at it would be that it is different from adding 3. So, it can't be stepping in the same direction as adding a positive number! Observe that if we add negative counters, then the resulting zero-pairs should be removed (Figure 6). What are the different strategies you can use for (c) Do you get the same result as proceeding one step at a time? (Figure 7)

So, **adding a negative number** is equivalent to:

(i) ______________ (adding/removing) negative counters

(ii) ______________ (adding/removing) positive counters

Is this similar to anything we have already done?

So, **adding a negative number** is equivalent to ______________ (adding/subtracting) **a positive number**.

How are the two numbers related?[1]

Try these:

IX. 7 + (-7) = ___

X. 3 + ___ = 0

XI. ____ + (-6) = 0

Think:

F. Tinku had a debt of ₹200 and had to borrow ₹500 more.

a. How much is the total debt now?

b. If debt of ₹200 is -2, what are the new debt and the total debt?

c. So, (-2) + ___ = ____ = (-2) – ____

G. Tintin had ₹400 and made a loss of ₹100.

a. If ₹400 is 4, then a loss of ₹100 is ___ (1/-1).

b. How much did Tintin have after the loss? And the loss of ₹___ is ___.

c. So, 4 + ____ = ____

d. If the loss was of ₹900, then what would Tintin have?

H. Tulu also had ₹400 and spent ₹100.

a. Using the same notation, is this 4 – 1 = ___?

b. So, is 4 + (-1) = 4 – 1?

c. If Tulu had to spend ₹900, then what would have happened?

d. If Tintin made a loss of ₹900 and Tulu had to spend ₹900, who is poorer?

7. Sums of integers

Consider the following pairs of sums: 5 + (-1) and (-1) + 5 (Figure 8) as well as 2 + (-5) and (-5) + 2 (Figures 9). What do you observe?

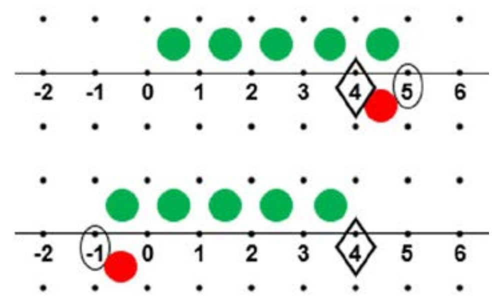


Figure 8

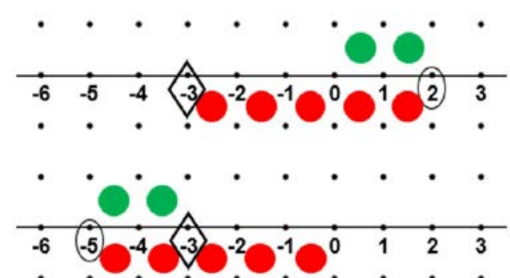


Figure 9

(a) How are the two pictures in Figure 8 related? Can you reflect the top one along some vertical line to get the bottom one?

(b) Where is this vertical line or the mirror for Figure 8?[2]

---

1 Adding -*n* is equivalent to subtracting *n*, its additive opposite

2 Figure 8: mirror is $x = 2 = (5 + (-1))/2$ … this can be observed by noting the common portion for both diagrams and therefore the common interval, and then finding the midpoint of that interval

(c) Observe the number this mirror is passing through. How can you get this number from 5 and -1?

(d) Can you find a similar mirror for Figure 9? Which number does it pass through? How can you get this number from 2 and -5?[3]

(e) Do observe similarly for (-2) + (-4) and (-4) + (-2). What about 3 + 4 and 4 + 3?

Try this with different pairs of numbers. What do you conclude?[4]

8. Subtracting negative numbers

When we subtract negative numbers, we take not one, but several steps to the _____ (right/left) and you remove ________ (red/green) counters.

[Now you should be knowing what to do when you run out of counters.]

(a) Subtract -3 from -5 (b) Subtract -4 from 3
(c) Subtract -6 from -2

(a) (-5) – (-3): Take ___ steps to the ______ (left/right) of -5, removing 1 counter at each step.

(b) 3 – (-4): Take ___ steps to the ______ (left/right) of 3, removing 1 counter at each step.

If we add zero-pairs, that is equivalent to adding which numbers? (See Figure 10)

Can you complete the steps? What comes after 3 in the last step?

3 – (-4) = 3 – (-4) + 0 = 3 – (-4) + ___ + (-___) = 3 _______________

Did you get the same answer both times?

(c) (-2) – (-6): Take ___ steps to the ______ (left/right) of -2, removing 1 counter at each step. What happens if we add zero-pairs instead? How many zero-pairs do we need? Can you complete the steps? [Hint: see Figure 11]
(-2) – (-6) = (-2) – (-6) + 0 = (-2) – (-6) + ___ + (-___) = (-2) _______________
Did you get the same answer?

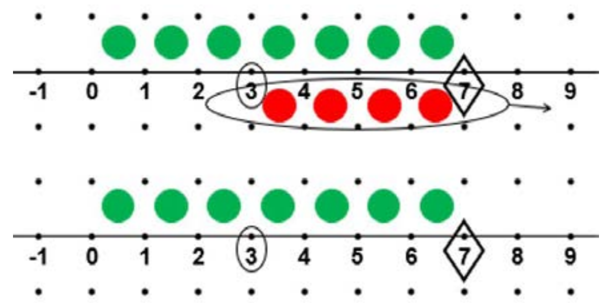


Figure 10

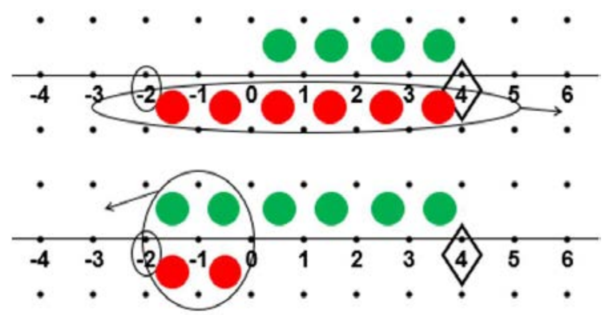


Figure 11

Therefore, we can add ___________ (positive/negative) counters as we take these steps.

So, **subtracting a negative number** is equivalent to:

(i) ______________ (adding/removing) negative counters

(ii) ______________ (adding/removing) positive counters

Is this similar to anything we have already done?

So, **subtracting a negative number** is equivalent to ______________ (adding/subtracting) **a positive number**.

How are the two numbers related?[5]

3 Figure 9: mirror is x = -1.5 = ((-5) + 2)/2

4 Commutative property of addition for integers

5 Subtracting *-n* is equivalent to adding *n*, its additive opposite

Try these:

XII. 6 – (-2)

a. Do you reach/cross 0 on the number line for this?

b. How many zero pairs do you need to add?

c. Try this one step at a time. How many positive counters do you add?

d. To which number do you add these positive counters?

e. So, adding ___ positive counters to ___ is ___ + ___. How is this related to 6 – (-2)?

f. How are the number of zero pairs and the number of positive counters related?

XIII. (-3) – (-4)

a. Do you reach/cross 0 on the number line for this?

b. How many (minimum) zero-pairs do you need?

c. Try this one step at a time. How many positive counters do you add?

d. To which number do you add these positive counters?

e. So, adding ___ positive counters to _____ is _____ + ___. How is this related to (-3) – (-4)?

f. How are the number of positive counters and the minimum number of zero pairs related?

Think:

I. On a chilly winter night, Srinagar's temperature was -6°C while Leh's was -17°C and Gangtok was 8°C

a. Which city was colder: Srinagar or Leh?

b. __________ was colder than ________ by (_____ – _____) = _____ °C.

c. Gangtok was warmer than Srinagar by (_____ – _____) = _____°C.

J. Titir and Tinku started the day with ₹100 each. Titir made a profit of ₹300 while Tinku made a loss of ₹200.

a. How much did each one have at the end of the day?

b. Who was richer and by how much?

c. Titir's profit is ___ and Tinku's loss is ___.

d. _______ (Titir/Tinku) is richer and by _____ – _____ = _____ .

K. Next week, they again started with ₹100 each, but both made losses – Titir ₹500 and Tinku ₹300.

a. How much did each one have at the end of the day?

b. Titir ended with ___ while Tinku with ___.

c. _______ (Titir/Tinku) is richer and by _____ – _____ = _____ .

The idea of this article was triggered by an exploration by Jauhar K M and Nagendra Singh, MA Education students at Azim Premji University, as part of their Curriculum Material Development – Mathematics course.

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Put Your Thinking Cap On!

It's not just computers which compress data, human beings do it very often too! How often have you used an acronym to remember an important list? Or jotted down points from a talk that you want to remember later? Data compression uses many aspects of computational thinking- decomposition, pattern recognition, abstraction and algorithms.

Suppose you saw a really cool knitting pattern and you had only a scrap of paper to write it on. Can you use computational thinking to fit the pattern on your piece of paper and be able to recreate it without losing any important instructions?

This is the original pattern.

Knitting a Rainbow Cap

Using No. 9 needles, cast on 120 stitches.

Knitting loosely, complete 7 inches of single rib (Knit 1 (K1), Purl 1 (P1)). {You can increase this length, if you want a longer cap.}

Shaping the cap:

Row 1: *(K2 together, (K1, P1) (**3** reps)) continue from * till end.

Pattern repeats (reps) every 8 stitches. (105 stitches)

Row 2: K1 P1 till end.

Row 3: *(K2 together, (K1, P1) (**2** reps), K1) continue from * till end taking care to preserve the single rib. **Pattern reps every 7 stitches.** (90 stitches)

Row 4: As Row 2 but taking care to preserve the single rib pattern.

Row 5: *(K2 together, (K1, P1) (**2** reps)) continue from * till end taking care to preserve the single rib. **Pattern reps every 6 stitches.** (75 stitches)

Row 6: As Row 2 but taking care to preserve the single rib pattern.

Row 7: *(K2 together, (K1, P1, K1)) continue from * till end taking care to preserve the single rib.

Pattern reps every 5 stitches. (60 stitches)

Row 8: As Row 2 but taking care to preserve the single rib pattern.

Row 9: *(K2 together, (K1, P1)) continue from * till end taking care to preserve the single rib.

Pattern reps every 4 stitches. (45 stitches)

Row 10: As Row 2 but taking care to preserve the single rib pattern.

Row 11: *(K2 together, (K1)) continue from * till end taking care to preserve the single rib.

Pattern reps every 3 stitches. (30 stitches)

Row 12: As Row 2 but taking care to preserve the single rib pattern.

Row 13: *(K2 together) continue from * till end taking care to preserve the single rib.

Pattern reps every 2 stitches. (15 stitches)

Draw a thread with a sewing needle through the remaining 15 stitches and sew the seam of the cap keeping the right-side in.

Reverse and you're done!

Do you see that there are several repeating patterns in the shaping of this cap?

Every even-numbered (alternate row) is the same, and the focus is on keeping the single rib (the vertical pattern) continuous.

.....contd on next page

Every odd-numbered row has a decreasing (number of reps and number of stitches) pattern which you can see in this table.

Original number of stitches:120			
Row Num-ber	**Number of stitches in each rep**	**Number of stitches decreased (number of times the pattern reps)**	**Remaining number of stitches**
1	8	15	105
3	7	15	90
5	6	15	75
7	5	15	60
9	4	15	45
11	3	15	30
13	2	15	15

Here are some questions!

- Why do you think the original number of stitches was 120?
- Would it work with 130 stitches?
- How can you make a bigger cap?
- What would you change?

This is the compressed pattern.

Shaping the cap: Call (K1, P1) Block 1 and let n be the number of reps, s be the number of stitches in each rep.

Set $n = 3$.

Rows 1, 5, 9, 13:*(K2 together, Block 1 (n reps)) continue from * till end.

Number of stitches in each rep: $2n + 2$.

Set $n = n - 1$

Rows 2, 4, 6, 8, 10, 12: Continue the vertical single rib pattern- no decrease.

Rows 3, 7, 11:*(K2 together, Block 1, (n reps), K1) continue from * till end.

Number of stitches in each rep: $2n+3$.

Draw a thread with a sewing needle through the remaining 15 stitches and sew the seam of the cap keeping the right-side in.

Reverse and you're done!

Reference: 1] https://classic.csunplugged.org/books/

1. (https://youtu.be/1vm6oaYzHyA)
(https://www.youtube.com/watch?v=Egp4NRhlMDg), the basic Knit and Purl stitches
(https://www.youtube.com/watch?v=7ePhLqw6HDM)
(https://www.youtube.com/watch?v=VSwjIUiQZlM), casting on and casting off stitches

A 'Mean' Question

MATHEMATICS
CO-DEVELOPMENT
GROUP

In the first three parts of this series, we unpacked the median and the mode formulas. Comparatively the formula for the mean is easier to understand and not as counter intuitive. However, while computing it for grouped data, we use the midpoints of the class intervals and compute the mean as if all data in each class interval is exactly the midpoint. Consider Table 1.

Class Intervals	Midpoints	Frequencies
0 – 10	5	5
10 – 20	15	10
20 – 30	25	25
30 – 40	35	30
40 – 50	45	20
50 – 60	55	10
Total		100

Table 1

Data values	Frequencies
5	5
15	10
25	25
35	30
45	20
55	10
	100

Table 2

The mean is computed as $(5 \times 5 + 15 \times 10 + 25 \times 25 + 35 \times 30 + 45 \times 20 + 55 \times 10)/100$. This is identical to computing the mean for the ungrouped data given in Table 2. Note that the actual data for the class interval 0-10 can be 1, 1, 2, 2, 4 (adding up to 10) or 5, 6, 8, 8, 9 (adding up to 36). But we are assuming that they sum to $5 \times 5 = 25$. Since we can't get the actual data values in each interval, we must approximate. So, **why do we choose the midpoints?** This article tries to unpack that.

Keywords: Mean, grouped, ungrouped, fair-share, fulcrum, moment, modelling, analysing.

But before answering that question, we considered two models of mean and could link the two for ungrouped data. The two models are (i) the fair-share model and (ii) the fulcrum model. In the **fair-share model**, the data values are all pooled together and then shared equally. Here is an example: 10 couples who are also parents were surveyed for number of children. Figure 1 shows the no. of children for each of these couples (C_1, C_2 … C_{10}). So, there are six couples with one child, viz., C_1, C_2, … C_6; three couples with two children, i.e., C_7, C_8 and C_9 and the last couple C_{10} with four children.

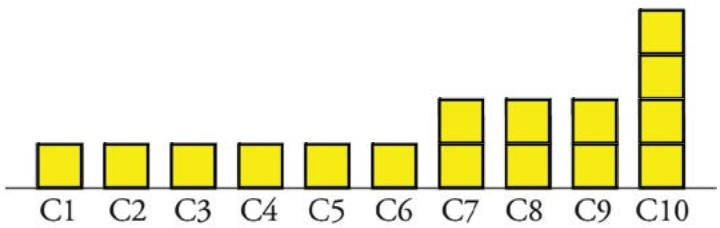


Figure 1

No. of children	No. of couples
1	6
2	3
3	0
4	1

Table .

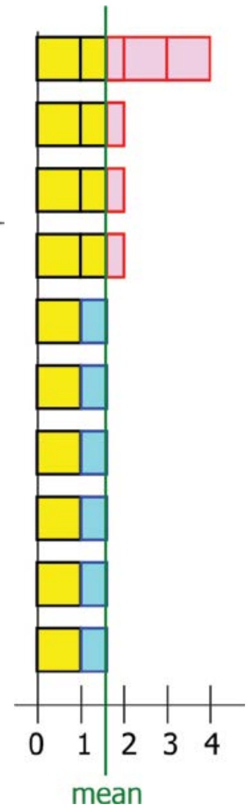


Figure 2

Now, in the fair-share model, all the yellow squares must be shared equally among the 10 couples. Then the resulting common height of the rectangle for each couple is the 'mean'. Figure 2 represents this mean (calculated to be 1.6) with the rows and columns flipped, i.e., each row now represents a couple. [The reason for the flip would become clear soon.] The pink parts have been redistributed to form the blue parts, so they have equal areas. Also, the mean is the common length of each row after redistributing the rectangles. So, the common length is the total yellow area redistributed equally among the rows, i.e., $(6 \times 1 + 3 \times 2 + 0 \times 3 + 1 \times 4) / 10 = 1.6$. Now the total blue area is $6 \times (1.6 – 1)$ and the total pink area is $3 \times (2 – 1.6) + 1 \times (4 – 1.6)$ when computed row by row.

Compare this to the stick representation (Figure 3) illustrating the fulcrum of the distribution. Now, in the **fulcrum model**, the 'mean' is where the fulcrum must be placed to balance the distribution (Figure 3 based on the frequency distribution given in Table 3). So, the total moments on either side of the fulcrum must be equal.

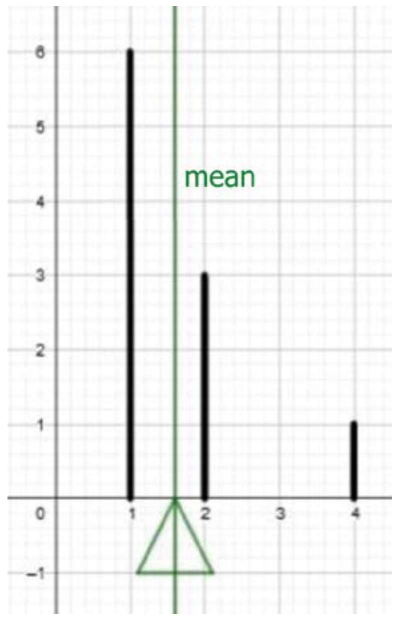


Figure 3

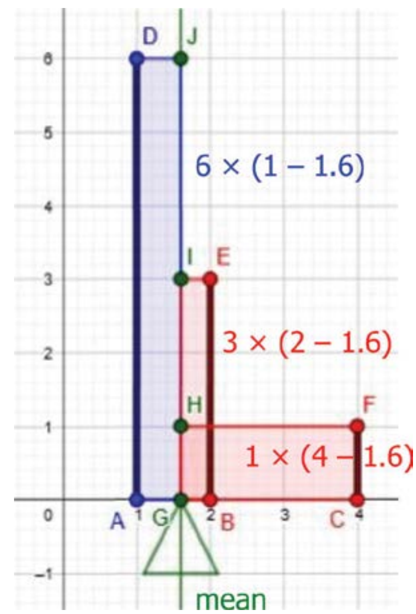


Figure 4

So, the total moment on the left of the fulcrum is indicated by the area of the blue rectangle, i.e., 6 × (1.6 – 1) in Figure 4. The height of each rectangle is the frequency while the base of each one is the difference 'mean – data value'. Similarly, the total moment on the right is given by the areas of the two pink rectangles, i.e., 3 × (2 – 1.6) and 1 × (4 – 1.6). Again, the heights are the frequencies, but the bases are 'data value – mean'. However, to combine the areas in one formula, we need to write them as 'data value - mean' (or 'mean - data value') for each rectangle. Therefore, in Figure 4, since the data value $= 1 < 1.6 =$ the mean, 'data value - mean' is negative. Thus, the area of the blue rectangle can be considered to be negative. Note that this area is of equal magnitude to the total pink area. In other words, $6 \times (1.6 - 1) = 3 \times (2 - 1.6) + 1 \times (4 - 1.6)$. Also observe the following:

- The blue rectangle AGJD in Figure 4 has the same area (3.6) as that of all the blue rectangles in Figure 2. In fact, if the blue rectangles are lined up by removing the gaps among them, then they would form the same blue rectangle AGJD.
- The pink rectangle GCFH in Figure 4 has the same area (2.4) as all the pink rectangles in top row of Figure 2. They in fact have the same dimensions.
- The pink rectangle GBEI in Figure 4 has the same area (1.2) as all the remaining pink rectangles in rows 7-9 in Figure 2. Like the blue one, if the pink rectangles are joined by removing the gaps among them, then they would form a rectangle with the same dimension as GBEI.

So, combining Figures 2 and 4, we observe that the fair-share and the fulcrum model have a deep connection. [This is why we flipped the graph in Figure 2.] We strongly encourage the reader to try to recreate Figure 2 and Figure 4 with any ungrouped data in order to understand this connection.

Algebraically speaking, if $x_1, x_2 \ldots x_k$ are the data values with frequencies $f_1, f_2 \ldots f_k$ respectively then the fulcrum is located at 'm' if the total moment from 'm' is 0, i.e., $\sum_{i=1}^{k} f_i (x_i - m) = 0$. Note that this generates $m = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$ which is the same formula derived from the fair-share model. For the above example, this is $6\,(1-m)+3\,(2-m)+1\,(4-m)=0$, which reduces to $6\times1+3\times2+1\times4=(6+3+1)\,m$, i.e., $m=\dfrac{6\times1+3\times2+1\times4}{6+3+1}=\dfrac{16}{10}=1.6$, i.e., what we got earlier.

However, for a grouped frequency distribution, it is impossible to know the individual data values as mentioned earlier. So, fair-share model can't be applied directly. However, the fulcrum model can be adopted. Instead of a stick diagram as in Figures 3 and 4, we would use the histogram and the fulcrum would be the point where it can be balanced.

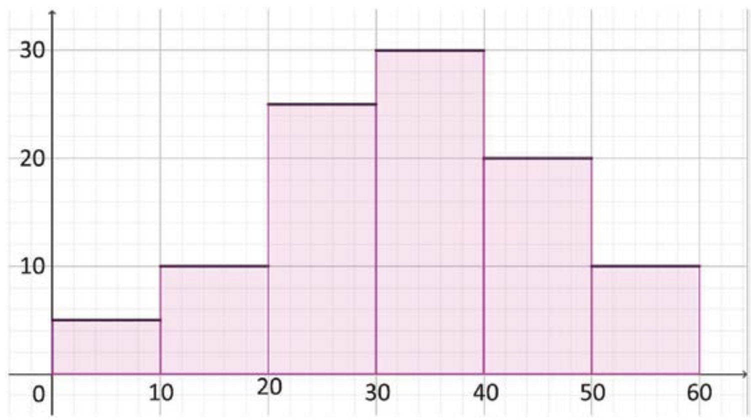


Figure 5

Let us consider the histogram corresponding to the grouped frequency distribution given in Table 1 (Figure 5). Now we can think of the corresponding step function $f(x)$ as follows (Figure 6):

$$
\begin{aligned}
f(x) &= 0 && \text{for } x<0\\
&= 5 = f_1 && \text{for } 0\le x<10\\
&= 10 = f_2 && \text{for } 10\le x<20\\
&= 25 = f_3 && \text{for } 20\le x<30\\
&= 30 = f_4 && \text{for } 30\le x<40\\
&= 20 = f_5 && \text{for } 40\le x<50\\
&= 10 = f_6 && \text{for } 50\le x<60\\
&= 0 && \text{for } x\ge 60
\end{aligned}
$$

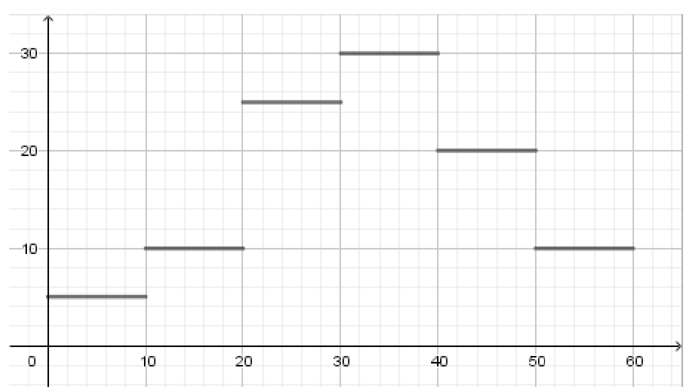


Figure 6

Then the sum for total moment becomes this integral

$$
\int_0^{60} f(x)\,(x-m)\,dx=\sum_{i=1}^{6}\int_{10(i-1)}^{10i} f_i\,(x-m)\,dx = 5\times\int_0^{10}(x-m)\,dx+10\times\int_{10}^{20}(x-m)\,dx+25\times\int_{20}^{30}(x-m)\,dx+30\times\int_{30}^{40}(x-m)\,dx+20\times\int_{40}^{50}(x-m)\,dx+10\times\int_{50}^{60}(x-m)\,dx.
$$

So, mean is 'm', the value that makes this integral zero.

Now, any general integral of this form, i.e., $\int_a^b (x - m)\, dx = \left[\left(\frac{b^2}{2} - bm\right) - \left(\frac{a^2}{2} - am\right)\right] = \frac{b^2 - a^2}{2} - (bm - am) = (b - a)\left(\frac{a + b}{2} - m\right)$ using $\frac{b^2 - a^2}{2} = (b - a)\frac{a + b}{2}$. Note that $b - a = 10 =$ common class width for each of these integrals.

So, this integral becomes

$$= 10\left\{5\left[\frac{10 + 0}{2} - m\right] + 10\left[\frac{20 + 10}{2} - m\right] + 25\left[\frac{30 + 20}{2} - m\right] + 30\left[\frac{40 + 30}{2} - m\right] + 20\left[\frac{50 + 40}{2} - m\right] + 10\left[\frac{60 + 50}{2} - m\right]\right\}$$

$$= 10 \times \{5\,[5 - m] + 10\,[15 - m] + 25\,[25 - m] + 30\,[35 - m] + 20\,[45 - m] + 10\,[55 - m]\}.$$

So, the integral is 0 if and only if

$$5\,[5 - m] + 10\,[15 - m] + 25\,[25 - m] + 30\,[35 - m] + 20\,[45 - m] + 10\,[55 - m] = 0 \ldots \qquad (1)$$

which is possible if and only if

$$m = \frac{5 \times 5 + 10 \times 15 + 25 \times 25 + 30 \times 35 + 20 \times 45 + 10 \times 55}{5 + 10 + 25 + 30 + 20 + 10} = \frac{3300}{100} = 33 \ldots \qquad (2)$$

Note that (1) is exactly like $\sum_{i=1}^{k} f_i\,(x_i - m) = 0$ while (2) resembles $m = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}$; both corresponding to the ungrouped distribution in Table 2 represented in Figure 7 by the stick representation. Note that the sticks are at the midpoint of each class interval and share the same height (i.e., frequency).

So, generally, when we consider the histogram of a grouped frequency distribution, the sum for total moment becomes the area under the histogram which is the integral $\int_{x_0}^{x_k} f(x)\,(x - m)\, dx = \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f_i\,(x - m)\, dx = \sum_{i=1}^{k} f_i \times \int_{x_{i-1}}^{x_i} (x - m)\, dx$ where $x_0 - x_1, x_1 - x_2 \ldots x_{k-1} - x_k$ are the class

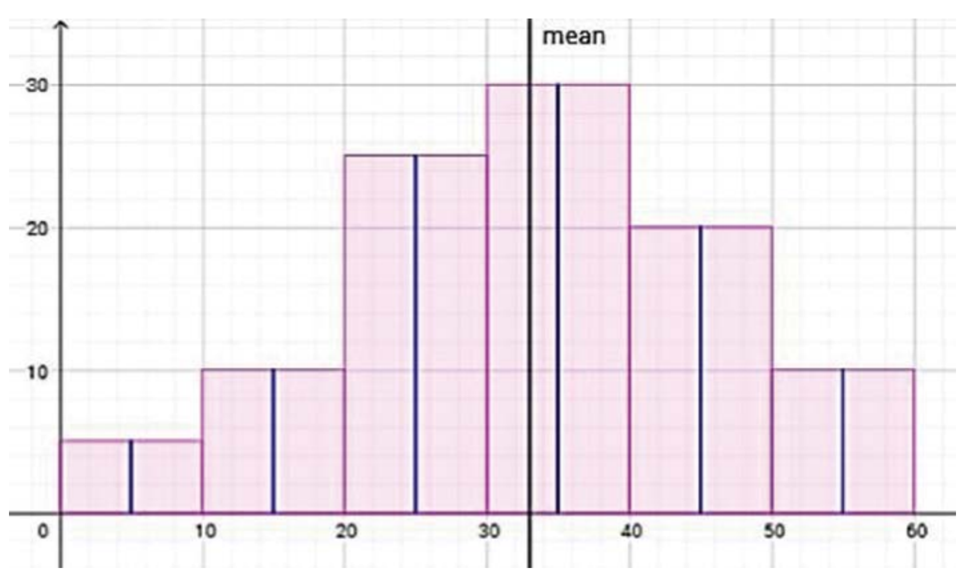


Figure 7

intervals with frequencies $f_1, f_2 \ldots f_k$ respectively where the mean is that value of 'm', which makes this integral zero. Therefore, the integral becomes

$$= \sum_{i=1}^{k} f_i \times \int_{x_{i-1}}^{x_i} (x - m)\, dx = \sum_{i=1}^{k} f_i \left[\left(\frac{x_i^{\,2}}{2} - m x_i\right) - \left(\frac{x_{i-1}^{\,2}}{2} - m x_{i-1}\right)\right]$$

$$= \sum_{i=1}^{k} f_i \left(x_i - x_{i-1}\right) \left[\frac{x_i + x_{i-1}}{2} - m\right]$$

$$= h \times \sum_{i=1}^{k} f_i \left[\frac{x_i + x_{i-1}}{2} - m\right]$$

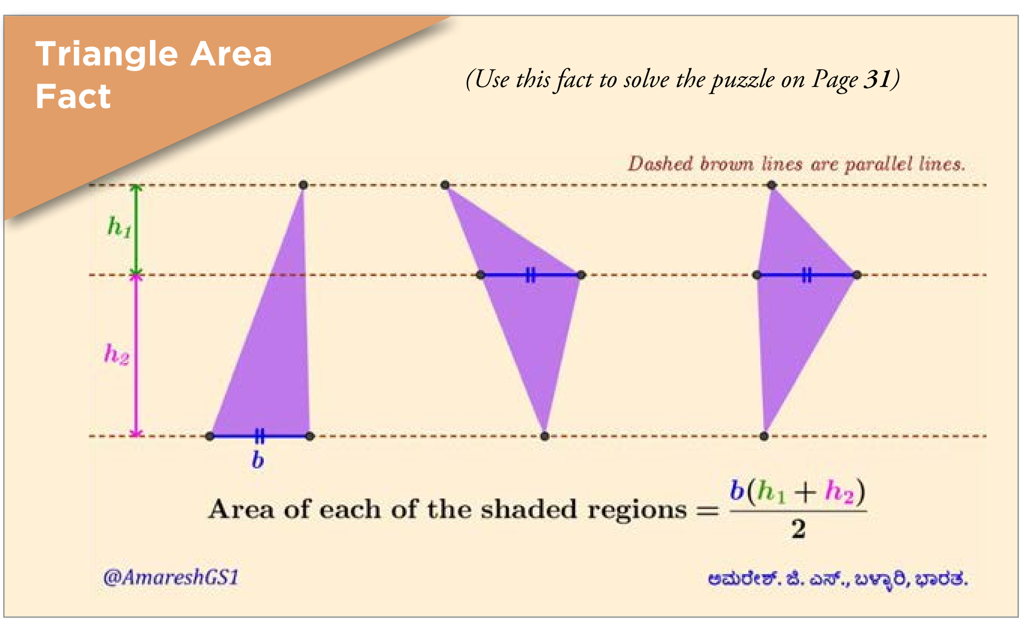
where h is the class interval (which is usually the same for all classes).

Now, $\frac{x_i + x_{i-1}}{2} = y_i$ is nothing but the midpoint of the class interval $x_{i-1} - x_i$ for $i = 1, 2 \ldots k$. So, the integral becomes $h \times \sum_{i=1}^{k} f_i \left(y_i - m\right)$ which is similar to ungrouped frequency distribution with the midpoints $y_1, y_2 \ldots y_k$ as the data values. This converts the histogram into a stick representation.

Note that if we find the fulcrum for each class, then that is also the midpoint of each class by the symmetry of rectangles. This can also be arrived at with integration.

It is interesting to observe that while the formulas for median and mode for grouped data looked very complicated, they required nothing beyond Class 10 syllabus to decipher. However, a much simpler looking process for calculating mean for group data requires a much more sophisticated tool like integration to understand it.

Math Co-dev Group or more elaborately **Mathematics Co-development Group** is an internal initiative of Azim Premji Foundation where math resource persons across states put their heads together to prepare simple materials for teachers to develop their understanding on different content areas and how to transact the same in their classrooms. It is a collaborative learning space where resources are collected from multiple sources, critiqued and explored in detail. Math Co-dev Group can be reached through ashish.gupta@azimpremjifoundation.org



A Special Class of Strong Prime Numbers – Krishnan's Primes

SASIKUMAR

It is well-known that there are infinitely many prime numbers. The 'Twin Prime Conjecture' states that there are infinitely many primes p for which $p+2$ is prime. We define a special class of prime numbers as follows and state a conjecture about them:

Definition 1. A prime number p is a *Krishnan prime* if both $p+2$ and p^2+4 are prime numbers.

For example, 3 is such a prime, since $3+2=5$ and $3^2+4=13$ are both prime numbers. There are 22 Krishnan prime numbers below 10000:

3, 5, 17, 137, 347, 827, 2087, 2687,
3557, 3917, 4517, 4967, 5477, 5657, 5867, 6827,
7457, 7547, 7877, 8087, 8537, 8597.

Conjecture (Sasikumar)**.** *There are infinitely many Krishnan primes.*

Keywords: Prime number, strong prime, Krishnan prime, twin primes

Discussion

Definition 2. A prime number p_n is said to be **strong** (see [1]) if it is greater than the arithmetic mean of its nearest prime neighbours, i.e., if $p_n > \frac{1}{2}(p_{n-1} + p_{n+1})$.

The first few strong primes are the following:

$$\begin{array}{cccccccccc} 11, & 17, & 29, & 37, & 41, & 59, & 67, & 71, & 79, & 97, \\ 101, & 107, & 127, & 137, & 149, & 163, & 179, & 191, & 197, & 223, \\ 227, & 239, & 251, & 269, & 277, & 281, & 307, & 311, & 331, & \ldots \end{array}$$

Our aim is to show that *all Krishnan primes* > 5 *are strong*. For that we start with a lemma:

Lemma. *Let p be a Krishnan prime greater than 5. Then the last digit of p is 7.*

Proof. Let p be a Krishnan prime greater than 5; then $p \geq 17$. We write

$$p = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^n a_n$$

for some natural number n. Since $p \geq 17$, $a_0 \neq 0, 2, 4, 5, 6, 8$, and $a_1 \geq 1$.

Since $p + 2$ is prime, $a_0 \neq 3$ (else $p + 2$ will be divisible by 5 and so cannot be a prime number since $p \geq 17$).

Next, $p^2 + 4 = a_n 10^{2n} + \cdots + a_0^2 + 4 \geq 293$ is a prime number (by definition), so $a_0 \neq 1, 9$. Hence $a_0 \neq 0, 1, 2, 3, 4, 5, 6, 8, 9$. Therefore $a_0 = 7$. That is, the last digit of a Krishnan prime beyond 5 is 7.

Now we prove our main theorem:

Theorem. A Krishnan prime greater than 5 is a strong prime number.

Proof. Let $p > 5$ be a Krishnan prime. By the lemma we know that $p - 3, p - 2, p - 1$ and $p + 1$ cannot be prime numbers. Let q and r be respectively the prime numbers just preceding and just succeeding p.

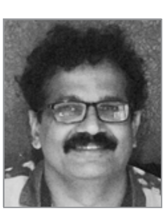
By the definition of p, we know that $p + 2$ is prime, so we conclude that $r = p + 2$.

Since $p - 3$, $p - 2$ and $p - 1$ are not prime, we conclude that $q < p - 2$. Hence $q + r < 2p$, i.e., $p > (q + r)/2$. Hence a Krishnan prime greater than 5 is a strong prime number. □

Conclusion. The set of Krishnan primes greater than 5 is a subset of the class of strong primes.

References

1. Wikipedia, "Strong prime", from https://en.wikipedia.org/wiki/Strong_prime
2. Tom M Apostol, *Introduction to Analytic Number Theory*, Springer (2010)



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On a Generalization of a Problem on Factorisation

SIDDHARTHA SANKAR CHATTOPADHYAY

In high school mathematics, we come across the topic of factorisation of polynomials with integer or rational coefficients. Usually, we only consider factorisation of polynomials of low degrees. Checking whether or not a polynomial of higher degree such as $X^{37} - 27X^{11} + 3$ is factorisable turns out to be a difficult problem. Although there are some methods available in higher mathematics to deal with such problems, the tools and techniques of high school mathematics seem to be of little use for addressing such problems. In this article, we discuss the factorisation of a particular infinite family of polynomials of arbitrary degree.

Statement of the problem

In [1], it is shown that for any integer $n \geq 1$ and distinct integers $a_1, a_2, \ldots, a_n$, the polynomial

$$(X - a_1)(X - a_2)\cdots(X - a_n) - 1 \tag{1}$$

is not factorisable. It is interesting to note that if we consider a variant of this polynomial, namely

$$(X - a_1)(X - a_2)\cdots(X - a_n) + 1, \tag{2}$$

then the polynomial is sometimes factorisable. For instance we have:

$$(X - 3)(X - 5) + 1 = (X - 4)^2.$$

Keywords: Polynomial, factorisation, integer, rational, coefficient, degree

However, keeping $+1$ in place of -1, we can show that a large number of such polynomials are not factorisable. More precisely, we prove the following theorem.

Theorem 1. Let $n \geq 1$ be an integer and let $a_1, a_2, \ldots, a_n$ be distinct odd integers. Then the polynomial

$$f(X) = (X-2)(X-a_1)(X-a_2)\cdots(X-a_n)+1 \tag{3}$$

is not factorisable over the integers.

Remark 1. *The presence of the factor* $(X-2)$ *in Theorem 1 is crucial, otherwise the following serves as a counter-example:*

$$(X-3)(X-5)+1=(X-4)^2.$$

Remark 2. *As* n *is an arbitrary positive integer, the degree of* f *can be made arbitrarily large. Since the integers* $a_1, \ldots, a_n$ *vary over the odd natural numbers, these polynomials form an infinite family.*

Proof of Theorem 1. If possible, suppose that

$$f(X) = (X-2)(X-a_1)(X-a_2)\cdots(X-a_n)+1$$

is factorisable over the integers. Let $f(X) = g(X)h(X)$ for some two polynomials g and h with integer coefficients. We note that

$$f(2) = f(a_1) = \ldots = f(a_n) = 1.$$

Consequently, we have

$$g(2)h(2) = g(a_1)h(a_1) = \ldots = g(a_n)h(a_n) = 1. \tag{4}$$

Since g and h are polynomials with integer coefficients, $g(2), h(2), g(a_1), h(a_1), \ldots, g(a_n), h(a_n)$ are all integers. Therefore, from (4),

$$\left.\begin{aligned} &g(2) = h(2) = \pm 1, \\ &g(a_i) = h(a_i) = \pm 1 \text{ for all } i \in \{1, \ldots, n\}. \end{aligned}\right\} \tag{5}$$

Let $P(X) = g(X) - h(X)$. Then P is a polynomial with integer coefficients, and $P(2) = P(a_i) = 0$ for all $i \in \{1, \ldots, n\}$. Now,

$$\begin{aligned} \deg P(X) &= \deg(g(X) - h(X)) \\ &\leq \max\{\deg(g(X)), \deg(h(X))\} \\ &< \deg(f(X)) = n+1. \end{aligned}$$

This means that $P(X)$ is a polynomial of degree $< n+1$, having at least $n+1$ distinct zeros (namely, $2, a_1, \ldots, a_n$). Hence $P(X)$ is identically zero, which implies that $g(X) = h(X)$.

Therefore, the relation $f(X) = g(X)h(X)$ becomes $f(X) = (g(X))^2$.

Let $g(0) = k$. Then from the equation $f(X) = (g(X))^2$, we get $f(0) = (g(0))^2 = k^2$. That is,

$$(-1)^{n+1} \cdot 2 \cdot a_1 a_2 \cdots a_n + 1 = k^2.$$

Since the a_i are all odd integers, k is odd. Hence $k^2 - 1 = (k-1)(k+1)$ is divisible by 8 (as it is the product of two consecutive even integers). This implies that $(-1)^{n+1} \cdot 2 \cdot a_1 a_2 \cdots a_n$ is divisible by 8, which is not possible as all the a_i are odd.

This contradiction shows that the stated factorisation is not possible.

This completes the proof of Theorem 1. $\square$

References

1. A. Engel, *Problem-Solving Strategies*, Springer.

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A Magic Trick

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A magician places some coins in a tray and calls a member of his audience on stage to support his act. He asks the spectator to take fewer than 10 coins in his right fist (he always looks away when the coins are taken).

Now he asks the spectator to count the remaining coins and find the sum of the digits of this number. (For example, if there are 13 coins remaining then the sum of the digits is 4.)

Now the magician tells the spectator to take as many coins as the sum of digits from the tray, again into his right fist. Then he asks him to take any number of coins from the remaining coins in the tray into his left fist.

The magician turns back to the tray and tells the audience how many coins the spectator has in his left and right fists.

And he is right!

How did he do this trick? *(See page **74** for the magic behind this trick)*

Contributed by: Kalpesh Akhani (Asst.Teacher)

Student Corner

Galaxy of Unit Fractions with Tom and Jerry

ADITHYA RAJESH

In Prithwijit De's article in the July 2021 issue of *At Right Angles* (page 59), Jerry had asked Tom the following question:

Problem 1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find all triples $a, b, c \in S$ with $a < b$, $c \neq a$, $c \neq b$, such that the following is true for all integers $n \geq 0$:

$$\frac{a}{b} = \frac{\overbrace{ccc\ldots cc}^{n}a}{b\underbrace{ccc\ldots cc}_{n}}. \tag{1}$$

(Here, $\overbrace{ccc\ldots cc}^{n}a$ denotes the $(n+1)$-digit number whose first n digits are c and last digit is a. Similarly, $b\underbrace{ccc\ldots cc}_{n}$ denotes the $(n+1)$-digit number whose first digit is b and last n digits are c.)

Tom is still looking for an answer!

I had written to Tom, stating that Problem 1 has no solutions. Tom, in response, decided to pose a problem of his own and wrote the following on the board.

Keywords: Unit fractions, base, recurring decimal

Challenge 1.

$$\frac{1}{3} = 0.\overline{01},$$
$$\frac{1}{5} = 0.\overline{0011},$$
$$\frac{1}{9} = 0.\overline{000111}, \quad \ldots$$

Can you identify the pattern and verify if this relationship is true for all integers $n \geq 0$?

Here the line over the 'decimal' indicates the recurring pattern: $0.\overline{01}$ means 0.01 01 01 01 …; $0.\overline{0011}$ means 0.0011 0011 0011 0011 …

Jerry could easily sort out the pattern on the LHS as $\frac{1}{2^n+1}$ but could not get things right on the RHS. But he had trained under Tom and knew all about his tactics. He was intensely observing the pattern to find the missing link.

Voila! He got it! The missing link is the base system. Tom had intentionally avoided mentioning that he was using the binary base system. (To understand non-decimal number bases better, the reader could refer to the 'Pullout' of the March 2022 issue of *At Right Angles*.) Quickly coming up with the proof was now a piece of cheesecake for Jerry.

Proof. We consider a typical positive integer, say $n = 3$. We wish to prove that

$$\frac{1}{2^3+1} = 0.\overline{000111} \quad \text{(in base 2).} \tag{2}$$

Let $x = 0.\overline{000111}$ in base 2. Then we have $8x = 0.\overline{111000}$ (remember that in base 2, multiplication by 8 results in moving the 'decimal point' 3 places to the right, just like what multiplication by 1000 does in base 10). Hence by addition we get

$$\begin{aligned} 9x &= 0.\overline{000111} + 0.\overline{111000} \quad \text{(in base 2)} \\ &= 0.\overline{111111} = 0.\overline{1} \quad \text{(in base 2)} \\ &= 1. \qquad \text{(Comment. This is the base 2 equivalent of the base 10 relation } 0.99999\ldots = 1.) \end{aligned}$$

Therefore $x = \frac{1}{9}$. This shows that (2) is true.

Though we have written the solution only for the case $n = 3$, this approach clearly works for all integers $n \geq 0$. □

Jerry did not stop with this finding but did some more research and extended this result to other bases:

$$\frac{1}{2^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{111\ldots11}_{n}} \quad \text{(in base 2),} \tag{3}$$

$$\frac{1}{3^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{222\ldots22}_{n}} \quad \text{(in base 3),} \tag{4}$$

$$\frac{1}{4^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\,\underbrace{333\ldots33}_{n}} \quad \text{(in base 4),} \quad \ldots \tag{5}$$

and:

$$\frac{1}{8^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{777\ldots77}_{n}} \quad \text{(in base 8)}, \tag{6}$$

$$\frac{1}{10^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{999\ldots99}_{n}} \quad \text{(in base 10)}, \tag{7}$$

$$\frac{1}{16^n+1} = 0.\overline{\underbrace{000\ldots00}_{n}\ \underbrace{FFF\ldots FF}_{n}} \quad \text{(in base 16)}, \quad \ldots \tag{8}$$

and so on.

Tom was unable to trap Jerry now and tried to challenge Jerry with a new question:

Challenge 2.

What is the corresponding relationship when $+1$ is replaced by -1 in the above relations (for all integers $n \geq 0$)?

Jerry in a split-second gave the following answer:

$$\frac{1}{2^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base 2)}, \tag{9}$$

$$\frac{1}{10^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base 10)}, \tag{10}$$

and in general:

$$\frac{1}{b^n-1} = 0.\overline{\underbrace{000\ldots00}_{n-1}\ \underbrace{1}_{1}} \quad \text{(in base } b\text{)}, \tag{11}$$

for any base $b > 1$.

Proof. The proof was very simple for Jerry. We consider a typical positive integer, say $n = 3$. We wish to prove that

$$\frac{1}{2^3-1} = 0.\overline{001} \quad \text{(in base 2)}. \tag{12}$$

Let $y = 0.\overline{001}$ in base 2. Then we have $2y = 0.\overline{010}$ and $4y = 0.\overline{100}$ (remember that in base 2, multiplication by 2 results in moving the 'decimal point' 1 place to the right, and multiplication by 4 results in moving the 'decimal point' 2 places to the right). Hence by addition we get

$$y + 2y + 4y = 0.\overline{111} = 0.\overline{1},$$

i.e., $7y = 1$. Hence $y = \frac{1}{7}$, so (12) is true. □

Jerry then came up with the following generic table where b is the base:

b	$\frac{1}{b^n+1}$	$\frac{1}{b^n}$	$\frac{1}{b^n-1}$
2	$0.\overline{\underbrace{000\dots00}_{n}\,\underbrace{111\dots11}_{n}}$	$0.\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}}$
3	$0.\overline{\underbrace{000\dots00}_{n}\,\underbrace{222\dots22}_{n}}$	$0.\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}}$
10	$0.\overline{\underbrace{000\dots00}_{n}\,\underbrace{999\dots99}_{n}}$	$0.\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}$	$0.\overline{\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}}$

And in general, for base $b > 1$:

$$\frac{1}{b^n+1} = 0.\overline{\underbrace{000\dots00}_{n}\,\underbrace{(b-1)(b-1)(b-1)\dots(b-1)(b-1)}_{n}},$$

$$\frac{1}{b^n} = 0.\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1},$$

$$\frac{1}{b^n-1} = 0.\overline{\underbrace{000\dots00}_{n-1}\,\underbrace{1}_{1}}.$$

Now it was Jerry's turn to revert. He posed the following challenge to Tom:

Challenge 3.

Can the relationship in the table be extended to non-unit fractions in base k ($k \geq 2$), i.e., to fractions whose numerator is not 1?

Tom is now looking to the readers to provide an answer.

Acknowledgements. I thank my mentor Rajesh Sadagopan (Director, *Aryabhatta Institute*) for providing guidance in writing this article. I thank my sister and mother for helping me edit this article.

ADITHYA RAJESH is 12 years old and loves exploring mathematics particularly, number patterns. He enjoys reading, solving puzzles, creating new games, playing the keyboard, chess, board games and listening to tunes. His research on mathematical aspects of Carnatic ragas may be read at Musical Math Recipe by Inspired Minds. His motivation and exploration of mathematics and related subjects are guided by Rajesh Sadagopan (Aryabhatta Institute), Dr Hemalatha Thiagarajan and Raising a Mathematician Foundation. He can be reached at adithehero07@gmail.com

A New Way of Looking at the Difference-of-Two-Squares Identity

BEDANTO BHATTACHARJEE & RIDDHI SARKAR

Day 1

Our maths teacher, Krittika Ma'am, gave us a very interesting topic to work on. She gave us a few patterns like this . . . .

1. $8^2 - 7^2 = 15$
2. $4^2 - 3^2 = 7$
3. $13^2 - 12^2 = 25$
4. $10^2 - 9^2 = 19 \ldots$

and told us to find a pattern between the LHS and RHS. We quickly figured out that we just need to add up the numbers to get to RHS. She pointed out that what we had on the LHS was a difference of squares of consecutive numbers (where the gap between the numbers is 1).

Day 2

Ma'am told us to work with a difference of squares where the gap is 2 instead of 1 and then with 3, and so on. We had to find a rule which holds true for all the cases. So, we all put on our thinking caps and came up with two different ideas.

Observations by the class. The difference of two squares is always the product of their sum and their difference.

$$\therefore a^2 - b^2 = (a+b)(a-b)$$

Example:

$$\text{(i)}\ 8^2 - 6^2 = (8-6)(8+6) = 2 \times 14 = 28$$
$$\text{(ii)}\ 13^2 - 12^2 = (13-12)(13+12) = 1 \times 25 = 25$$
$$\text{(iii)}\ 4^2 - 2^2 = (4-2)(4+2) = 2 \times 6 = 12$$
$$\text{(iv)}\ 10^2 - 5^2 = (10-5)(10+5) = 5 \times 15 = 75$$

Observations by Bedanto and Riddhi (for examples 1 & 3)

Bedanto	**Riddhi**
(i) $8^2 - 6^2 = 2 \times (8-6) \times (8-1) = 2 \times 2 \times 7 = 28$ (iii) $4^2 - 2^2 = 2 \times (4-2) \times (4-1) = 2 \times 2 \times 3 = 12$	(i) $8^2 - 6^2 = 2 \times (8-6) \times (6+1) = 2 \times 2 \times 7 = 28$ (iii) $4^2 - 2^2 = 2 \times (4-2) \times (2+1) = 2 \times 2 \times 3 = 12$

We then understood that we were arriving at the same product.

Now we tried our theory with the other examples, but we saw that they were not working.

For eg- $13^2 - 12^2 \neq 2 \times (13-12) \times (13-1)$; Also $13^2 - 12^2 \neq 2 \times (13-12) \times (12+1)$;

$\therefore$ We had to generalize it. We observed that in example (i) $8 - 1 = 6 + 1$ and in (iii) $4 - 1 = 2 + 1$

We realised, $8^2 - 6^2 = 2 \times (8-6) \times (6+1) = 2 \times \text{difference} \times \text{mean}$

And $4^2 - 2^2 = 2 \times (4-2) \times (4-1) = 2 \times \text{difference} \times \text{mean}$

After we had a generalized form, we tested our theory with other examples-

$$\therefore\ \text{(ii)}\ 13^2 - 12^2 = 2 \times \text{difference} \times \text{mean} = 2 \times (13-12) \times (12.5) = 25$$
$$\text{(iv)}\ 10^2 - 5^2 = 2 \times \text{difference} \times \text{mean} = 2 \times (10-5) \times (7.5) = 75$$

So, our conclusion was that- For, the difference of two squares, the difference is always twice the product of their difference and their mean.

$$\therefore a^2 - b^2 = 2 \times (a-b) \times (\text{mean of } a \text{ and } b)$$

We had a question to answer! How are the two methods related?

One of our classmates, Anurag, figured it out. . . Identity the class arrived at:

$$a^2 - b^2 = \underline{(a+b)(a-b)}$$

Our method-

$$a^2 - b^2 = 2 \times (a-b) \times (\text{mean of } a \text{ and } b) = 2 \times (a-b)\left(\frac{a+b}{2}\right) = (a+b)(a-b)$$

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Searching for Pythagorean Quadruples

ATHARV TAMBADE

A *Pythagorean quadruple* is a 4-tuple of positive integers a, b, c, d such that

$$a^2 + b^2 + c^2 = d^2. \qquad (1)$$

In this article we will try to find a way to generate Pythagorean quadruples.

Let $d = a + m$. Since $(a + m)^2 = a^2 + 2am + m^2$, if we can find integers such that

$$b^2 + c^2 = 2am + m^2, \qquad (2)$$

then the relation $a^2 + b^2 + c^2 = d^2$ will be satisfied.

We consider separately the cases when $b^2 + c^2$ is odd and when it is even.

Case 1: $\mathbf{b^2 + c^2}$ is odd. Then one of b^2 and c^2 is odd and other is even.

From (2) we get $2am + m^2 = b^2 + c^2$, hence

$$am = \frac{b^2 + c^2 - m^2}{2}. \qquad (3)$$

Since $b^2 + c^2$ is odd, by assumption, m^2 and therefore m is odd.

Keywords: Pythagorean quadruple, 4-tuple, odd, even

Next we have $d = a + m$, so $dm = am + m^2$, and so

$$dm = \frac{b^2 + c^2 - m^2}{2} + m^2 = \frac{b^2 + c^2 + m^2}{2}. \tag{4}$$

The quantity on the right side is not necessarily a multiple of m, so we scale up the 4-tuple (a, b, c, d) by a factor of m. That is, we consider instead the 4-tuple (am, bm, cm, dm). We may now generate infinitely many such 4-tuples (am, bm, cm, dm) using the identity

$$\left(\frac{b^2 + c^2 - m^2}{2}\right)^2 + (mb)^2 + (mc)^2 = \left(\frac{b^2 + c^2 + m^2}{2}\right)^2. \tag{5}$$

Example 1. Take $b = 2$, $c = 5$, $m = 3$. Then (5) yields:

$$10^2 + 6^2 + 15^2 = 19^2.$$

Example 2. Take $b = 3$, $c = 8$, $m = 5$. Then (5) yields:

$$24^2 + 15^2 + 40^2 = 49^2.$$

Case 2: $\mathbf{b^2 + c^2}$ is even. Essentially the same analysis works here too; we use identity (5) again. The only difference is that m must now be an even integer. If both b and c are even, then there will be a common factor which can be divided out from the 4-tuple.

Example 3. Take $b = 3$, $c = 7$, $m = 4$. Then (5) yields:

$$21^2 + 12^2 + 28^2 = 37^2.$$

Example 4. Take $b = 3$, $c = 11$, $m = 8$. Then (5) yields:

$$33^2 + 24^2 + 88^2 = 97^2.$$

Example 5. Take $b = 4$, $c = 14$, $m = 10$. Then (5) yields:

$$56^2 + 40^2 + 140^2 = 156^2.$$

We see that all the terms have a common factor of 4 which we may divide out. This yields:

$$14^2 + 10^2 + 35^2 = 39^2.$$

ATHARV TAMBADE is currently doing his B-Tech in engineering physics at IIT Bombay. He has a deep interest in both mathematics and physics. He has his own telescope which he uses for star gazing. He wishes to pursue a research-oriented career in pure science. He is also an NTSE scholar. He may be contacted at atharvtambade@gmail.com.

Problem 61 from KVPY 2016

KIAN SHAH

In this note, we look at Problem 61 from the Kishore Vaigyanik Protsahan Yojana (KVPY) entrance examination of 2016. Its statement is short, but finding a straightforward solution is a challenge. Here is a possible approach.

Problem. Suppose a is a real number such that $a^5 - a^3 + a = 2$. Show that $3 < a^6 < 4$.

Solution. From $a^5 - a^3 + a = 2$ we obtain $a\left(a^4 - a^2\right) = 2 - a$, i.e.,

$$a^4 - a^2 = \frac{2-a}{a}. \qquad (1)$$

Again from $a^5 - a^3 + a = 2$ we obtain:

$$a^6 - a^4 + a^2 = 2a. \qquad (2)$$

From (1) and (2) we obtain

$$a^6 = 2a + \frac{2}{a} - 1. \qquad (3)$$

Therefore, a is the real root of the equation $g(x) = f(x)$ where

$$g(x) = x^6, \qquad f(x) = 2x + \frac{2}{x} - 1. \qquad (4)$$

Keywords: KVPY, continuous function, real root

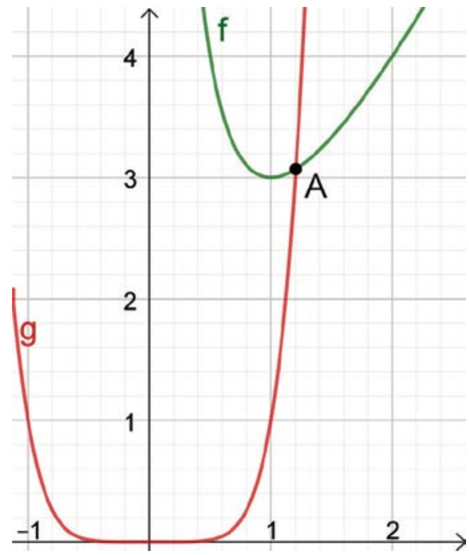

Figure 1. Graphs of g and f, over the interval $-1 < x < 2$

Now note the following.

- For $x > 0$, both $f(x)$ and $g(x)$ are continuous functions.
- $g(1) = 1, f(1) = 3$, and $g(2) = 64, f(2) = 4$; so $g(1) < f(1)$ while $g(2) > f(2)$. Hence $1 < a < 2$.
- The derivative of $f(x)$ is

$$f'(x) = 2 - \frac{2}{x^2}, \tag{5}$$

  which is strictly positive for $x > 1$. This implies that $f(x)$ is an increasing function for $x > 1$. (We can see this visually on the graph, but it needs to be verified rigorously. This can be done using the derivative, as we have just done.)

- Therefore $f(1) < f(a) < f(2)$, i.e., $3 < f(a) < 4$.
- But by the definition of a we have $f(a) = a^6$.
- Hence $3 < a^6 < 4$. □

About the Kishore Vaigyanik Protsahan Yojana (KVPY). For information about the KVPY, please refer to http://www.kvpy.iisc.ernet.in/main/index.htm.

KIAN SHAH is currently studying in 12th grade in Sahyadri School, Pune. He has a passion for mathematics, especially number theory. Apart from that he has an active interest in physics and enjoys solving problems from both these subjects. His curiosity for science was sparked by flipping through books on the history of science. He also occasionally plays the guitar. He may be contacted at kian.shah@sahyadrischool.org.

TechSpace

The Spaghetti Problem

JONAKI GHOSH

The topic of probability forms an important part of the school mathematics curriculum. It is introduced in middle school and is revisited in detail in senior secondary school. Though the concepts of probability can be taught using practical examples, these are usually restricted to tossing of coins, rolling of dice and shuffling playing cards. There are many interesting problems, which can be introduced to enliven the teaching of probability and to enable students to explore the fundamental concepts. In this article we shall explore such a problem called *The Spaghetti Problem* using the spreadsheet MS Excel.

In a previous article titled *The Game of Craps*, which appeared in the November 2019 issue, we explained the importance of simulations as a tool for modeling practical problems. In particular we mentioned *Monte Carlo Simulation,* a technique used to approximate the probability of outcomes of an experiment by running multiple trials generated by random numbers. Simple simulations of real world problems can be explored through spreadsheets such as MS Excel as these are equipped with inbuilt functions for generating random numbers.

The Problem

The problem is fairly simple and is stated as follows:

Let us assume that we are able to randomly break a spaghetti stick of length L into three pieces. What is the probability that the three pieces will form a triangle?

It would be worthwhile to take a spaghetti stick, break it into three pieces and arrange them to form a triangle. This may be repeated a few times using different spaghetti

Keywords: Probability, simulation, spreadsheets

sticks. In some cases the three pieces will form a triangle, while in other cases they will not.

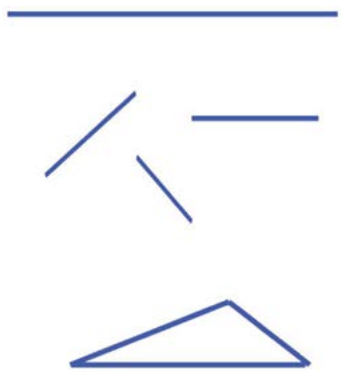

Figure 1. A Spaghetti stick of length L, broken into three pieces, forms a triangle.

In order to estimate the probability, we will need to break a large number of spaghetti sticks, each time checking if the three pieces form a triangle. After a sufficiently large number of trials, we would be in a position to calculate the empirical probability of a triangle being formed. However, it would be tedious and impractical to keep breaking spaghetti sticks. Simulation comes to the rescue here by allowing us to simulate the breaking of sticks using random numbers.

Before describing the process of simulation, let us analyse the problem mathematically.

Consider a stick of length L units, which is broken into three pieces. Let the length of the pieces be x units, y units and $(L - x - y)$ units. The triangle inequality tells us that these *three pieces will form a triangle if the sum of the lengths of any two pieces is greater than the third.* This leads us to the following three inequalities.

$$x + y > L - x - y \quad \text{or} \quad x + y > L/2 \qquad (1)$$

$$x + (L - x - y) > y \quad \text{or} \quad y < L/2 \qquad (2)$$

$$y + (L - x - y) > x \quad \text{or} \quad x < L/2 \qquad (3)$$

Also we observe that both x and y must lie between 0 and L. Let us consider the piece of length x units. If the stick is not broken at all, then x equals L units. If the stick is broken into two pieces, then one will be of length x units and the other will be y units. In this case x is less than L units. The same arguments may be repeated for the piece of length y units. This leads us to conclude that $0 < x, y < L$. Thus, every time a spaghetti stick is broken into three pieces, an ordered pair of numbers (x, y) is generated which can be easily plotted as a point on the coordinate plane. Also since x and y are greater than 0, all these points will lie in the first quadrant. Now, the inequalities (1), (2) and (3) represent those ordered pairs (x, y) which lead to the formation of a triangle. Graphing the linear equations $x + y = L/2$, $x + y = L$, $x = L/2$ and $y = L/2$ in the first quadrant leads to Figure 2.

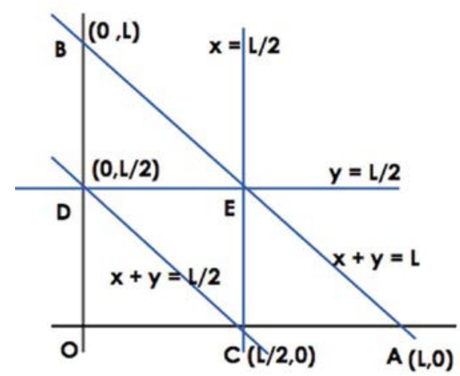

Figure 2: Graphical representation of the linear equations arising out of Spaghetti Problem.

Every time a spaghetti stick is broken, the corresponding ordered pair (x, y) will lie on or within the triangle OAB. This is primarily because of the condition $0 < x, y < L$. Now if we graph the inequalities (1), (2) and (3) simultaneously, the region common to them is represented by the triangle CDE. Hence, if we break a sufficiently large number of sticks such that their corresponding points (x, y) fill up triangle OAB, then the probability of a randomly broken stick (into three pieces) forming a triangle will be given by

Area of triangle CDE/Area of triangle OAB

$$= \frac{{}^{1}\!/_{2} \times {}^{L}\!/_{2} \times {}^{L}\!/_{2}}{{}^{1}\!/_{2} \times L \times L} = \frac{{}^{L^2}\!/_{8}}{{}^{L^2}\!/_{2}} = \frac{1}{4}$$

The above analysis confirms that the *theoretical probability of a spaghetti stick broken randomly, into three parts, forming a triangle is ¼.*

The spreadsheet simulation

In order to compute the empirical probability of a triangle being formed out of a randomly broken spaghetti stick, we shall simulate it on MS Excel using its random number generator.

Let us assume that the length of the stick, *L*, equals 100 units. Suppose that the stick is broken into three parts of lengths *a, b* and *c*. We will use the **RANDBETWEEN** command to simulate the lengths *a, b* and *c*. Note that =**RANDBETWEEN(1,** ***n***) generates a random integer between 1 and ***n***.

The steps of simulation are as follows.

Step 1: We begin by creating a column of numbers from 1 to 100 in column A. Enter **1** in cell **A2** and = **A2** + 1 in cell **A3**. Take the cursor to the corner of cell **A3** and drag till cell **A101**. This will create a column of numbers from 1 to 100 (as shown in Figure 3) and will help us keep track of the simulations.

	A
1	S.No
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8
10	9
11	10
12	11
13	12
14	13
15	14
16	15
17	16
18	17
19	18
20	19
21	20

Figure 3: A column of numbers to indicate the number of simulations.

Step 2: We will generate two random numbers between 1 and 100 and call them X and Y. These represent the points at which the stick is broken. For this we enter =**RANDBETWEEN(1, 100)** in cells **B2** and **C2**. The number in cell **B2** represents X and that in **C2** represents Y. Selecting cells **B2** and **C2** simultaneously and double clicking on the corner of cell **C2** generates 100 pairs of numbers X and Y.

	A	B	C
1	S.No	X	Y
2	1	29	55
3	2	18	20
4	3	23	46
5	4	37	3
6	5	18	21
7	6	35	46
8	7	70	1
9	8	27	45
10	9	15	17
11	10	27	16
12	11	69	79
13	12	4	86
14	13	99	47
15	14	56	76
16	15	60	37
17	16	35	88
18	17	95	35
19	18	32	64
20	19	23	55
21	20	17	88

Figure 4: The **RANDBETWEEN** command generates two numbers X and Y.

Step 3: The smaller of the two numbers X and Y may be assigned to **a**. To simulate this we enter the conditional formula

=IF(B2<C2, B2, C2) in **D2** (to simulate the length of piece '**a**') and double click in the corner of cell **D2**.

	A	B	C	D
1	S.No	X	Y	a
2	1	29	55	29
3	2	18	20	18
4	3	23	46	23
5	4	37	3	3
6	5	18	21	18
7	6	35	46	35
8	7	70	1	1
9	8	27	45	27
10	9	15	17	15
11	10	27	16	16
12	11	69	79	69
13	12	4	86	4
14	13	99	47	47
15	14	56	76	56
16	15	60	37	37
17	16	35	88	35
18	17	95	35	35
19	18	32	64	32
20	19	23	55	23
21	20	17	88	17

Figure 5: The conditional statement **=IF(B2<C2, B2, C2)** is used to simulate the piece of length '**a**'.

Step 4: Once X or Y has been assigned to **a**, the difference X – Y or Y – X will be assigned to **b.** This can be achieved as follows

Enter **=IF(B2<C2, C2-B2, B2-C2)** in **E2** (to simulate the length of piece **'b'**) and double click in the corner of cell **E2.**

	A	B	C	D	E
1	S.No	X	Y	a	b
2	1	29	55	29	26
3	2	18	20	18	2
4	3	23	46	23	23
5	4	37	3	3	34
6	5	18	21	18	3
7	6	35	46	35	11
8	7	70	1	1	69
9	8	27	45	27	18
10	9	15	17	15	2
11	10	27	16	16	11
12	11	69	79	69	10
13	12	4	86	4	82
14	13	99	47	47	52
15	14	56	76	56	20
16	15	60	37	37	23
17	16	35	88	35	53
18	17	95	35	35	60
19	18	32	64	32	32
20	19	23	55	23	32
21	20	17	88	17	71

Figure 6: The conditional statement **=IF(B2<C2, C2-B2, B2-C2)** is used to simulate the piece of length **'b'**.

Step 5: Finally, to simulate the length of the third piece c, we enter **=100 - (D2 + E2)** in **F2** and double click in the corner of cell **F2.** Note that each row of the spreadsheet represents one broken spaghetti stick of length 100 units.

	A	B	C	D	E	F
1	S.No	X	Y	a	b	c
2	1	29	55	29	26	45
3	2	18	20	18	2	80
4	3	23	46	23	23	54
5	4	37	3	3	34	63
6	5	18	21	18	3	79
7	6	35	46	35	11	54
8	7	70	1	1	69	30
9	8	27	45	27	18	55
10	9	15	17	15	2	83
11	10	27	16	16	11	73
12	11	69	79	69	10	21
13	12	4	86	4	82	14
14	13	99	47	47	52	1
15	14	56	76	56	20	24
16	15	60	37	37	23	40
17	16	35	88	35	53	12
18	17	95	35	35	60	5
19	18	32	64	32	32	36
20	19	23	55	23	32	45
21	20	17	88	17	71	12

Figure 7: The piece of length 'c' is simulated in column F.

Step 6: We now have a simulation of 100 spaghetti sticks, each broken into three pieces of lengths **a, b** and **c.** We need to identify those simulations in which a triangle is formed. The condition which we shall use is that a triangle will be formed only if the maximum length among **a, b** and **c** is less than 50 (half the length of the original stick).

Enter **=IF(MAX(D2,E2,F2)<50, "YES", "NO")** in **G2** (to check if the triangle is formed). The output in cell **G2** will be **"YES"** if a triangle is formed else it will be **"NO".**

	A	B	C	D	E	F	G
1	S.No	X	Y	a	b	c	Forms a triangle?
2	1	29	55	29	26	45	YES
3	2	18	20	18	2	80	NO
4	3	23	46	23	23	54	NO
5	4	37	3	3	34	63	NO
6	5	18	21	18	3	79	NO
7	6	35	46	35	11	54	NO
8	7	70	1	1	69	30	NO
9	8	27	45	27	18	55	NO
10	9	15	17	15	2	83	NO
11	10	27	16	16	11	73	NO
12	11	69	79	69	10	21	NO
13	12	4	86	4	82	14	NO
14	13	99	47	47	52	1	NO
15	14	56	76	56	20	24	NO
16	15	60	37	37	23	40	YES
17	16	35	88	35	53	12	NO
18	17	95	35	35	60	5	NO
19	18	32	64	32	32	36	YES
20	19	23	55	23	32	45	YES
21	20	17	88	17	71	12	NO

Figure 8: The conditional statement **=IF(MAX(D2,E2,F2)<50**, "YES", "NO") is used to verify if each row of the simulation represents the formation of a triangle.

Step 7: Finally we need to count the number of cases in which a triangle is formed (indicated by **"YES"** in column G) to compute the empirical probability. For this we enter **=COUNTIF(G2:G101,"YES")/100** in any cell, say **J2.**

The empirical probability = Number of broken spaghetti sticks which form a triangle / total number of spaghetti sticks that are broken

Every time we click on a cell on the spreadsheet, a new set of 100 simulations of broken spaghetti sticks is generated. Each time, the value in cell **J2** gives us the empirical probability.

The reader may explore the problem by generating several sets of 100 simulations, each time computing the empirical probability of obtaining a triangle. In some cases the empirical probability may be less than 0.25 and in other cases it may exceed 0.25. It would be worthwhile to generate 1000 simulations (instead of 100) and compute the empirical probability. After

	A	B	C	D	E	F	G	H	I	J
1	S.No	X	Y	a	b	c	Forms a triangle?			
2	1	29	55	29	26	45	YES		Probability	0.26
3	2	18	20	18	2	80	NO			
4	3	23	46	23	23	54	NO			
5	4	37	3	3	34	63	NO			
6	5	18	21	18	3	79	NO			
7	6	35	46	35	11	54	NO			
8	7	70	1	1	69	30	NO			
9	8	27	45	27	18	55	NO			
10	9	15	17	15	2	83	NO			
11	10	27	16	16	11	73	NO			
12	11	69	79	69	10	21	NO			
13	12	4	86	4	82	14	NO			
14	13	99	47	47	52	1	NO			
15	14	56	76	56	20	24	NO			
16	15	60	37	37	23	40	YES			
17	16	35	88	35	53	12	NO			
18	17	95	35	35	60	5	NO			
19	18	32	64	32	32	36	YES			
20	19	23	55	23	32	45	YES			

Figure 9: The empirical probability of a triangle being formed in this set of 100 simulations is equal to 0.26.

several simulations it will be evident that the value of the empirical probability gets closer to the theoretical probability, ¼, as the number of simulations increases.

An extension

A natural extension to the problem is to find the probability that a spaghetti stick of length L units, broken into four parts, forms a quadrilateral. We will analyse this situation using a method different from the three pieces case discussed in the previous sections of this article.

Let us break the stick of length L into four pieces such that one piece is of length $L/2$ and the sum of the lengths of the other three pieces is $L/2$. Clearly these four pieces, when joined together, will not form a quadrilateral. In fact a quadrilateral will be formed only when the length of the largest piece *is less than the sum of the other three pieces*. Thus we need to find the probability that no piece (among the four pieces) has length greater than or equal to $L/2$.

To find the required probability let us consider a circle of circumference L units. (We may imagine that our spaghetti stick is bent into a circle.) Let the four broken parts of the stick be n_1n_2, n_2n_3, n_3n_4 and n_4n_1 and let the four points n_1, n_2, n_3 and n_4 be randomly placed (in that order) on the circumference of the circle as shown in Figure 10.

Note that the circumference of the circle can be divided into two semi-circles. If the four points n_1, n_2, n_3 and n_4 lie on the same semi-circle then the four broken parts will not form a quadrilateral (as in such a case the length of the largest part will be greater than or equal to $L/2$). So, what is the probability of this happening?

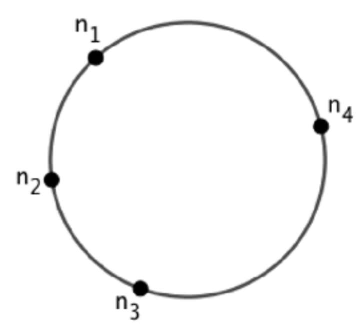

Figure 10. The circle represents a spaghetti stick broken into four parts n_1n_2, n_2n_3, n_3n_4 and n_4n_1.

Consider any one of the four points (say n_1) and the diameter of the circle passing through it, which divides the circumference into two semi-circles (as shown in Figure 11). Now the probability of n_2 lying on either semi-circle is ½ each. Similarly the probabilities of n_3 and n_4 lying on either semi-circle are also ½ each. Thus the probability that all three points, n_2, n_3 and n_4 *lie on the same semi-circle* is ½ × ½ × ½ = ⅛. Also, to begin with, we could have chosen any of the four points (n_1, n_2, n_3 or n_4) to draw a diameter and thus we have $4_{C_1} = 4$ choices.

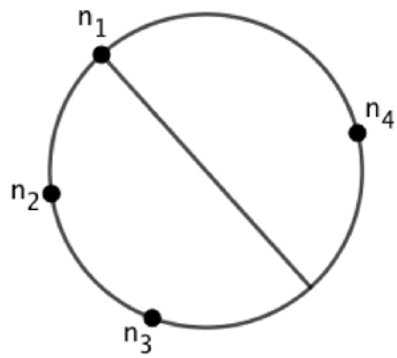

Figure 11. The diameter of the circle passing through n_1 divides the circumference into two semi-circles.

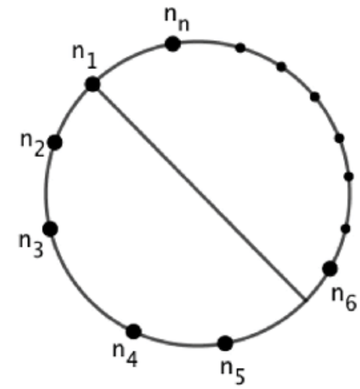

Figure 12. The circle represents a spaghetti stick randomly broken into n parts. The diameter through any one of the n points divides the circumference into two semi-circles.

This leads to the conclusion that the probability of all four points lying on the same semi-circle (in which case a quadrilateral cannot be formed) is $4_{C_1} \times \frac{1}{2^3} = 1/2$.

Hence, *the probability that all four points do not lie on the same semi-circle* (that is, *a quadrilateral is formed*) is 1 – ½ = ½.

The Generalisation

We are now in a position to generalize the spaghetti problem.

If a spaghetti stick is randomly broken into 'n' parts, what is the probability that these parts will form an n-sided polygon?

Clearly (extending from the four parts case discussed in the previous section), the polygon will be formed if the length of the largest part is less than the sum of the remaining (n – 1) parts. Once again, to obtain the solution, we need to find the probability that none of the n parts is of length greater than or equal to $L/2$. We can randomly place the n points on the circumference of a circle (of perimeter L units) as shown in Figure 12.

The reader is urged to prove the following result:

The probability that the n points (shown in Figure 12) lie on the same semicircle is $\frac{n_{C_1}}{2^{n-1}}$.

Hence, the probability that these n points do not lie on the same semi-circle (and that the n broken parts form an n-sided polygon) is $1 - \frac{n_{C_1}}{2^{n-1}}$.

This generalized formula is indeed useful. We can use it to verify the probability of the $n = 3$ case (three spaghetti pieces form a triangle with probability 1/4) and the probability of the $n = 4$ case (four spaghetti pieces form a quadrilateral with probability 1/2). The reader is encouraged to work out the probabilities for different values of n. It is interesting to note that as n increases, the probability of the n pieces forming an n-sided polygon also increases!

An interesting follow-up activity is to design a simulation of the $n = 4$ case on MS Excel and verify that the empirical probability of a quadrilateral being formed approaches the theoretical probability ½ as the number of trials increases. Students of secondary and senior secondary school can easily explore this problem. Students of grades 9 and 10 will be able to appreciate the initial analysis of the $n = 3$ case using the triangle inequality and also the spreadsheet simulation. Students of grades 11 and 12 with a stronger grounding in probability, are likely to be curious about the $n = 4$ case and the generalization of the problem. The discussion of the general case is well within the reach of most students.

Integrating Computational thinking and Mathematical Thinking

In recent years computational thinking (CT) has been identified as one of the key analytical abilities required for mathematics and science learning. The rapidly changing nature of scientific and mathematical disciplines and the need to prepare students for careers in these disciplines have been the primary motivation for bringing computational thinking into classroom practices. While many definitions of CT may be found in the literature, Seymour Papert [5,6,7] was the first to emphasise the importance of computational thinking and its connection to mathematics learning. Papert vividly talked about children using computers as instruments for learning and for enhancing creativity, innovation, and was actually responsible for "concretizing" the term computational thinking. With turtle geometry and logo programming he introduced a computational style of learning geometry. Later, in 2006, the term computational thinking was greatly popularized by Jeanette Wing [9] in her seminal article in which she advocated that

> Computational thinking is a fundamental skill for everyone, not just for computer scientists. To reading, writing, and arithmetic (the three R's), we should add computational thinking to every child's analytical ability.

She further elaborated on what encompasses computational thinking. According to her

> Computational thinking involves solving problems, designing systems, and understanding human behavior, by drawing on the concepts fundamental to computer science. Computational thinking includes a range of mental tools that reflect the breadth of the field of computer science.

Weintrop et.al. [8] attempt to define computational thinking in the context of school mathematics and science education and also suggest a theoretical grounding for the same. They propose a taxonomy comprising four categories: data practices, modeling and simulation practices, computational problem solving practices and systems thinking practices. All definitions or frameworks that define CT emphasise a certain set of skills. These include the ability to deal with challenging problems, representing ideas in computationally meaningful ways, creating abstractions for the problem at hand, breaking down problems into simpler ones, assessing the strengths and weaknesses of a representation system and engaging in multiple paths of inquiry. These skills are also critical for mathematics learning and there is a common consensus on the understanding that CT skills have to be developed in mathematics classrooms right from the early school years. It is also evident that among all school subjects, mathematics can provide ample contexts to integrate CT and computation and can in turn enrich mathematics learning through technology. Mathematical thinking (MT) skills and Computational thinking (CT) practices are distinct and yet mutually supportive.

However, operationalizing CT in mathematics classrooms is a pedagogical challenge. It requires identifying tasks that are both mathematically and computationally rich – tasks that allow students to explore concepts, develop and employ CT practices and also foster MT. Digital tools such as dynamic software, computer algebra systems and spreadsheets can play a significant role in mediating CT and MT. In fact spreadsheets can be very powerful tools for CT-MT based explorations as they can aid in identifying patterns, generating graphical representations and creating simulations. They are very suited for the inquiry based approach to learning and do not require high level coding skills. The spaghetti problem and other similar problems, which connect to various topics in the mathematics curriculum and are amenable to exploration via simulations, can be easily modeled on spreadsheets. The Spaghetti problem exploration may be divided into three stages. We may identify the role of MT and CT in each of these stages.

1. *Visualising the problem:* Applying the triangle inequality to model the problem and arriving at the inequalities (1) to (3) requires MT whereas modifying the problem for simulating it on the spreadsheet entails CT.
2. *Solving the problem:* Graphing the inequalities (1) to (3), arriving at the feasible region and computing the theoretical probability requires MT while simulating the problem on the spreadsheet (using random number generators and conditional statements) to obtain the empirical probability entails CT.
3. *Generalising the problem:* Extending the problem to 4 or n pieces and calculating the theoretical probability entails MT whereas extending the spreadsheet simulation to larger number of pieces requires CT.

We are now in a position to claim that exploration of the Spaghetti problem and other similar problems can help to foster both CT and MT. The reader is encouraged to read the articles *The Birthday Paradox* [3] and *The Monty Hall Problem* [4], which appeared in earlier issues of the magazine, to deliberate on possibilities offered by these problems for integration of CT and MT in the mathematics classroom.

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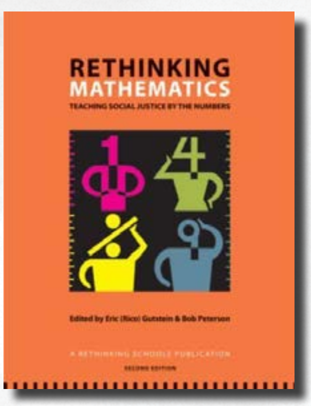
Review

Rethinking Mathematics: Teaching Social Justice by the Numbers

Reviewed by Prof. Parvin Sinclair



The first edition of the book under review was published in 2005[2], as part of the series, 'Rethinking Schools'. This was the same year that a new National Curriculum Framework (NCF) [3] was brought out in India. This NCF propagated the constructivist view of learning, which was a radical shift for most of the Indian schooling system and for Indian society. The NCF also stressed the fact that no learning is culture-free, including mathematics learning. This perspective has been spelt out by several other authors (e.g., see [1] and [4]). Rethinking Mathematics follows the same philosophy of learning. Of course, there have been many books and articles written across the world in the last few decades propagating the constructivist view of mathematics teaching. But this book goes a step further. The different points discussed in the chapters pertain specifically to examples built around various aspects of social and economic inequity. Through these articles, we see active embedding of social justice issues in the math teaching-learning process.

The articles in this compendium focus on helping teachers develop and transact a critical mathematics education curriculum intertwining mathematics with social justice. The contributors are mathematics student teachers, experienced teachers, teacher educators and education researchers. This makes for an interesting mix of theory and practical examples found in the book.

Keywords: social justice, stereotypes, school mathematics, mathematical thinking

Though the subtitle of the book says it is about teaching number, the examples pertain to developing algebraic, analytic, geometric and statistical thinking, along with learning to understand the society one lives in. In fact, the variety of examples cover mathematics learning at all levels, including in informal settings like out-of-school learning situations and adult education.

This book is placed in the US context of increasing inequity due to several reasons, particularly privatisation of school education, high-stakes evaluation and a greater digital divide, which we find in India too! It comprises 32 chapters, divided into three parts. In the first part, the authors look at what one of them, Marilyn Frankenstein, calls 'Reading the World with Math'. The discussion in this part is about the broad societal issues, and why they need to be linked with the math curriculum. Here you will also find a discussion on using math to uncover social biases, using it to understand the racial issues in the US, using it to go beyond newspaper headlines and to read between the lines.

In Part Two, the articles look at ways of engaging learners with issues of social justice while teaching them mathematics. For instance, in one of the chapters, an activity requires the learners to consider the current unemployment situation in different categories, and using this the teacher introduces them to rates, percentages, proportion, and different stages of data handling. In another article in this part, the author tells us about a unit on proportional reasoning in which the learners look at whether an equal amount of contribution to a kitty is fair to all those contributing to it.

The third part comprises articles that consider ways of going beyond teaching math using social justice contexts to infuse this into other curricular areas. For instance, in Chapter 29, Peterson has written about a unit developed around action research undertaken by the learners on which presidents of the US were involved with slavery, and to what extent. He tells us how the students' understanding developed while studying math fed into their understanding of the social studies curriculum. In fact, the math classroom discussion helped students notice how the social science textbooks had deliberately omitted facts about the presidents being slave owners!

The book has a fourth part, which gives quite a collection of resources, including different websites. Of course, as this book is already about a decade old, the references need updating.

This book actively challenges the stereotype of mathematics—math is neutral, math is not really connected with our everyday lives, every problem has only one solution, etc. —through examples that force the learners to think and critique the social, economic and political environment. I see this book as being a good resource, not just for teacher development programmes, but also for supporting any adult who wants to teach mathematics. Any of the examples in this book taken up in a workshop discussion, could help teachers generate a variety of examples. For us in India, the huge social, economic, and digital divides throw up so much that needs to be questioned and discussed by learners, including those studying mathematics.

There are other books that have come out more recently in this area too (e. g., [5] and [6]). We need to see some books or website materials like these developed for the local contexts here.

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The Magic Behind the Trick

Source: https://images.app.goo.gl/7WuUvS2RhpNmAeLd9

This is an act (see page **53**) that I saw in the program of a magician and the mathematics behind it is very beautiful.

In this act, there are always 20 coins in the tray but the audience does not know this.

Suppose the spectator takes x coins. (Remember, $x < 10$)

So, the number of coins remaining in the tray $= 20 - x$. Now, $20 - x$ is a number between 10 and 20. So, it can be written as $10 + (10 - x)$. The sum of the digits of this number will be $= 1 + 10 - x = 11 - x$

In the next step, the spectator therefore adds $11 - x$ coins to the coins already in his right fist.

So, the number of coins in his right fist $= 11 - x + x = 11$, and this always happens, provided he takes less than 10 coins the first time.

So the magician knows that there will be 9 coins left in the tray. After the spectator takes a random number of coins (say y coins) in his left fist, then there are $9-y$ coins left in the tray. So the magician subtracts the number of coins he sees are left in the tray from 9, and gives the number of coins in the left fist. Of course, the number of coins in the right fist is fixed at 11.

Contributed by: Kalpesh Akhani (Asst.Teacher)

Manipulative Review: Geoboard

Reviewed by Math Space

Unlike many other teaching learning materials (TLM) for math, which we would like to call mat(h)erials, geoboard is very well known among teachers and referred to in teacher education programs, pre-service in particular. It is a board – wooden or plastic, with many pegs or nails stuck on it. One can stretch a rubber band along some of these pegs to create many polygons. The pegs can be arranged in a rectangular array or in a few concentric circles (Figure 1).

Geoboards can be used to create shapes. The rectangular array version is a good precursor to the rectangular dot sheet. Check the Geometry and Geometry II Pullouts (At Right Angles, November 2014 and March 2015) for activities transitioning from tactile experiences, to drawing on paper, which are appropriate for young children to explore shapes (Figure 2). Geoboard is a good precursor to dot sheets because it provides a tactile experience, in which shapes can be changed much faster, and polygons are guaranteed straight sides. Moreover, it can be lifted up and displayed to the whole class.

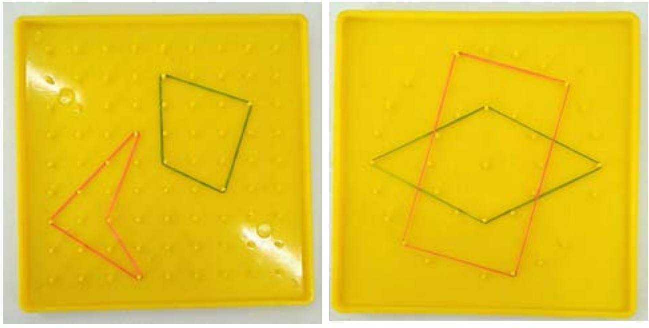

Figure 1

Keywords: Geoboard, shapes, identification, exploration, reasoning, justification

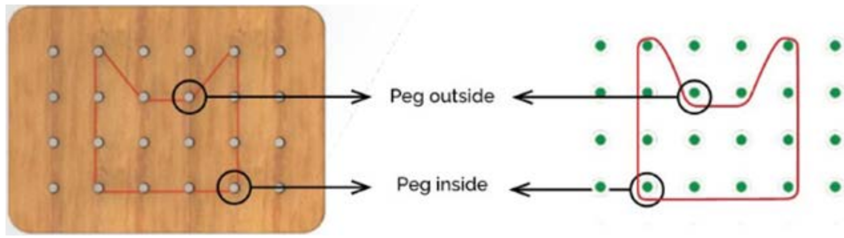

Figure 2 from Sikkim textbook

At the primary level, i.e., Class 1-5, it can help with the following concepts:

- Polygons, i.e., closed 2D shapes with only straight edges
- Corners and edges – especially arriving at the relation between number of corners and number of sides of a polygon
- Identifying and creating rectangles in different orientations
- Exploring angles – possibly with two rubber bands acting as the two arms of the angle
- Comparing angles

Similarly, at the upper primary or middle school level, i.e., Class 6-8, geoboard can be used for the following concepts:

- Types of triangles – children can be asked to identify and/or make different types of triangles; these can be done on the circular geoboard as well
- Types of quadrilaterals – similar to the above
- Diagonals – can be shown with different coloured rubber bands
- Concave and convex polygons – Are all pegs inside the rubber band? What is special about the peg or vertex that is outside the rubber band?

Stretchability of regular rubber bands may be a limitation, and many rubber bands may snap while playing with a geoboard. Also, the pegs, because they are fixed, may come in the way. This can be resolved by using a softboard with pushpins replacing the pegs.

However, a geoboard illustrates the notion of concave and convex shapes perfectly. In a geoboard one can make a polygon and its envelope using two differently coloured rubber bands. Then it becomes clear if the envelope is congruent to the polygon in question. For a concave polygon, some of the sides of the enveloping polygon (shown in pink) will not match those of the concave one (shown in green). These unmatched sides of the envelope are diagonals of the concave polygon lying outside (Figure 3). Also, the vertex untouched by the envelope is special. The internal angle at the vertex is a reflex angle. In a physical geoboard, the peg corresponding to such a vertex lies outside the boundary of the concave polygon created by the rubber band, while the remaining vertices corresponding to angles < 180° lie inside.

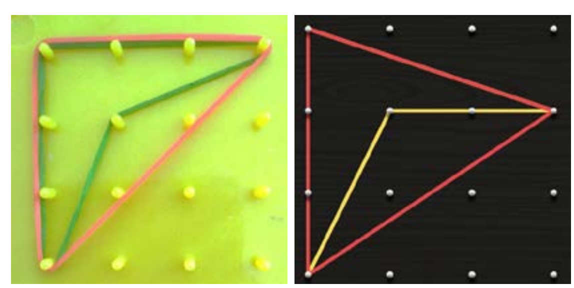

Figure 3

Virtual geoboards,such as https://apps.mathlearningcenter.org/geoboard/, are also available and a good substitute for the physical ones. They resemble the rectangular dot sheet more and the tactile aspect is reduced a bit. However, there is no restriction on the elasticity of the rubber bands and no chance of tearing those. They are available in multiple colours and can be stretched between just two points resembling a line segment much better than in

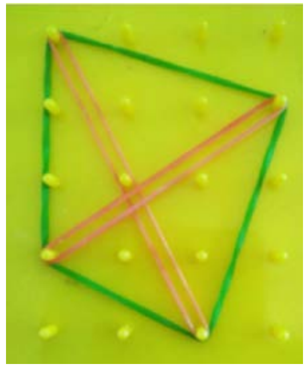

Figure 4

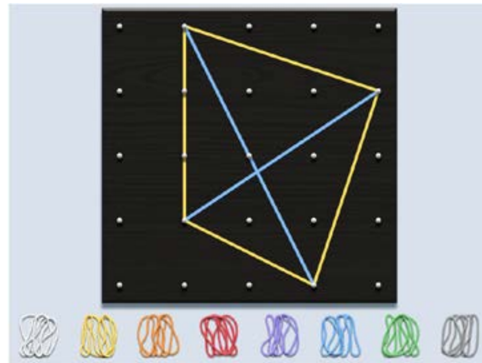

Figure 5

a physical geoboard. So, it can show the diagonals of a polygon much better than its physical counterpart. Also, the pegs don't come in the way (Figures 4 and 5).

However, the virtual geoboard doesn't distinguish a vertex corresponding to a reflex angle in a concave polygon as the physical geoboard does (Figure 6). Also, the rubber bands overlap exactly, hiding the one below completely. Check how the red envelope hides some of the (yellow) sides of the concave quad (Figure 3). It is uncertain if the virtual geoboard can be reprogrammed to address these issues.

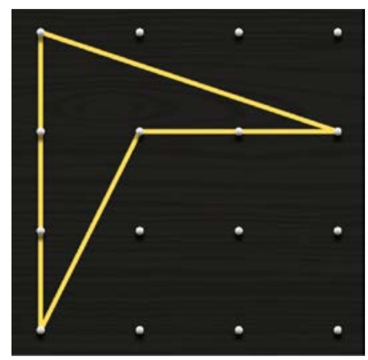

Figure 6. (w/o red envelope)

Reference

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- Geometry II pullout: http://publications.azimpremjifoundation.org/1644/1/17_Teaching%20geometry%20-%20II.pdf
- Sikkim SCERT Math textbook, Class 2, pp.172-174: https://online.fliphtml5.com/iuwdn/kgob/#p=172

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Worksheet based on Geoboard (Class 6)

1. **Use the rectangular array geoboard and make the following triangles**
   - Make a square. Now release any one corner to get a triangle. Which type of triangle did you get? Consider both types of triangles, i.e., those classified by either side or angle.
   - Now make a rectangle. Again, release any one corner to get a triangle. How is this triangle similar to the earlier one? How are the two different?
   - Use different rubber bands to make as many isosceles triangles as you can keeping the base fixed. How does the angle opposite to the base change as you increase the height of the triangle?
   - Can you make an equilateral triangle on this geoboard? Why?
2. **Take the circular geoboard.**
   - Can you make an equilateral triangle here? Can you justify using symmetry?
   - Keep one vertex of the equilateral triangle fixed along with the line of symmetry passing through it. Change the remaining pair of vertices so that you get an isosceles triangle with the same line of symmetry.
   - Use different rubber bands to make as many isosceles triangles as you want with the same line of symmetry. Write down your observations.
3. **Use the rectangular geoboard and make the following squares and rectangles**
   - A square whose sides are tilted
   - A rectangle whose sides are tilted
   - A square whose sides are tilted by an angle less than 45°
   - A rectangle whose sides are tilted by an angle more than 45°
4. **Use the rectangular geoboard and make the following trapeziums**
   - An isosceles trapezium
     i. What shape do you get if you extend the sides?
     ii. Is it possible that no pair of opposite sides ever meet? When?
   - A trapezium with a right angle and an acute angle
     i. What do you observe about the remaining angles?
     ii. How many right angles are there?
   - A trapezium with two acute angles which are opposite to each other
     i. What shape do you get if you extend the sides?
     ii. Which quadrilateral do you get if the acute angles are equal?
5. **Use the rectangular geoboard and make the following quadrilaterals**
   - A kite whose halving diagonal is shorter than the halved diagonal
   - A kite whose equal angles are right angles [Hint: Can you use 3D cleverly?]
   - A kite with exactly one right angle
   - A kite with two unequal obtuse angles
     i. What type of angles are the equal ones?
     ii. If the unequal angles are acute, what type of angles are the equal ones?
   - A concave quadrilateral that has line symmetry
   - A concave quadrilateral without line symmetry

Send in your pictures and narratives (or those of your students) to MathSpace@apu.edu.in. We'd love to hear from you!

A Geometry Problem from IMO 2016

SHREYAS ADIGA

In this article, we solve a geometry problem from the International Mathematics Olympiad (IMO) 2016 (Hong Kong). It was proposed by Art Waeterschoot from Belgium, who received an honourable mention during the IMO 2015 (Thailand). The problem was given to the problem-solving group of our school.

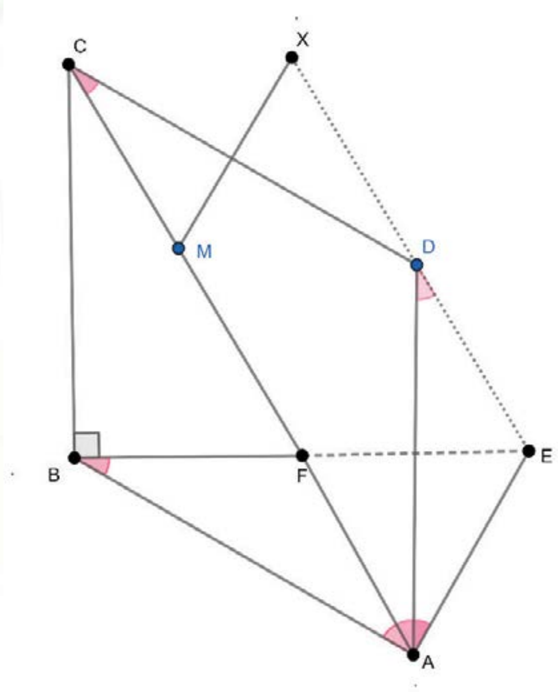

Figure 1

Keywords: Triangle, parallelogram, concurrent, radical axis, concyclic, IMO, INMO, USAMO

Problem 1. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen such that $DA = DC$ and AC is the bisector of $\measuredangle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\measuredangle EAC$.

Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$).

Prove that lines BD, FX and ME are concurrent.

Solution

We find three circles such that BD, FX, ME are the respective radical axes associated with pairs of these circles. Then we apply the radical axis theorem.

From the data of the problem, we can denote $\measuredangle CAB = \measuredangle FAD = \measuredangle DAE = \measuredangle EDA = \theta$.

The main claims in the proof are the following:

- Points D, F, B, C, X are concyclic.
- Points B, A, E, D, M are concyclic.
- Points E, F, M, X are concyclic.
  - **Claim 1**: Points E, D, X are collinear.

    Proof: Since AD is the angle bisector of $\measuredangle EAC$, we have $\measuredangle CAD = \measuredangle EAD$.

    Since $ED = AE$, $\triangle AED$ is isosceles and hence $\measuredangle EAD = \measuredangle EDA$. Thus $\measuredangle CAD = \measuredangle EDA$.

    This implies that $ED \parallel AC$.

    Now, as $AMXE$ is a parallelogram, $EX \parallel AC$. It follows that E, D, X are collinear (Figure 1).

  - **Claim 2**: Points D, F, B, C are concyclic.

    Proof: From the observation that $\triangle ABF$ is similar to $\triangle ACD$, we have,

    $$\frac{AB}{AC} = \frac{AF}{AD}.$$

    Next, observe that $\measuredangle BAF = \measuredangle FAD$. It therefore follows that $\triangle ABC$ is similar to $\triangle AFD$.

    We further note that $\measuredangle AFD = \measuredangle ABC = 90^\circ + \theta$. Since $\measuredangle DCF = \theta$ and $\measuredangle AFD$ is the exterior angle of $\triangle CFD$, we conclude that $\measuredangle FDC = 90^\circ$ and hence B, C, D, F are concyclic with CF as diameter. Denote this circle by Γ_1 (Figure 2).

  - **Claim 3**: B, E, F are collinear.

    Proof: Observe that $\triangle CDA$ is isosceles, with $\measuredangle DCA = \measuredangle DAC = \theta$.

    From the previous claim, we know that B, C, D, F are concyclic. Hence $\measuredangle FBD = \measuredangle FCD = \theta$ (as these are angles on the same segment FD).

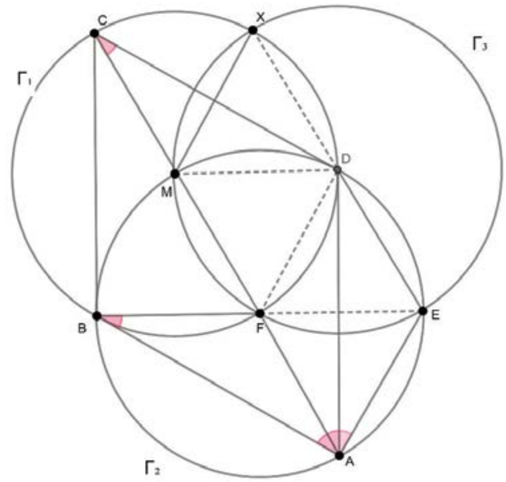

Figure 2

From the observation that $\triangle ABF$ is similar to $\triangle ADE$ we have:

$$\frac{AB}{AD} = \frac{AF}{AE}.$$

Since $\measuredangle BAD = \measuredangle FAE$, it follows that $\triangle ABD$ is similar to $\triangle AFE$.

Now $\measuredangle AFE = \measuredangle ABD = \theta + \measuredangle FBD = \theta + \measuredangle FCD = \theta + \theta = 2\theta$.

But in $\triangle ABF$, $\measuredangle AFE = 2\theta = 180 - \measuredangle BFA$. This proves the claim (Figure 1).

- **Claim 4**: A, B, M, D are concyclic.

  Proof: Since B, C, D, F are concyclic (from Claim 2) with M as centre of the circumcircle of the $\triangle FBC$, $\measuredangle AMD = \measuredangle FMD = 2\measuredangle ACD = 2\theta$.

  Again, B, C, D, F being concyclic implies that $\measuredangle EBD = \theta$, and hence $ABD = \measuredangle ABE + \measuredangle EBD = \theta + \theta = 2\theta$.

  We conclude that A, B, M, D are concyclic. Denote this circle by $\mathbf{\Gamma}_2$ (Figure 2).

- **Claim 5**: E lies on $\mathbf{\Gamma}_2$.

  Proof: From Claim 2, we know that B, C, D, F are concyclic, so $\measuredangle FBD = \measuredangle FCD = \theta$.

  Since E, F, B are collinear, $\measuredangle EBD = \measuredangle FBD = \theta$.

  But $\measuredangle EAD = \theta$. Thus E, A, B, D are concyclic.

  Now, from Claim 4, A, B, M, D are concyclic and hence E lies on $\mathbf{\Gamma}_2$.

- **Claim 6**: X lies on $\mathbf{\Gamma}_1$.

  Proof: From the observation that $MDEA$ is cyclic, it follows that $\measuredangle MDX = \measuredangle EAM$. Since $AMXE$ is a parallelogram, $\measuredangle EAM = \measuredangle DXM$.

  Therefore, $\triangle MDX$ is isosceles and so $MD = MX$, thus proving the claim (Figure 2).

- **Claim 7**: M, F, E, X are concyclic.

  Proof: We observe that $\angle AFE = 2\theta$ (this is the exterior angle for the triangle AFB). So $EF = EA$.

  Since $AEXM$ is a parallelogram, $EA = MX$. Therefore $EF = MX$ and so $MFEX$ is an isosceles trapezium. It follows that M, F, E, X are concyclic (an isosceles trapezium is always cyclic). Let us denote circle $MFEX$ by Γ_3.

  Now observe that BD, FX, ME are the radical axes of the circle pairs (Γ_1, Γ_2), (Γ_3, Γ_1), (Γ_2, Γ_3). By applying the radical axis theorem, we get the desired result. (The Radical Axis Theorem states: *The three pairwise radical axes of three circles concur at a point*. The point where the lines meet is called the 'radical centre' of the three circles.)

Two problems for the reader

Problem 2 (USAMO 1997)**.** Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

(Here, USAMO refers to the Mathematics Olympiad held in USA.)

Problem 3 (IMO 1995)**.** Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. Line XY meets BC at Z. Let P be a point on line XY other than Z. Line CP intersects the circle with diameter AC at C and M, and line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

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Roots in Coefficients

PRITHWIJIT DE

We start with a simple observation. The quadratic $x^2 + x - 2$ factors as

$$x^2 + x - 2 = (x-1)(x+2) \qquad (1)$$

and we see that its roots are 1 and -2 which are same as the coefficient of x and the constant term in $x^2 + x - 2$. This begets the question:

Is this the only quadratic polynomial with integer coefficients whose roots are the same as the coefficient of the linear term and the constant term?

Let's see how we might answer this. A general quadratic polynomial has the form

$$ax^2 + bx + c$$

where a, b and c are the coefficients and $a \neq 0$. It is *monic* if $a = 1$. (More generally, a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is called *monic* if $a_n = 1$.) If the roots of $ax^2 + bx + c$ where $a \neq 0$ are α and β then

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

Observe that if a, b and c are integers and we want the roots of $ax^2 + bx + c$ to be integers, then a must divide both b and c. This would reduce the quadratic to

$$a(x^2 + px + q)$$

which is a constant multiple of a monic quadratic polynomial. Here the integers p and q satisfy

$$ap = b, \qquad aq = c.$$

Keywords: Quadratic, cubic, factor, roots, coefficient, monic

Thus, finding integers a, b, c such that the roots of $ax^2 + bx + c$ are integers reduces to finding integers p and q such that the roots of $x^2 + px + q$ are integers. So henceforth we will limit our search to monic quadratic polynomials. The question stated earlier may thus be restated more formally as

For which integers p and q is it true that the roots of $x^2 + px + q$ are p and q?

Here the sum of the roots is $p + q$ and their product is pq. But then

$$p + q = -p, \qquad pq = q,$$

whence $(p, q) = (0, 0)$ or $(1, -2)$. Thus there are two such quadratic polynomials, namely, x^2 and $x^2 + x - 2$, whose roots are the same as the coefficient of x and the constant term.

Here we make another observation. If we multiply x^2 and $x^2 + x - 2$ by x^m where m is a positive integer, we obtain two new polynomials of degree $m + 2$ with their roots belonging to the set of coefficients. But each of them has 0 as a root and many terms with zero coefficients. To avoid zero coefficients, let us consider only polynomials with non-zero coefficients. This also rules out zero as a root since the roots are required to be in the set of coefficients.

One is tempted to ask the same question for monic cubic and higher degree polynomials. Let us deal with the cubic case first.

We are looking for non-zero integers a, b, c such that the roots of $x^3 + ax^2 + bx + c$ are a, b, c. This means that the relation

$$x^3 + ax^2 + bx + c = (x - a)(x - b)(x - c) \qquad (2)$$

is an identity in x. Equating the coefficients of like terms, we get

$$a = -(a + b + c), \qquad b = ab + bc + ca, \qquad c = -abc. \qquad (3)$$

Since $c \neq 0$, $ab = -1$, and as a, b are integers, $a, b \in \{-1, 1\}$ and they have opposite signs. Hence $a + b = 0$. Therefore

$$b = ab + c(a + b) = -1 + 0 = -1$$

and $a = 1$, which shows that $c = -(2a + b) = -1$. Thus the cubic is

$$x^3 + x^2 - x - 1 = (x - 1)(x + 1)^2$$

with roots $(a, b, c) = (1, -1, -1)$.

What happens if the degree of the polynomial is greater than 3? Can we find such a polynomial of degree n for every natural number $n > 3$? Let us investigate.

Let

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \cdots + a_1x + a_0$$

be a polynomial of degree n with non-zero integer coefficients, $n > 3$. Suppose the roots are $a_0, a_1, \ldots, a_{n-2}, a_{n-1}$. Then, by Vieta's theorem,

$$\left.\begin{array}{lcl} a_0 + a_1 + \cdots + a_{n-2} + a_{n-1} & = & -a_{n-1}, \\ a_0a_1 + a_0a_2 + \cdots + a_{n-2}a_{n-1} & = & a_{n-2}, \\ a_0a_1 \ldots a_{n-2}a_{n-1} & = & (-1)^n a_0. \end{array}\right\} \quad (4)$$

Since $a_0 \neq 0$,we have $a_1a_2 \ldots a_{n-2}a_{n-1} = (-1)^n$, and as each a_k for $k \in \{1, 2, \ldots, n-2, n-1\}$ is an integer, it must be that $a_k \in \{-1, 1\}$.

Now observe that

$$\begin{aligned} & a_0^2 + a_1^2 + \cdots + a_{n-1}^2 \\ & = (a_0 + a_1 + \cdots + a_{n-1})^2 - 2(a_0a_1 + a_0a_2 + \cdots + a_{n-2}a_{n-1}) \\ & = a_{n-1}^2 - 2a_{n-2} && (5) \\ & \leq 3. && (6) \end{aligned}$$

But $a_k^2 = 1$ for $k \in \{1, 2, \ldots, n-1\}$ yields $n \leq 4 - a_0^2$. As $n > 3$ we get $a_0^2 < 1$ forcing $a_0 = 0$, a contradiction.

This shows that there does not exist such a polynomial of degree n for $n > 3$.

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Printed at Suprabha ColorGrafix (P) Ltd., No. 10, 11, 11-A, J.C. Industrial Area, Yelachenahalli, Kanakapura Road, Bengaluru 560062.
Published at Azim Premji University, Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562 125;
Editor: Shailesh Shirali

Tech Space

A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
AtRiA.editor@apu.edu.in

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

The Closing Bracket . . .

This column has typically featured teachers who have been innovative problem solvers. The introduction of the element of computational thinking in NEP 2020 has made several teachers begin to explore why and how computational thinking can improve their pedagogy, apart from the value addition to their students' thinking and problem-solving skills.

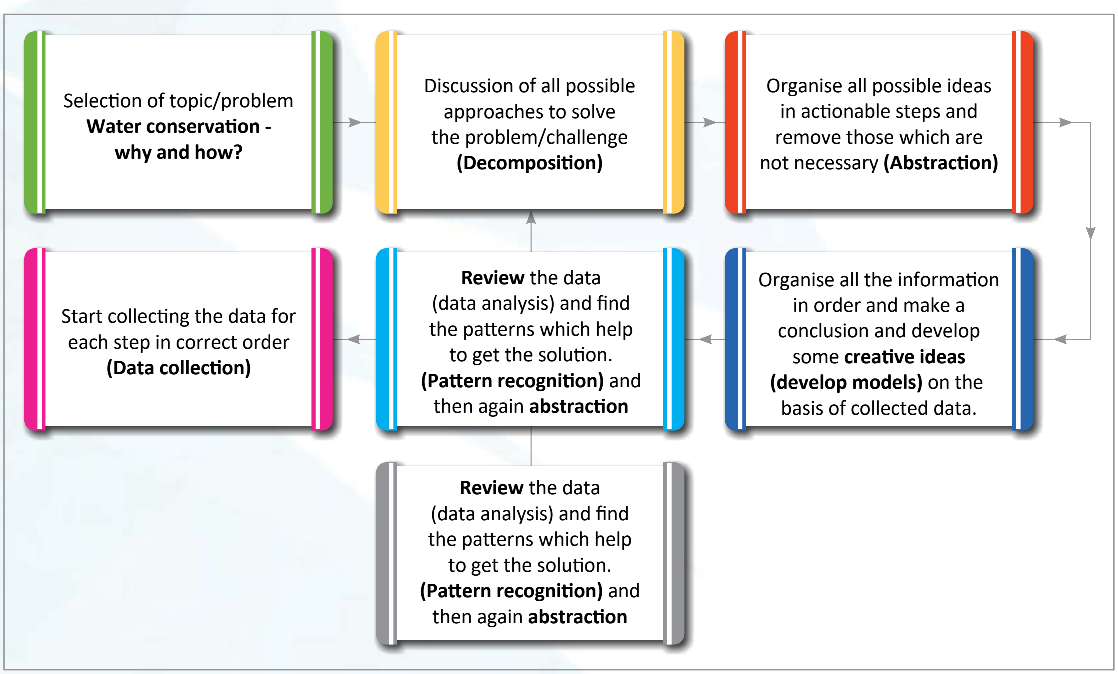
CSpathshala (https://cspathshala.org/) is an ACM India (Association for Computing Machinery India) initiative to bring a modern computing curriculum to Indian schools with 400,000 students learning CT (2/3rd from govt schools in rural areas). The CTiS (Computational Thinking in Schools https://india.hosting.acm.org/CTiS/) conference is an annual event organised by the CSpathshala community to provide a platform for teachers, educators and experts to share their best practices as well as challenges faced in implementing computational thinking in education. The 4th edition -CTiS 2022- will be held in Pune on 8th and 9th July. The conference has received 180+ abstracts from educators and teachers across the country and also from Singapore, Thailand and Brazil. The teacher accounts in this narrative are based on some of these abstracts. There are many pioneers in this effort both in India and abroad and clearly, teachers have risen to the challenge to push new frontiers and set new standards in pedagogy and student empowerment.

Sunita Maurya and Deepa Sharma are both teachers from the School of Scholars, Nagpur. Their abstract was based on the class project on the topic of "**Water Conservation: Why and How?**" based on the lesson Water- the precious resource of Std -VII.

In their own words, *we followed the cycle of* ***discussion, experimentation,*** *and* ***reviews*** *until we got the outcome. In every discussion, we discussed the outcomes followed by abstraction (to pick the necessary information from the collected data) and analysis of it, and again reworked.*

What is heartening was that the anticipated outcome was not the development of computational thinking. From their abstract- *In this project-based learning, our objective was to* ***sensitise*** *all the students about* ***water conservation by developing awareness about different sources of water, how people around the world are facing water crisis, and what we can do at the individual level.***

Clearly, they met their objective - *On an average, every student was able to save around* ***100 to 150 litres of water daily*** *and they concluded that if everyone saves a small amount of water then we can minimise*



Student Project: Water Conservation- Why and How?

the global issues of shortage of water. Look at this list drawn up by the students at the end of the project.

Suggestions- Where to minimise the use of water

1. Do not keep the tap on while brushing teeth or washing utensils.
2. Use a bucket and mug instead of a shower.
3. Minimise the use of washing machines.
4. Repair a leaking tap immediately.
5. Reuse the water wherever possible, such as: water used for mopping can be used for washing floors, drained water from water filters can be used for watering plants and other household work.

Suggestions for new models which help to save water

1. **Model-1** Use of plastic bottles for watering plants - we can hang one at some height and fix a drip tube near the roots or
2. **Model-2** Fix any bottle with holes in it into the soil and fill water in it for watering.
3. **Model-3** They also suggested developing some devices like a regulator in every tap and shower which will help to control the usage of water as per the need.

At the same time, these teachers were mindful of the value of computational thinking.

- *Students followed all the steps of the CT while doing this project. The CT approach improves the student's engagement in the learning process in very systematic ways.*
- *Activity 3 was actually time-consuming. And here we applied the CT skill to complete the project by decomposing the task into smaller tasks.*
- *We discussed and motivated the students to focus on patterns, such as how many times activities are repeated in a day, which container is used multiple times, and for what purpose. All these activities were very helpful in estimating the usage of water.*
- *With the help of algorithms learned in CT, they tried to write an algorithm on how to work on any project-based concepts.*

Clearly, computational thinking is intertwined with several disciplines, as other abstracts also showed. Teacher S. Sreedevi of Dr. B.R. Ambedkar Gurukulam, Palnadu, and teachers M Purna Bhavani and V Syamala Gowri of Dr. B.R. Ambedkar Gurukulam, Bapatla used the steps of computational thinking in student explorations of the Triangle Inequality theorem and in understanding the formula for the sum of the first n odd numbers, respectively. Both these schools come under the fold of APSWREIS (Andhra Pradesh Social Welfare Residential Educational Institutions Society) which serves students from severely disadvantaged backgrounds. On the other hand, teacher Sheetal Marwade (again from the School of Scholars) used CT to teach Sanskrit grammar and vocabulary, decomposing words phonetically and observing patterns in forming new words. A most interesting project on Quiet Time was carried out in Dr. Kalmadi Shama Rao High School in Pune by teachers Pallavi Naik and Pallavi Iyer. The problems of teaching and functioning in a noisy school was decomposed into factors over which they had no control (the school was in a busy traffic zone) and those over which they did (ambient noise). As students analysed patterns in both factors, they understood how they could work towards a quiet and peaceful environment. The school implemented a Quiet Time at the end of the working day- *One of the strongest takeaways from the implementation of quiet time was that the noise levels that were initially at 85-90 dB at the end of the day dramatically dropped to around 72-75 dB during the QT. One Std 9 student said that the Quiet time helped her to introspect about her day and think about ways to improve. She said that it helped her become more aware of what she was doing. A Std 6 student said that there is so much noise all day, that they look forward to the quiet time at the end of school and like to carry that feeling of peace home with them.*

Let's think about this!

Sneha Titus
Associate Editor

At Right Angles
A Resource for School Mathematics

SPATIAL THINKING WITH 3-D OBJECTS

PADMAPRIYA SHIRALI

Azim Premji University
A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

SPATIAL THINKING WITH 3-D OBJECTS

Our brains have adapted over the long period of evolution to a 3-D world and we are able to assess our location and position in space in relation to other objects. When we look at an object depicted on paper (2-D), we automatically construct a 3-D image. We often fail to notice that several actions that we perform in our daily life involve spatial thinking and understanding. As I navigate through the map on my phone or fill up a refrigerator with containers of various sizes, I use spatial understanding. All of us rely on spatial thinking much more than we realise.

A lot of the geometry we study in school is about 2-D shapes and relationships amongst these shapes and their attributes. However, we live in a 3-D world which is even richer in intricate geometric facts and relations. To work with these spaces requires a good understanding of the properties of these objects and ways of visualisation and abstraction. Some concepts we use in 2-D geometry may not apply to 3-D shapes. For example, the shortest path connecting two points on a sphere is not the straight line connecting the two points, and this has implications for air travel.

What is spatial thinking? It is the way the brain processes the position and shape of an object in space. It is through spatial thinking that we understand the location and dimensions of objects, and how different objects relate to each other. It is through such thinking that we construct mental images of objects and visualise them.

The double helix is a famous example of a result of spatial thinking which meets certain requirements. It is a complex 3-D structure with two parallel but displaced spiralling chains.

What does spatial thinking involve? Is it a single skill? Or is it multiple skills? The following are clearly involved:

1. Abstracting the necessary and crucial information (distance, length, coordinates, dimension).
2. Focusing on a certain object embedded in a complicated background and noticing the relationships between the objects and suppressing information not relevant to a task.
3. Representing a design (different views, knowledge of projections, graphs, maps).
4. Scaling an object up or down, or manipulating it in some way.
5. Visualising rotations or symmetry.
6. Visualising an object with a single fold or double fold.
7. Navigating.
8. Memory, synthesis, filling a missing link (closure).
9. Making deductions, evaluating (for actions like taking a detour).

It is thinking which involves the concept of space, tools for representation and a process of reasoning.

Figure 1

HOW IMPORTANT IS SPATIAL THINKING?

MRI-based research of the brain has revealed that the part of the brain which becomes active during tasks involving spatial thinking is the same as the part used while solving mathematics problems. Spatial thinking can be improved through training and exposure to related tasks.

Many specialists use spatial thinking in their work areas. A civil engineer or an architect performs this feat while designing a building. It is the same capacity that helps a surgeon to navigate the human body and a pilot to fly an aircraft.

While spatial perception and spatial understanding is fundamental to the human thinking process, it poses a challenge in many ways. Our spatial perception can be fooled quite easily, and there are many such puzzles which challenge our perception of objects. Here are two such disturbing examples!

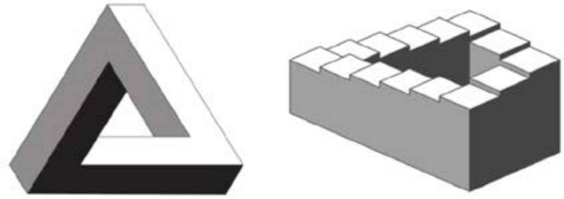


Figure 2

BUILDING SPATIAL THINKING

At primary school level, various activities that involve usage of spatial language, gestures for directional movements, symmetry patterns, reading maps, playing with tangrams are all part of an attempt to build spatial thinking abilities in 2-D space.

Spatial thinking in 3-D must similarly involve handling of 3-D objects, manipulating and studying them. Understanding the location (position) and dimensions (such as the length and size) of 3-D objects, and studying how different objects are related to each other is an important part of this study. It involves building or constructing these objects with blocks or plasticine and clay, studying the nets of such structures, working on paper and pencil tasks, and using geometric software.

While we manage our daily lives with the spatial sense we have developed over time, coping with the complex spatial problems of the world today requires us to use GIS or 'Geographic Information System' technology.

Spatial thinking can be integrated into various subjects as it has relevance in many areas. However, it is good to study it in a 3-D context as a separate unit. Also, regular polyhedra have such beauty to them that it would be a great pity not to make their study a part of the geometry curriculum.

Da Vinci was one person from history who had a tremendous capacity for visualisation. Sculptors like Michelangelo used it when they visualized a future sculpture trapped inside a lump of stone.

Note: Various activities involving usage of 3-D objects can be given to students in a scaffolded manner to build their visualisation skills. The activities incorporate several minor skills like determining and comparing direction, orientation, location, distance, size, colour, shape, and other attributes. Some initial activities can also be used in primary school.

More advanced activities will use major skills like changing perspective (reference frame), changing orientation (mental rotation), transforming shapes, changing size, moving wholes and reconfiguring parts.

Spatial thinking involves visualizing relations, imagining transformations from one scale to another, mentally rotating an object to look at its other sides, creating a new viewing angle or perspective, and remembering images in places and spaces. Spatial thinking also allows us to externalize these operations by creating representations such as a map.

This article focuses on simple mathematical objects in 3-D to develop the capacity to visualize and abstract out their properties.

ACTIVITY 1

Objective: Building a model of a given 3-D construction
Materials: Interlocking cubes, complex structure made with cubes

Let students work in pairs. One student builds a complex 3-D structure. The other student must observe the construction carefully and build a similar one (same size, colour combination and orientation as the one built by the partner).

Can they describe their shape using spatial language (top, left, at right angles, parallel to, …)?

Figure 3

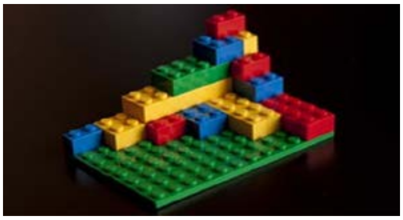


Figure 4

Building complex models helps the students to stay focused on the subtler features of a problem. They need to notice the colour, the length, perceive the parts and the whole and their relationships.

They will also need to notice the angles and the orientation.

ACTIVITY 2

Objective: Building a model using a picture of a 3-D construction

The ability to interpret a drawing, visualise the hidden cubes and reconstruct a model takes the student to the second level of the spatial understanding process.

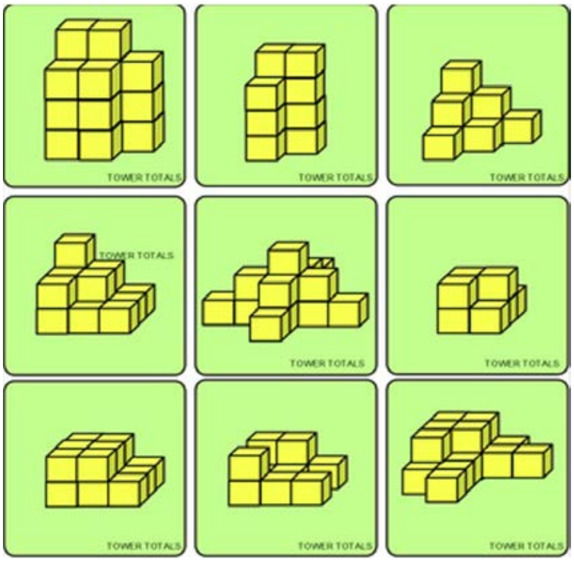


Figure 5

Given some pictures of models, can the students reproduce the models with blocks accurately (assuming no missing pieces)?

Can they figure out the number of blocks they would need for each of these models before building them?

How accurate is their reasoning?

What are the special features of each model?

Does it taper upwards? Does it have symmetry?

Will it look the same if it is rotated?

Figure 6

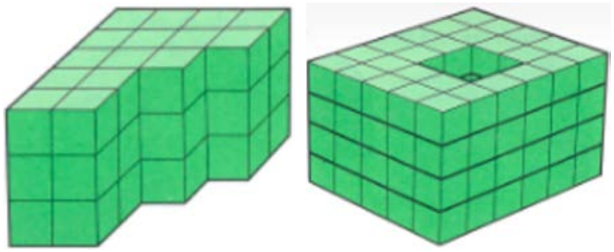


Figure 7

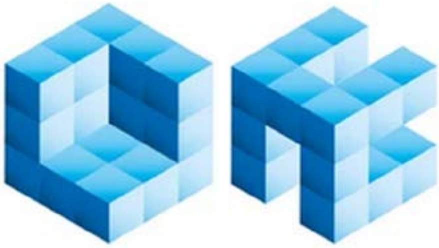


Figure 8

What is the length of the base? What is the breadth of the base? What is the height at its highest point?

How would they figure out the volume of an irregular object?

Can the students compute the volume for these irregular models? What are the different approaches they might use?

ACTIVITY 3

Objective: Building 3-D structures with plasticine/ clay and straw/ toothpicks
Materials: Straw/Toothpicks and Plasticine/Clay, Chart displaying prisms and pyramids with names.

Let students build 3-D structures in pairs to create different 3-D objects. Let them explore the shapes and record the data about vertices, edges and faces.

Figure 9

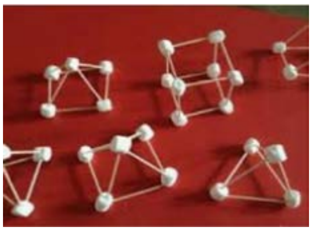


Figure 10

At the end the student pairs can record and consolidate their findings in a table as follows:

Object	Vertices	Edges	Faces
Pyramid with a square base	5	8	5
Triangular prism			
…			

Students should discuss their findings at the end.

How do the numbers of vertices, edges and faces relate to each other? Is there a pattern?

Let students explore ways of figuring out the surface areas of these objects; e.g., using their nets.

How would one find the volumes of these objects? Can the students come up with some ideas?

Project

Pyramids have held a great fascination for many civilisations. Egyptians used them as burial tombs. One such pyramid is the great pyramid of Giza. It measures nearly 480 ft in height and 750 feet at the base and has a slope of 50°. Students can build a scale model of this pyramid and study its features.

If you were to walk around the base, how far would you walk?

What shape would emerge if you sliced a pyramid in half horizontally? Vertically through the apex?

Figure 11

ACTIVITY 4

Objective: Designing the nets for simple 3-D objects
Materials: Prism and pyramid shaped objects

Figure 12

Given the pictures of some 3-D objects, can the students draw the nets?

Here are some possible nets of a square pyramid.

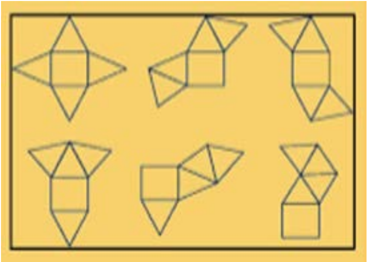


Figure 13

Students will need to clearly know the difference between a pyramid and a prism. A pyramid has one pointed end; slant edges connect it to all the vertices of a polygonal base. The base can be a triangle, producing a *tetrahedron;* it can be a square, giving rise to a *square pyramid;* or it can be a pentagon, or a hexagon, …. There are therefore infinitely many types of pyramids. There is no limit to the number of sides the base can have.

A prism has the same face at both ends. The sides of the face can vary from 3 to any number.

Students tend to have a fixed idea of prisms as they see prisms mainly in the physics lab. However, a prism can have a multi sided base. It can even be L-shaped as shown here.

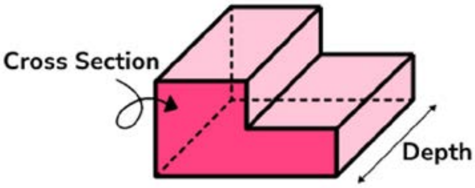


Figure 14

A prism can be cut into layers parallel to one side and all the layers will be exactly the same as shown in the figure.

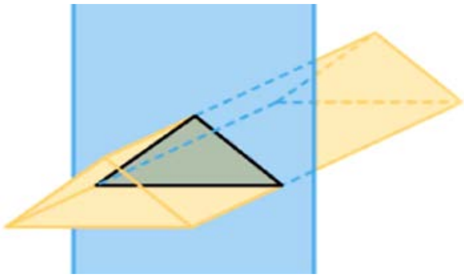


Figure 15

In contrast, a pyramid cannot be cut into layers which are identical to one another. The green square in the figure is not the same as the base of the pyramid.

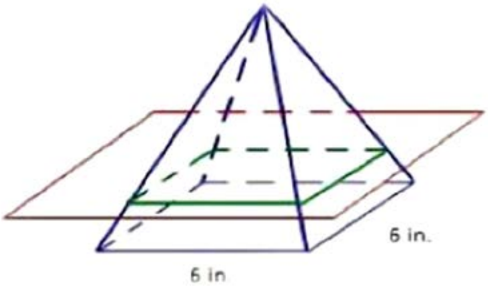


Figure 16

ACTIVITY 5

Objective: Drawing 3-D objects or constructions on isometric/triangular paper

Let the students make drawings (isometric sketches) of different solids on isometric paper.

Here are two such isometric sketches:

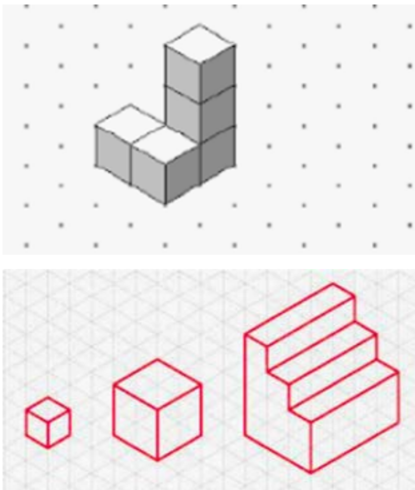


Figure 17

Students can build a variety of different structures and draw them on dot paper. The skill of representing 3-D in 2-D form needs to be built up gradually and students will need hand holding.

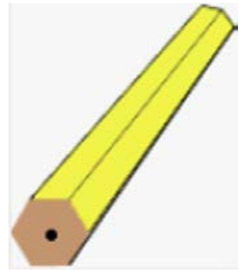


Figure 18

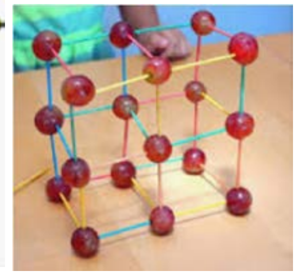


Figure 19

After making the isometric sketch, the students should compare it with the object to check whether the two correspond exactly. A few more complex ones:

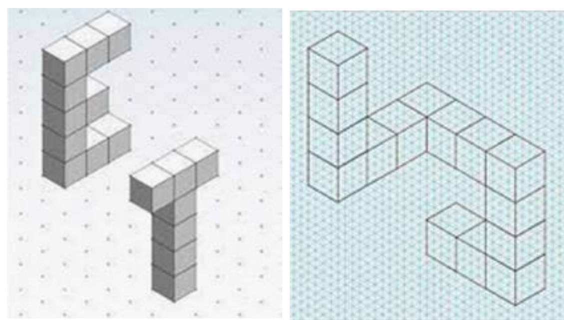


Figure 20

ACTIVITY 6

Objective: Visualising shapes from nets
Materials: Different nets with colour patterns or numbering

If this net is folded, what shape will the object have?

To come up with the answer, students need to form a mental picture of the prism being folded.

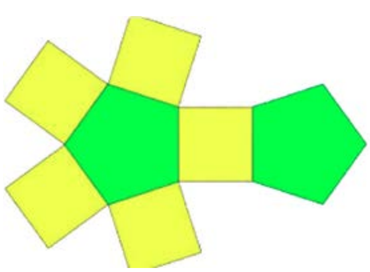


Figure 21

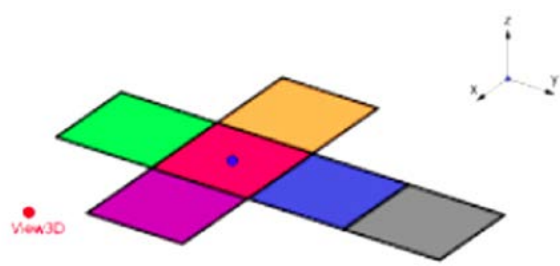


Figure 22

While doing so, they must keep track of the relative positions of the different coloured sides. What shape will be opposite the pentagon?

If the net in Figure 22 is folded, what coloured square will be opposite the pink square? What coloured squares will be adjacent to the green square?

What shape will the net in Figure 23 give rise to?

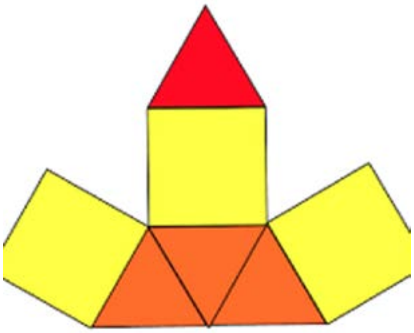


Figure 23

ACTIVITY 7

Objective: Sketching views of simple 3-D structures with two objects
Materials: Couple of blocks arranged touching each other

Drawing the top view, front view and side view is a skill which develops gradually and it is important to start with a few simple objects. (Note: Isometric sketches are made on isometric dot sheets, as shown above, while the views are generally made on square or rectangular grid sheets.)

Here are a few sample drawings.

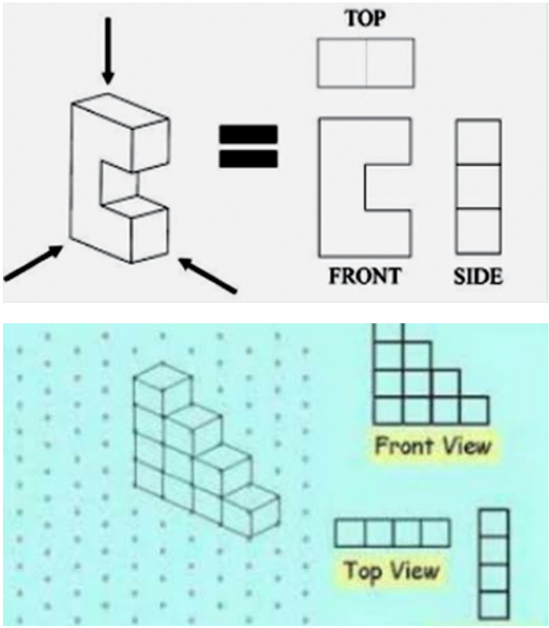


Figure 24

Let the student look at it from the top and draw a top view. It can be followed by a front view and a side view. They can use either isometric or square grid paper to aid in the drawing process.

Figure 25

One can increase the complexity of the structure gradually. A few more examples of drawings on dot paper are shown in Figure 26.

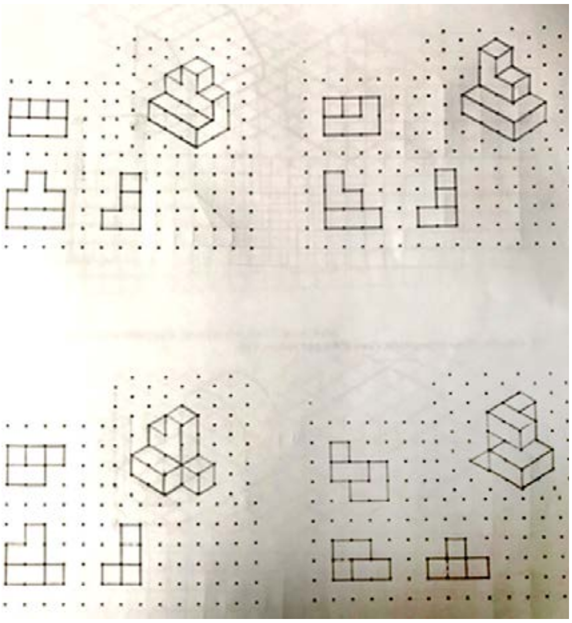


Figure 26

Views

They can study various views of cube structures as shown here.

A Rubik cube will make a very good model for such drawings. Many matching exercises can be created.

Challenge!

Construct a structure made up of 8 identical cubes and having the largest possible surface area.

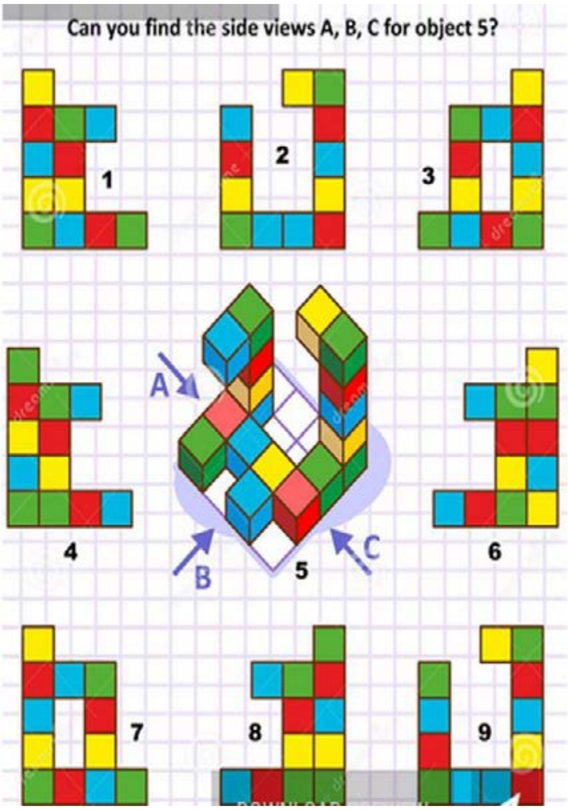


Figure 27

ACTIVITY 8

Objective: Understanding polyhedra and regular polyhedra
Materials: Various mathematical 3-D objects of different sizes or a chart containing pictures of various 3-D mathematical objects
Vocabulary: face, edge, vertex, polygon, polyhedron, regular, convex

Initiate a discussion on sorting the collection into two sets. Students may sort them on the basis of curved surface and plane surface.

Discuss why the word 'polyhedron' is used to refer to an object with polygonal faces.

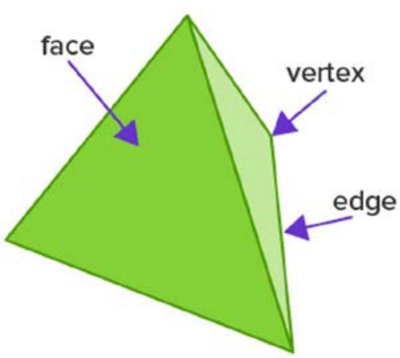


Figure 28

The word 'Poly' refers to *many* and 'hedra' refers to *faces*. Polyhedron means a 'many-faced object'. (The word polygon similarly refers to a many-angled shape.)

Polyhedra are objects which are 3-D, have polygonal flat faces, straight edges and vertices where three or more faces meet. We will consider only *convex polyhedra;* they have no surface indentations or holes.

Cubes and prisms are examples of polyhedra. Cylinders and spheres are not polyhedra.

Can the students now attempt at sorting the polyhedra into different categories? They will notice that in addition to prisms and pyramids, there are other objects which are also polyhedra.

They will see that some of these objects are highly symmetric: they look the same looking down at each face and looking down at each vertex. Their faces are regular, congruent polygons. These are the *regular polyhedra*. They are also known as *platonic solids*.

A polyhedron can fail to be regular in many ways. For example, its faces may not all be congruent copies of one another; rather, the faces may be regular polygons with different numbers of sides (there are many such polyhedra, highly symmetric in appearance). Or the polyhedron may not be convex, i.e., it may have indentations.

(Note: It is not necessary to bring in the notion of polyhedral angles here.)

Discuss, experiment and discover: Raise questions which help students discover that a minimum of 3 faces need to meet at a vertex to form a closed shape.

How many equilateral triangles can meet at a vertex?

Students will notice that if they have six equilateral triangles meeting at a vertex, the triangles will lie flat. Can they justify why it is so?

How many squares can come together at a vertex?

How many regular pentagons can come together at a vertex?

Can regular hexagons come together at a vertex? Why or why not?

This can lead to the discovery that at any vertex of a convex 3-D polyhedron, the sum of all the angles will always be less than 360 degrees. Can they generalise the result?

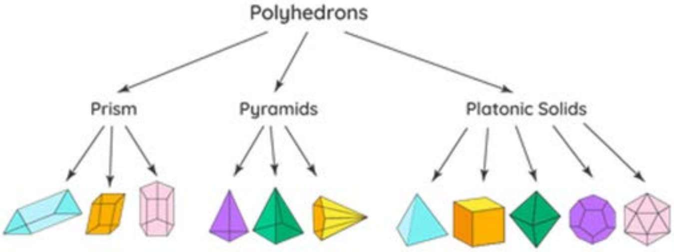


Figure 29

Rotating the objects

The students will need to be given explicit instructions while rotating an object initially. They first start the process by physically rotating them.

At the second stage, they try to rotate the object using their mind's eye.

What would the object look like if it was tilted by 45°? 90°? 120°?

Give students some pictures of pairs of rotated objects.

Ask: Are the two objects different? Or are they actually the same, merely oriented differently?

It's great fun to turn these objects into art pieces by colouring them or drawing patterns on them.

Plane symmetry and rotational symmetry

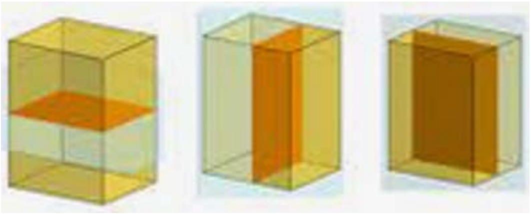


Figure 30

Discuss plane symmetry with examples.

An object has *plane symmetry* if it can be divided into two halves by a plane so that each half is a reflection of the other across the plane.

Such a plane is called a *plane of symmetry*.

A cuboid has 3 such planes of symmetry.

How many planes of symmetry does a cube have?

Discuss *rotational symmetry* with examples.

Practical demonstration is advisable, by piercing holes and passing a straw (or a taut thread, or a wire) through the faces or corners of the paper models.

If a 3-D figure is turned around a fixed line, it is called a *rotation*.

Objects that look the same after a certain amount of rotation are said to have *rotational symmetry*.

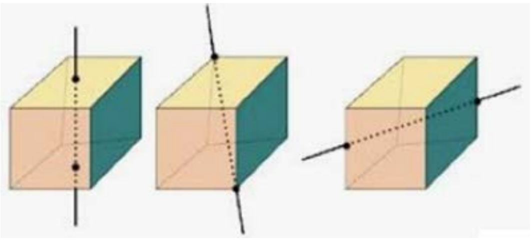


Figure 31

Rotational symmetry is measured in terms of 'order'. When we rotate an object like a cube through 360° about the axis connecting the centres of a pair of opposite faces, the cube fits exactly onto itself four times: after rotations of 90°, 180°, 270° and 360°; so this is called rotational symmetry of order 4. The axis about which it is rotated is called the axis of rotational symmetry. The order of rotational symmetry of a triangular pyramid would be 3, as it fits onto itself after rotations of 120°, 240° and 360°.

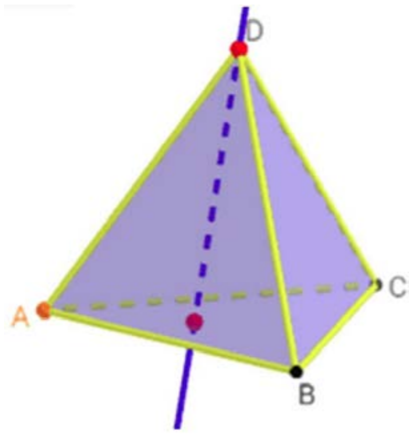


Figure 32

ACTIVITY 9

Objective: Study of regular polyhedra (tetrahedron)
Materials: Straws and plasticine/thread

Thus, the chief reason for studying regular polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense.

– H S M Coxeter

What closed structure can be built with equilateral triangles where every vertex has 3 adjacent triangles?

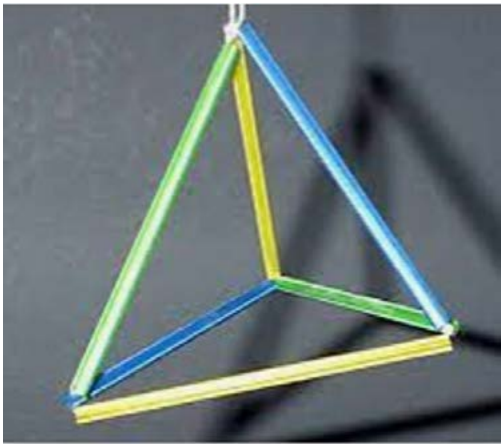


Figure 33

Let students build 3 equilateral triangles about a vertex, using straws.

They will see that they have built a regular tetrahedron (4-faced polyhedron).

Has it formed a closed figure?

Verify that there are 3 triangular faces at each vertex.

What types of symmetry does it have?

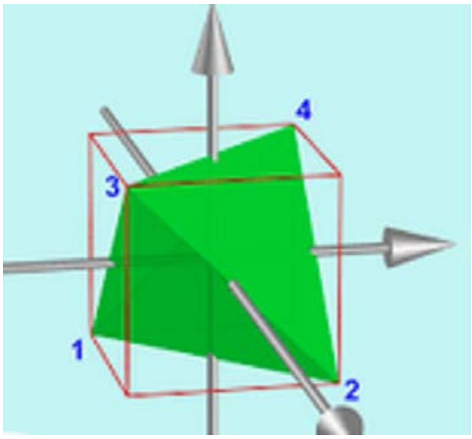


Figure 34

Does it have any plane symmetry? What will the plane pass through? How many such planes can you find in the regular tetrahedron?

Net: Students should be encouraged to design nets for a tetrahedron. They can design and fold the net to make a solid shape.

Pass straws (or wires or taut threads) through holes to check for rotational symmetry.

Does it have any rotational symmetry? Through which points does the axis of symmetry pass?

What is its order?

A **Dihedral meter** (a flexible L-shaped angle measure) can be used to measure dihedral angles (this is the angle at which adjacent faces meet). Students should be able to make such devices for themselves.

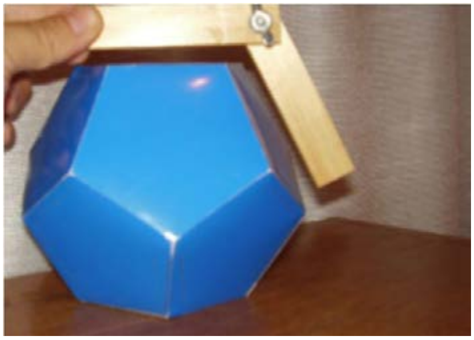


Figure 35

In Figure 35 it is being used to measure the angle between the faces of a dodecahedron.

Students can explore such objects in a variety of different ways, bringing different skills to these explorations:

- They can measure the angles between surfaces using a dihedral meter.
- They can study the relationship of the side to the surface area.
- They can generate different views of the object and sketch them.

There are many resources available online which reinforce their skills and knowledge (e.g., Figure 36).

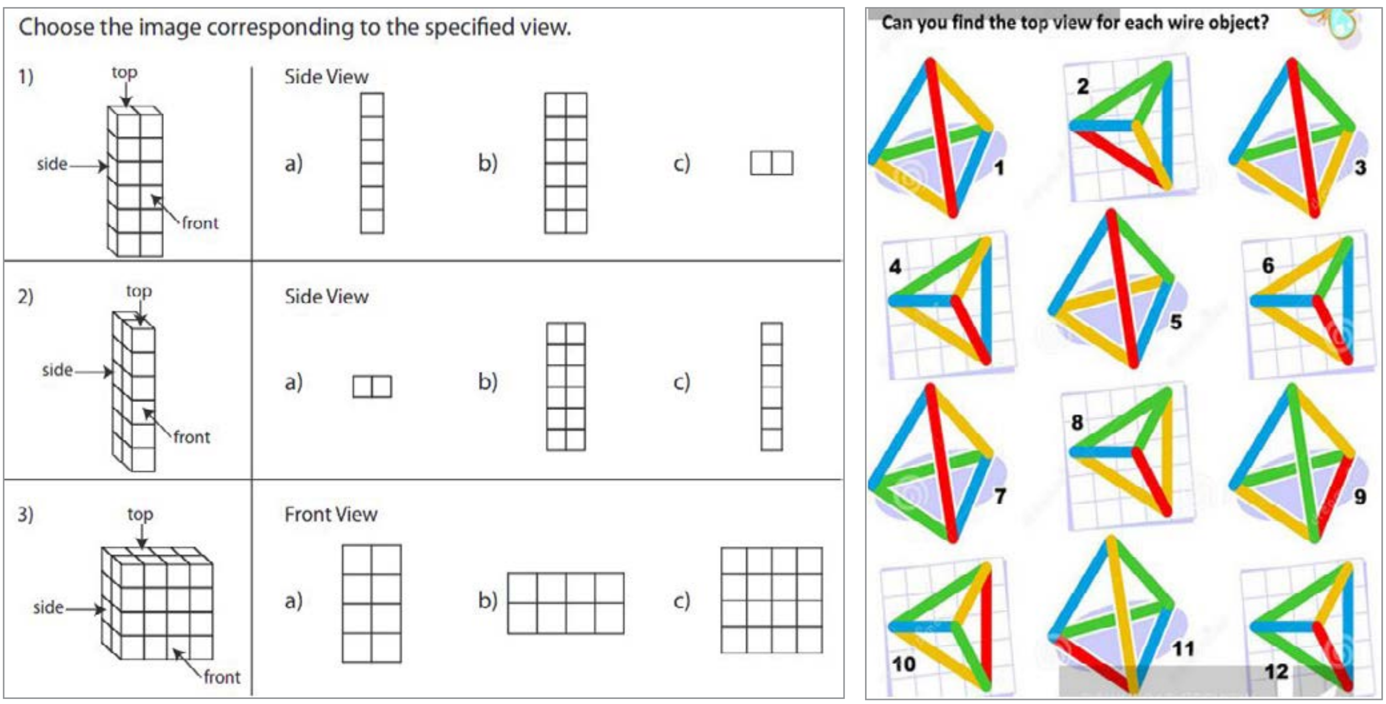


Figure 36

Optional exploration

Cross sections: Students can also explore horizontal or vertical cross sections of the objects

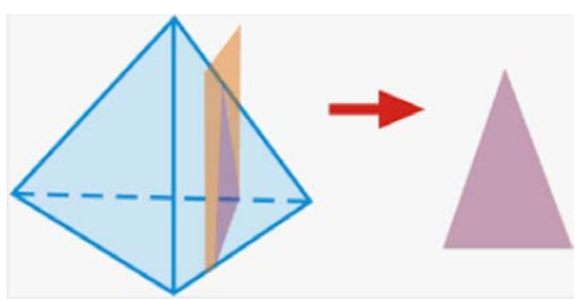


Figure 37

ACTIVITY 10

Objective: Study of regular polyhedra (cube)
Materials: Straws and plasticine/thread

What closed structure can be built using identical squares where every vertex has 3 adjacent squares?

Figure 38

Let students build 3 adjacent squares with straws to form a corner. Let them build more squares with three squares at each vertex.

They have formed a *Hexahedron* (cube).

Does it form a closed figure? Verify that there are 3 square faces at each vertex.

They can explore it and discover the relationship of the side to the surface area and volume of the cube.

They can look for a relationship between the side and any interior diagonal, and between the side and any face diagonal.

Does the cube have any plane symmetry?

Does it have any rotational symmetry? Through which points does the axis of symmetry pass through?

They can bring their understanding of coordinates to a 3-D object and describe various points in terms of coordinates.

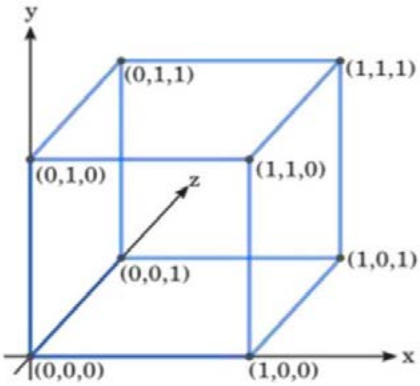


Figure 39

They can also design the net for a cube and fold it to make a solid shape.

ACTIVITY 11

Objective: Study of regular polyhedra (Octahedron)
Materials: Straws and plasticine/thread

What closed structure can be built with equilateral triangles where every vertex has 4 adjacent triangles?

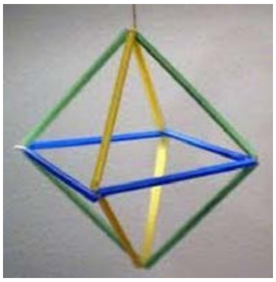


Figure 40

Let students build four equilateral triangles around a vertex. Let them build more triangles with four triangles meeting at each vertex.

A *regular octahedron* has been created. Verify that 4 faces meet at each vertex.

Does it have any plane symmetry?

Does it have any rotational symmetry? Through which points does the axis of symmetry pass through? How many such axes of symmetry does a regular octahedron have?

Cross sections

How would the cross sections (vertical and horizontal) of an octahedron look?

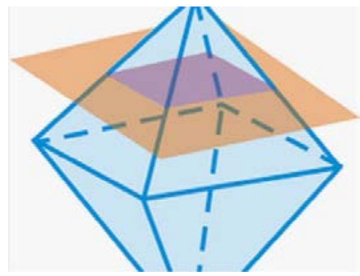


Figure 41

ACTIVITY 12

Objective: Study of regular polyhedra (Icosahedron)
Materials: Straws and plasticine/ thread

What closed structure can be built with equilateral triangles where 5 triangles meet at every vertex?

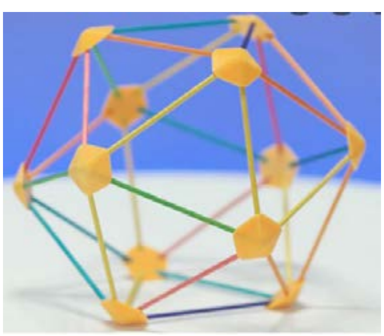


Figure 42

Let students build and join together 5 equilateral triangles, forming a convex shape. In successive steps, at each new vertex they can build 3 more equilateral triangles as shown in Figure 42.

Figure 43

The figure closes to form an *Icosahedron*.

Explore the structure for symmetry.

Building an icosahedron (indeed, building any of the regular polyhedra) using modular origami can be great fun (see Figure 43).

ACTIVITY 13

Objective: Study of regular polyhedra (Dodecahedron)
Materials: Straws and plasticine/thread

What closed structure can be built with regular pentagons where 3 pentagons meet at every vertex?

Let students build a regular pentagon and join together 5 regular pentagons on all its sides.

Figure 44

Figure 45

Three pentagons together can be joined to form a polyhedral vertex, but four pentagons together have a vertex angle that adds up to more than 360° making a concave vertex.

In successive steps, at five new vertices they can build five more pentagons as shown in Figures 44-45. The shape closes to form a *Dodecahedron*.

Findings

How many regular polyhedra are possible to build?

We have seen that it is possible to build regular polyhedra with either 3 or 4 or 5 equilateral triangles meeting at each vertex, but not with 6; with 3 squares meeting at each vertex, but not with 4; and with 3 regular pentagons meeting at each vertex, but not with 4 or more.

Is it possible to build a convex shape using only regular hexagons? Three regular hexagons make 360° which then create a flat vertex. Therefore, this is not possible.

Is it possible to build a convex shape using only regular polygons having more than 6 sides? Polygons with more than 6 sides have angles which exceed 120°, so it is not possible to join three of them together at a vertex. Therefore, this too is not possible.

Hence, it is possible to have only 5 regular solids.

Students can now create a table for the five regular polyhedra recording the number of faces, edges and vertices, and describe the face of each of the platonic solids.

Name of Polyhedron	Faces (F) Vertices (V)	Edges (E)	Tetrahe-dron
Tetrahedron	4	4	6

Students can now verify the relationship which they had noticed earlier between the vertices, faces and edges of a Polyhedra structure.

This is Euler's formula: $F + V = E + 2$ where F, V and E stand for the number of faces, vertices and edges of the polyhedron respectively.

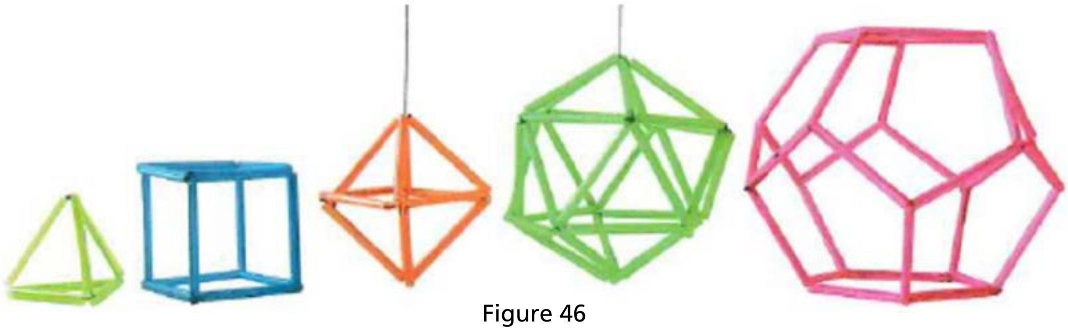


Figure 46

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- Teaching Mathematics with Art (ldlewis.com)
- Make Space: The Importance of Spatial Thinking for Learning Mathematics · Frontiers for Young Minds (frontiersin.org)
- How to teach ... 3D shapes | Teacher Network | The Guardian

Acknowledgements

I acknowledge with thanks feedback and suggestions received from Ms Swati Sircar.

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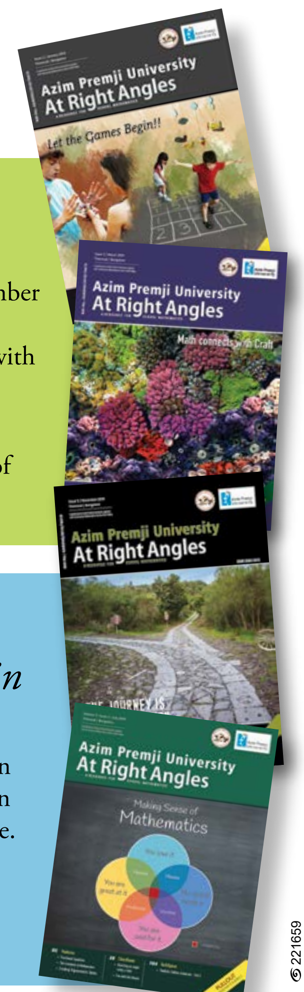

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